A Misuse of "Maxwell Theory"

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(February 10, 2022; updated February 12, 2022)

1 Problem

In 1989, in a paper on "relativity and electromagnetism", Rindler [1] considered the example where "an electric charge moving through the **B** field of a stationary magnet ... experiences a (Lorentz) force; referring our observations next to the instantaneous rest frame of the charge, we see that a stationary charge must experience a force when a magnet is moved in its vicinity".[1] He then added: "It is now natural to pose the problem whether these ... conclusions can be reached from Maxwell (*i.e.*, standard electromagnetic) theory *without* appeal to relativity."

Rindler did not answer this question, but a subsequent paper by Bartocci and Capria [4] claimed that, for Rindler's example, "Maxwell theory" is inconsistent with special relativity.

Can this be so, in view of the conventional wisdom that "Maxwell theory" implies the theory of special relativity?[2]

2 Solution

2.1 The Standard View

Neither of [1, 4] explicitly stated the standard view. In the lab frame where charge q has velocity $\mathbf{v}$ in a stationary magnetic field $\mathbf{B}$ (for which there is nominally no associated electric field $\mathbf{E}$)[3] the (Lorentz) force is (in Gaussian units, as used in [1]),

$$\mathbf{F} = q\frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{1}$$

where c is the speed of light in vacuum.

In the instantaneous inertial rest frame of charge q, the $'$ frame, the force on the charge is, with $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1$ at order v/c,

$$\mathbf{F}' = q\mathbf{E}' = \gamma q\frac{\mathbf{v}}{c} \times \mathbf{B} \approx \mathbf{F}, \tag{2}$$

recalling the Lorentz transformation of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$; the approximation holds for $v \ll c$ as considered in [4] and in the rest of this note.

[1]Rindler's nominal theme was the form of electromagnetism if magnetic monopole existed. For comments by the present author on this topic, see [2, 3].

[2]Maxwell's electrodynamics was the acknowledged inspiration to Einstein in his 1905 paper [5]. And in an unpublished review of the theory of relativity from 1920 [6], he stated "The special theory of relativity is nothing but a contradiction-free amalgamation of the results of Maxwell-Lorentz electrodynamics and those of classical mechanics".

[3]If the charge carriers of the conduction currents that generate the magnetic field have speed u, there is, in general, a tiny electric field in the lab frame, of order u^2/c^2. See, for example [7, 8].

In the standard view that the theory of special relativity follows from Maxwell's theory, eq. (2) follows from Maxwell's theory.

2.2 Rindler's "Trick"

Rindler argued in sec. II of [1] that the electric field experienced by the electron in its rest frame can be computed ("without appeal to relativity") from the lab-frame vector potential $\mathbf{A}$ via a "trick," *i.e.*, use of the convective derivative (called the Eulerian derivative by Rindler),

$$\mathbf{E} = -\frac{1}{c}\frac{d\mathbf{A}}{dt} = -\frac{1}{c}\left(\frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}\right) = -\frac{1}{c}(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}, \tag{3}$$

where the last form holds for a stationary magnet in the lab frame.

Now, identifying Rinlder's $\mathbf{E}$ of eq. (3) as the electric field $\mathbf{E}'$ in the instantaneous rest frame of the charge q is a kind of appeal to a theory of relativity, although this theory is not the theory of special relativity. In the latter (and in "Maxwell theory"), the electric field is related to the electromagnetic potentials V and $\mathbf{A}$ by,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{E}' = -\boldsymbol{\nabla}' V' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'}, \tag{4}$$

and the potentials V' and $\mathbf{A}'$ are related to the lab-frame V and $\mathbf{A}$ by a Lorentz transformation of the 4-vector $(V, \mathbf{A})$. While the scalar potential V is zero (to order u/c) in the lab frame for a stationary magnet, at order v/c where $\gamma \to 1$ the potentials V' and $\mathbf{A}'$ due to the moving magnet in the instantaneous rest frame of charge q are,[4]

$$V' = \gamma\left(V - \frac{\mathbf{v}}{c}\cdot\mathbf{A}\right) \qquad \Rightarrow \qquad V' = -\frac{\mathbf{v}}{c}\cdot\mathbf{A}, \tag{5}$$

$$\mathbf{A}'_\perp = -\mathbf{A}_\perp, \qquad \mathbf{A}'_\parallel = \gamma\left(\mathbf{A}_\parallel - \frac{\mathbf{v}}{c}V\right) \qquad \Rightarrow \qquad \mathbf{A}' = \mathbf{A}. \tag{6}$$

