Rocket Car

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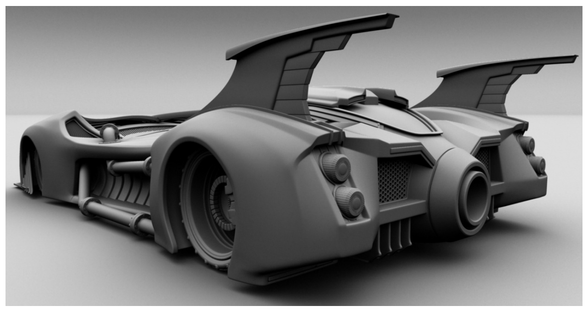
(October 1, 2012)

1 Problem

What is the maximum (linear) acceleration of a rocket-propelled car (Batmobile?), traveling on the Earth's surface, such that all four of its wheels stay in contact with the ground and roll without slipping? Compare with the case of a car with a rear-wheel motor drive.



You may ignore air resistance.

For the related issue of maximal deceleration, see prob. 2 of
http://kirkmcd.princeton.edu/examples/ph205set1.pdf

2 Solution

2.1 Rocket-Propelled Car

The car has total mass M, and the rocket produces (horizontal) thrust $\mathbf{F} = F\,\hat{\mathbf{x}}$ on the car, which is applied at height $\mathbf{H} = H\,\hat{\mathbf{y}}$ above the road. The center of mass of the car is at height $\mathbf{h} = h\,\hat{\mathbf{y}}$ above the road, and is at horizontal distances d_F and d_R from vertical axes through the front and rear wheels, respectively (and between them). These wheels have masses m_F and m_R, radii r_F and r_R, and moments of inertia $I_F = k_F m_F r_F^2$ and $I_R = k_R m_R r_R^2$. The coefficients of static friction of the wheels and the road are μ_F and μ_R.

When the car has linear acceleration $\mathbf{a} = a\,\hat{\mathbf{x}}$, and the wheels roll without slipping, they have angular accelerations,

$$\boldsymbol{\alpha}_F = -\frac{a}{r_F}\,\hat{\mathbf{z}}, \qquad \text{and} \qquad \boldsymbol{\alpha}_R = -\frac{a}{r_R}\,\hat{\mathbf{z}}. \tag{1}$$

For the wheels to roll without slipping they must be subject to torques about their axles,

$$\boldsymbol{\tau}_F = I_F\,\boldsymbol{\alpha}_F = r_F F_F\,\hat{\mathbf{z}}, \qquad \text{and} \qquad \boldsymbol{\tau}_R = r_R F_R\,\hat{\mathbf{z}}, \tag{2}$$

where the forces of static friction of the road on the wheels are $\mathbf{F}_F = F_F\,\hat{\mathbf{x}}$ and $\mathbf{F}_R = F_R\,\hat{\mathbf{x}}$. Hence,

$$F_F = -\frac{I_F\,a}{r_F^2} = -k_F m_F\,a, \qquad \text{and} \qquad F_R = -k_R m_R\,a. \tag{3}$$

The total horizontal force on the (four-wheeled) car has magnitude,

$$F_{\rm tot} = F + 2F_F + 2F_R = F - 2(k_F m_F + k_R m_R)a = Ma, \tag{4}$$

so the rocket thrust F is related to the acceleration a by,

$$F = [M + 2(k_F m_F + k_R m_R)]a. \tag{5}$$

The magnitudes of the frictional forces are bounded by,

$$|F_F| \le \mu_F N_F, \qquad \text{and} \qquad |F_R| \le \mu_R N_R, \tag{6}$$

where N_F and N_R are the (upward) normal forces on the front and rear wheels. Of course, the total upward normal force balances the downward force of gravity on the car,

$$2N_F + 2N_R = Mg, \tag{7}$$

where g is the acceleration due to gravity at the Earth's surface. For the car not to "flip" (and for all wheels to stay in contact with the road), the total torque about its center of mass must be zero,

$$\tau_{\rm cm} = 0 = F(H - h) + 2N_R d_R - 2N_F d_F - 2(F_F + F_R)h, \tag{8}$$

or,

$$2N_F d_F - 2N_R d_R = Ca, \tag{9}$$

where the constant C,

$$C = H[M + 2(k_F m_F + k_R m_R)] - hM, \tag{10}$$

could have either sign. Thus, the normal forces are,

$$N_F = \frac{d_R Mg + Ca}{2(d_F + d_R)}, \qquad \text{and} \qquad N_R = \frac{d_F Mg - Ca}{2(d_F + d_R)}. \tag{11}$$

These must both be positive for the wheels to be in contact with the road, which implies a limit on the acceleration a,

$$a \le \begin{cases} d_R Mg/C & (C > 0), \\ d_F Mg/|C| & (C < 0). \end{cases} \tag{12}$$

The bounds (6) on the frictional forces can now be written as,

$$|F_F| = k_F m_F a \le \mu_F \frac{d_R Mg + Ca}{2(d_F + d_R)}, \qquad \text{and} \qquad |F_R| = k_R m_R a \le \mu_R \frac{d_F Mg - Ca}{2(d_F + d_R)}. \tag{13}$$

That is,

$$a \le \frac{\mu_F d_R M}{2(d_F + d_R)k_F m_F - \mu_F C}\, g, \qquad \text{and} \qquad a \le \frac{\mu_R d_F M}{2(d_F + d_R)k_R m_R + \mu_R C}\, g. \tag{14}$$

In the special case that $C = 0$, the limits on the acceleration are,

$$a \le \frac{\mu_F d_R}{2(d_F + d_R)k_F} \frac{M}{m_F}\, g, \qquad \text{and} \qquad a \le \frac{\mu_R d_F}{2(d_F + d_R)k_R} \frac{M}{m_R}\, g \qquad (C = 0), \tag{15}$$

so that the acceleration is limited to g times a quantity of the order of the ratio of the total mass of the car to the mass of a wheel, which is large compared to g, in general.

