Rayleigh's Spinning Ring

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1 Problem

A uniform steel wire in the form of a circular ring is made to revolve in its own plane about its center of figure. Show that the greatest possible linear velocity is independent of both the cross section of the wire and of the radius of the ring, and find roughly this velocity, the breaking strength of the wire being given as 400 MPa and the density of steel being 7.8 g/cm^3.

Posed by Lord Rayleigh on the 1876 Cambridge Mathematical Tripos[1] *(J.H. Poynting Third Wrangler),*[2] *and recalled by P. Nahin in the Introduction to his forthcoming book* Nastyglass and Other Puzzles of Mathematical Physics *(Princeton U. Press, 2017).*[3]

2 Solution

Denoting the tension in the spinning ring by T, the force holding two half-rings together is $2T$. This force provides the centripetal acceleration $a = \omega^2 b$ of a half ring, where ω is the angular velocity of the ring, and b is the distance from the center of the ring to the center of mass of a half-ring.

A half-ring has mass $m \approx \pi r A\rho$, where r is the radius of the ring, A is the cross sectional area of the wire/ring, and ρ is its mass density. The distance b of the center of mass of a half-ring from the center of the ring is related by,

$$b = \frac{1}{m}\int r\sin\theta\, dm \approx \frac{1}{m}\int_0^\pi (r\sin\theta) rA\rho\, d\theta = \frac{2}{\pi}\frac{\pi r^2 A\rho}{m} = \frac{2}{\pi} r, \tag{1}$$

where θ is the angle between the diameter of the half-ring and a volume element $rA\,d\theta$. Hence,

$$2T = m\omega^2 b \approx \pi r A\rho\omega^2 \frac{2r}{\pi} = 2A\rho(\omega r)^2 = 2A\rho v^2, \qquad v \approx \sqrt{\frac{T}{\rho A}} = \sqrt{\frac{T}{\lambda}}, \tag{2}$$

where $v = \omega r$ is the linear velocity of a point on the spinning ring, and $\lambda = \rho A$ is the linear mass density of the wire. The various approximations hold for $A \ll r^2$.

The velocity of a point on the ring equals the velocity $\sqrt{T/\lambda}$ of transverse waves on the ring. A perturbation to the shape of the ring leads to a pulse that propagates with respect

[1] J.W. Strutt, *Scientific Papers* Vol. 1 (Cambridge U. Press, 1899), Question vi, p. 280, http://kirkmcd.princeton.edu/examples/EM/strutt_papers_v1.pdf

[2] See also O. Lodge, *The Flying Pieces of a Whirling Ring*, Nature **43**, 439 (1891), http://kirkmcd.princeton.edu/examples/mechanics/lodge_nature_43_439_91.pdf.

[3] See also, P.J. Nahin, *Rayleigh's Rotating Ring*, Math. Intell. **39** (2), 72 (2017), http://kirkmcd.princeton.edu/examples/mechanics/nahin_mi_39(2)_72_17.pdf.

to the lab frame at angular velocity 2ω, and another pulse that appears to be at rest in the lab while the (perturbed) ring rotates through it. The latter phenomenon is well known in rope tricks.[4]

Returning the problem as posed, the breaking strength (with dimensions of pressure) is $T_{\rm max}/A$, so the maximum velocity before breaking is,

$$v_{\rm max} = \sqrt{\frac{T_{\rm max}/A}{\rho}} = \sqrt{\frac{4 \times 10^8}{7.8 \times 10^3}} \approx 225 \text{ m/s}. \tag{3}$$

In general, the velocity v of eq. (2) is a function of T, ρ and A, but not r. For the limiting case of maximal tension $T_{\rm max}$ before breaking, the ratio $T_{\rm max}/A$ is a constant = the breaking strength of the material, and hence $v_{\rm max}$ is independent of A (and of r).[5]

The above analysis was made in a frame in which the axis of the ring is at rest. For a wheel that rolls without slipping with velocity (of its axis) v, the velocity of the rim in the frame in which the axis is at rest is also v.

The largest drive wheels on steam locomotives had radii of about 1 meter, so an implication of eq. (3) is that these wheels would break apart at speeds of about 500 mph.

A rocket car[6] designed for speeds of 1,000 mph has aluminum wheels (density 2.7 g/cm^3, nominal breaking strength $\approx$ 300 MPa) of radius 0.46 m; according to eq. (3) such wheels, if made from typical aluminum, would fail at speed 333 m/s = 750 mph. However, special Al alloys, such as 7068, can have breaking strengths of 700 MPa, corresponding to maximum speed of 1750 mph.

[4]See, for example, http://kirkmcd.princeton.edu/examples/lasso.pdf.

[5]The maximum velocity depends on the type of material; for example, $v_{\rm max} \approx 333$ m/s for aluminum.

[6]http://www.bloodhoundssc.com/project/facts-and-figures