We must also consider the Lorentz transformation of spacetime derivatives, which involves the derivative 4-vector,[5]

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right), \tag{7}$$

for which the transformations are, to order v/c,[6]

$$\frac{1}{c}\frac{\partial}{\partial t'} = \gamma\left(\frac{1}{c}\frac{\partial}{\partial t} - \frac{\mathbf{v}}{c}\cdot(-\boldsymbol{\nabla})\right) \qquad \Rightarrow \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}, \tag{8}$$

$$-\boldsymbol{\nabla}'_\perp = -\boldsymbol{\nabla}_\perp, \qquad -\boldsymbol{\nabla}'_\parallel = \gamma\left(-\boldsymbol{\nabla}_\parallel - \frac{\mathbf{v}}{c}\frac{1}{c}\frac{\partial}{\partial t}\right) \qquad \Rightarrow \qquad \boldsymbol{\nabla}' = \boldsymbol{\nabla} + \frac{\mathbf{v}}{c^2}\frac{\partial}{\partial t}. \tag{9}$$

[4] The second form of our eq. (5) is Maxwell's eq. (6) in Art. 600 of [15].

[5] See, for example, p. 219 of http://kirkmcd.princeton.edu/examples/ph501/ph501lecture18.pdf

[6] The transformation (8) is essentially the convective derivative, but the theory of relativity (*i.e.*, "Maxwell theory") also includes the transformation (9) that was not considered by Rindler.

Hence, for stationary $\mathbf{A}$ (to order v/c),

$$\mathbf{E}' = -\boldsymbol{\nabla} V' - \frac{1}{c}\frac{\partial \mathbf{A}'}{\partial t'} = \frac{\boldsymbol{\nabla}(\mathbf{v}\cdot\mathbf{A})}{c} - \frac{(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}}{c} = \frac{(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}}{c} + \frac{\mathbf{v}}{c}\times(\boldsymbol{\nabla}\times\mathbf{A}) - \frac{(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}}{c}$$
$$= \frac{\mathbf{v}}{c}\times\mathbf{B}, \qquad (10)$$

in agreement with the Lorentz transformation of the fields $\mathbf{E}$ and $\mathbf{B}$ that was used in eq. (2).

Thus, we should expect that Rindler's "trick", which supposes $\mathbf{E}' = -(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}/c$, omitting the term $-\boldsymbol{\nabla} V'$, will not lead to the result that $\mathbf{F}' = \mathbf{F}$ (at order v/c) for his example.

2.3 Bartocci and Capria

Bartocci and Capria [4] pursued the use of Rindler's "trick", and concluded that it indeed implies $\mathbf{F}' \neq \mathbf{F}$ for Rindler's example (with $v \ll c$).[7] They also claimed that Rindler's "trick" is "the Maxwell solution" (as implied by Rindler in [1]).

Then, Bartocci and Capria noted that Rindler's "trick" should be augmented by $-\boldsymbol{\nabla} V$, seeming not to consider this part of "Maxwell theory", but rather an insight dependent on special relativity.[8]

Bartocci and Capria also claimed that "Maxwell theory" is inconsistent with special relativity in [11].

2.4 Is Rindler's "Trick" Part of "Maxwell Theory"?

In the view that "Maxwell theory" is Maxwell's equations (plus the Lorentz force law), Rindler's "trick" (which supposes that the electric field of a moving magnet is not related to a scalar potential but only to a vector potential) is not obviously part of that theory.