2.2 Rear-Wheel-Drive Car

When the car is driven through the rear (or front or all) wheels, friction plays a different role. Ignoring air resistance, the only external, horizontal forces on the care are those due to friction between the wheels and the road. The total frictional force equals the acceleration of the car times its total mass,

$$2F_F + 2F_R = Ma, \tag{16}$$

so the friction of the drive wheels must point forward, while the friction at the free wheels points backwards (as considered in sec. 2.1). Each wheel must still be subject to a torque $I\boldsymbol{\alpha}$, which is in the opposite direction to the torque $r_R F_R\,\hat{\mathbf{z}}$ caused by the friction on the (rear) drive wheels. There must be an additional torque, of opposite sign, on the drive wheels, provided by the transmission that links the rear axle to the motor.

As found in sec. 2.1, the maximum acceleration of the car without slipping of free wheels is much larger than g, so we anticipate that friction at the (rear) drive wheels is the limit in the present case. That is, we can assume that the limit is,

$$\mathbf{F}_{R,\mathrm{max}} = \mu_R N_R\,\hat{\mathbf{x}}, \tag{17}$$

while the front (free) wheel obeys eq. (3),

$$\mathbf{F}_F = -\frac{I_F\,a}{r_F^2}\,\hat{\mathbf{x}} = -k_F m_F\,a\,\hat{\mathbf{x}}. \tag{18}$$

To determine the normal force N_R on a rear wheel in eq. (17) we again note that (ignoring effects of air friction), the total upward normal force balances the downward force of gravity on the car,

$$2N_F + 2N_R = Mg, \tag{19}$$

For the car not to “flip” (and for all wheels to stay in contact with the road), the total torque about its center of mass must be zero,

$$\tau_{\mathrm{cm}} = 0 = 2N_R d_R - 2N_F d_F - 2(F_F + F_R)h, \tag{20}$$

or,

$$2N_R(d_R - \mu_R h) - 2N_F d_F = -2k_F m_F a h. \tag{21}$$

Thus, the normal forces are,

$$N_F = \frac{(d_R - \mu_R h)Mg - 2k_F m_F a h}{2(d_F + d_R - \mu_R h)}, \qquad \text{and} \qquad N_R = \frac{d_F Mg - 2k_F m_F a h}{2(d_F + d_R - \mu_R h)}. \tag{22}$$

These must both be positive for the wheels to be in contact with the road, which is readily satisfied for N_R, but which requires $d_R - \mu_R h$ to be not too small to keep $N_F > 0$. That is, the center of mass should not be too close to the rear wheels or the front wheels will lift off the ground during a large acceleration a.

Assuming that the wheels stay on the ground,[1] we finally obtain the maximum acceleration for a rear-wheel drive car from eq. (16) using eqs. (17)-(18) and (22),

$$a \leq \frac{\mu_R d_F M}{M(d_F + d_R - \mu_R h) - \mu_R k_F m_f h} g \approx \mu_R g. \tag{23}$$

Comparing with sec. 2.1, we see that the maximum possible acceleration (when all wheels roll without slipping) is much higher for a rocket-propelled car than for one with rear-wheel drive.[2]

Appendix A: Single, Rocket-Propelled Wheel


Key features of the solution given in sec. 2.1 can be understood from consideration of a single wheel of mass m, radius r, moment of inertia $I = kmr^2$, coefficient of static friction μ, with horizontal acceleration a of its center of mass due to rocket propulsion. For rolling without slipping, the wheel has angular acceleration $\alpha = a/r$, and the torque equation is,

$$\tau = I\alpha = kmra \leq \mu N = \mu rmg, \tag{24}$$

so the acceleration is limited to $a \leq \mu g/k$, which is of order g.

[1] If the front wheels leave the ground the analysis of Appendix B holds, $a_{\rm max} = \mu_R g$, which may be advantageous so long as the car doesn't flip over.

[2] Since forward acceleration reduces the normal force on the front wheels, a front-wheel-drive car has lower maximum acceleration (with wheels that roll without slipping) than a rear-wheel-drive car.

If the wheel were part of a unicycle of total mass M, the torque equation would be,

$$\tau = I\alpha = kmra \leq \mu N = \mu rMg, \tag{25}$$

and the acceleration would be limited to $a \leq \mu Mg/km$, which is of order g times the ratio of the total mass to the mass of the wheel.

Appendix B: Single, Motor-Driven Wheel

Key features of the solution given in sec. 2.2 can be understood from consideration of a single wheel + motor of total mass M, coefficient of static friction μ, with horizontal acceleration a of its center of mass due to the motor drive on the wheel (which motor is somehow stabilized as in a Segway i2, shown below).

The acceleration a is due to the frictional force $F \leq \mu N = \mu Mg$, so,

$$F = Ma \leq \mu mg, \tag{26}$$

and the acceleration is limited to $a \leq \mu g$, which is of order g.