In his first paper on electromagnetism, Maxwell (1855), p. 64 of [12], did consider the use of the convective derivative to compute the force on a moving charge.[9] But, already in his second set of papers (1861), pp. 340-341 of [14], Maxwell gave a different argument that included the gradient of the electric scalar potential in computation, essentially arriving at the Lorentz force law.[10] Maxwell's argument was hard to follow, in part because his notation did not distinguish between $d\mathbf{A}/dt$ and $\partial\mathbf{A}/\partial t$.[11]

Maxwell gave somewhat different arguments in secs. 64-65 of [19], and in Arts. 598-600 of [15], which confirmed his view that computation of the force on an electric charge involves

[7] A subsequent paper by Redžić [9] noted that Bartocci and Capria made some technical errors in their argument, which did not affect their main conclusion.

[8] Further commentary on the paper of Bartocci and Capria was given by Rosser [10], who emphasized relativity rather than "Maxwell theory".

[9] See also Appendix A.28.1.7 of [13].

[10] See also Appendices A.28.2.6, A.28.3.7 and A.28.4.7 of [13].

[11] Helmholtz [eq. (5^d), p. 309 of [16] (1874)], Larmor [p. 12 of [17] (1884)], Watson [p. 273 of [18] (1888)), and J.J. Thomson [in his editorial note on p. 260 of [15] (1892)] argued that Maxwell's earlier view emphasizing use of the convective derivative was more correct than his later views (*i.e.*, the Lorentz force law) based on use of $\partial\mathbf{A}/\partial t$ rather than $d\mathbf{A}/dt$ in certain places. Rindler's "trick" seems in the spirit of these early, misguided commentaries.

derivatives of both the scalar potential V and the vector potential $\mathbf{A}$ (essentially in the form of the "Lorentz" force law).

That is, Maxwell understood Rindler's "trick" is only part of the story of the magnetic force on a moving charge.

A Appendix: Galilean Relativity

SI units are employed in this Appendix.

Maxwell himself did not arrive at the theory of special relativity, but did consider Galilean relativity in Art. 600-601 of [15] under the heading: *On the Modification of the Equations of Electromotive Intensity when the Axes to which they are referred are moving in Space.*[12]

The notion of Galilean electrodynamics, consistent with Galilean relativity, *i.e.*, coordinate transformations such as $x' = x - vt$, $y' = y$, $z' = z$, $t' = t$, seems to have been fully developed only in 1973 [21].

In Galilean electrodynamics there are no electromagnetic waves, but only quasistatic phenomena, so this notion is hardly compatible with Maxwellian electrodynamics as a whole. In contrast, electromagnetic waves can exist in the low-velocity approximation to special relativity, and, of course, propagate in vacuum with speed c.

In Galilean electrodynamics the symbol c does not represent the speed of light (as light does exist in this theory), but only the function $1/\sqrt{\epsilon_0\mu_0}$ of the (static) permittivity and permeability of the vacuum.

In fact, there are two variants of Galilean electrodynamics:

1. Electric Galilean relativity (for weak magnetic fields) in which the transformations between two inertial frames with relative velocity $\mathbf{v}$ are (sec. 2.2 of [21]),

$$\rho'_e = \rho_e, \qquad \mathbf{J}'_e = \mathbf{J}_e - \rho_e \mathbf{v}, \qquad (c\,|\rho_e| \gg |J_e|), \qquad V'_e = V_e, \qquad \mathbf{A}'_e = = \frac{\mathbf{v}}{c^2} V_e, \tag{11}$$

$$\mathbf{E}'_e = \mathbf{E}_e, \qquad \mathbf{B}'_e = \mathbf{B}_e - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_e \qquad \mathbf{f}_e = \rho_e \mathbf{E}_e \qquad \text{(electric)}, \tag{12}$$

where ρ and $\mathbf{J}$ are the electric charge and current densities, V and $\mathbf{A}$ are the electromagnetic scalar and vector potentials, $\mathbf{E} = -\boldsymbol{\nabla} V - \partial \mathbf{A}/\partial t$ is the electric field, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ is the magnetic (induction) field,.

2. Magnetic Galilean relativity (for weak electric fields, sec. 2.3 of [21]) with transformations,

$$\rho'_m = \rho_m - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}_m, \quad \mathbf{J}'_m = \mathbf{J}_m, \quad (c\,|\rho_e| \ll |J_e|), \quad V'_m = V_m - \frac{\mathbf{v}}{c^2} \cdot \mathbf{A}_m, \quad \mathbf{A}'_m = \mathbf{A}_m, \tag{13}$$

$$\mathbf{E}'_m = \mathbf{E}_m + \mathbf{v} \times \mathbf{B}_m, \qquad \mathbf{B}'_m = \mathbf{B}_m \qquad \mathbf{f}_m = \rho_m (\mathbf{E}_m + \mathbf{v} \times \mathbf{B}_m) \quad \text{(magnetic)}. \tag{14}$$

For comparison, the low-velocity limit of special relativity has the transformations,

$$\rho'_s \approx \rho_s - \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}_s \qquad \mathbf{J}'_s \approx \mathbf{J}_s - \rho_s \mathbf{v}, \qquad V'_s \approx V_s - \mathbf{v} \cdot \mathbf{A}_s, \qquad \mathbf{A}'_s \approx \mathbf{A}_s - \frac{\mathbf{v}}{c^2} V_s, \tag{15}$$

$$\mathbf{E}'_s \approx \mathbf{E}_s + \mathbf{v} \times \mathbf{B}_s, \qquad \mathbf{B}'_s \approx \mathbf{B}_s - \frac{\mathbf{v}}{c^2} \times \mathbf{E}_s \qquad \text{(special relativity, } v \ll c). \tag{16}$$

In the example of Rindler, where the issue is the transformation of the fields of a magnet between two frame, one should consider magnetic Galilean relativity, not electric Galilean

[12]The term Galilean relativity was first used in 1911 [20].

relativity. However, Rindler's "trick" does not correspond to either form of Galilean relativity (nor with special relativity).

In contrast, Maxwell's consideration of Galilean relativity in Arts. 600-601 is consistent with magnetic Galilean relativity (as well as with special relativity).

Furthermore, in Arts. 769-770 of [15], Maxwell elaborated on a remark by Faraday, Art. 1644 of [22], that a moving electric charge has the effect of an electric current, (*i.e.*, generates a magnetic field). Maxwell's argument can be transcribed as,

$$\mathbf{B} = \frac{\mathbf{v}}{c^2} \times \mathbf{E}, \tag{17}$$

for the magnetic field experienced by a fixed observer due to a moving charge. Maxwell noted that this is a very small effect, and claimed (1873) that it had never been observed.[13]

If $\mathbf{v}$ represents the velocity of a moving observer relative to a fixed electric charge, then eq. (17) implies that the magnetic field experienced by the moving observer would be,

$$\mathbf{B}' = -\frac{\mathbf{v}}{c^2} \times \mathbf{E}, \tag{18}$$

This corresponds to the low-velocity limit (4) of special relativity, and to form (2) of electric Galilean relativity).

It remains that while Maxwell used Galilean transformations as the basis for his considerations of fields and potentials in moving frames, he was rather deft in avoiding the contradictions between Galilean electrodynamics and his own vision.[14]

Maxwell did not note the incompatibility (at order v^2/c^2) of his use of Galilean transformations in his Arts. 601-602 and 769-770 with his system of equations for the electromagnetic fields, but if he had, he might have mitigated this issue by deduction of self-consistent transformations for both $\mathbf{E}$ and $\mathbf{B}$ between the lab frame and a uniformly moving frame (*i.e.*, the full field transformations of special relativity), as in Appendix C of [29].

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[3] K.T. McDonald, *Comay's Paradox: Do Magnetic Charges Conserve Energy?* (June 1, 2015), http://kirkmcd.princeton.edu/examples/comay.pdf

[13]The magnetic field of a moving charge was detected in 1876 by Rowland [23, 24] (while working in Helmholtz' lab in Berlin). The form (8) was verified (in theory) more explicitly by J.J. Thomson in 1881 [25] for uniform speed $v \ll c$, and for any $v < c$ by Heaviside [26] and by Thomson [27] in 1889 (which latter two works gave the full special-relativistic form for $\mathbf{E}$ as well).

[14]For a contrast, in which use of Galilean transformations for electrodynamics by J.J. Thomson [28] (1880) led to a result in disagreement with Nature (unrecognized at the time), see Appendix B of [29].

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