

# Rolling Water Pipe

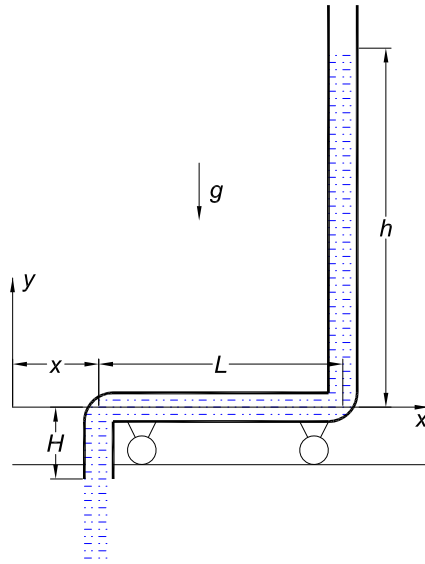
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## 1 Problem

Discuss the motion of a zig-zag-shaped pipe of mass  $m$  that can roll (or slide) without friction on a horizontal surface, supposing that the pipe is initially filled with water of density  $\rho$  to height  $h_0$  above the horizontal segment (of length  $L$ ) of the pipe. The water exits the pipe from a vertical segment of height  $H$  below the horizontal segment, where  $H$  is large enough that the exit velocity of the water from the pipe is vertical in the rest frame of the pipe.



*This variant of the leaky-tank-car problem [1] was suggested by Johann Otto.<sup>1</sup>*

## 2 Solution

Energy and horizontal momentum are conserved in this problem. The center of mass of the pipe + water is initially at rest, and the horizontal coordinate,  $x_{\text{cm}}$ , of the center of mass remains constant as all times.

As the water first begins to flow, water moves from the right vertical pipe to the left vertical pipe, from which it exits. This action moves the center of mass to the left with respect to the pipe. Hence, the initial horizontal velocity of the pipe must be to the right so that the center of mass remains fixed.

As the water exits the pipe with downward vertical velocity in frame of the pipe, the exiting water has horizontal velocity to the right. And, the pipe has horizontal velocity to the right. Meanwhile, water flows to the left in the horizontal segment of the pipe, such that the velocity of the center of mass of the system remains zero.

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<sup>1</sup>Feb. 8, 2022. A version of this problem was first posed (and solved) by E.B. Wilson in 1910 [2].

However, if this qualitative situation remained until all the water exited the pipe, after which no more water flows to the left inside the pipe, the center of mass of the system would be moving to the right, rather than being at rest.

Hence, the velocity of the pipe must reverse direction by the time that all the water has drained out of the pipe.

We now deduce equations of motion for the pipe + water using conservation of momentum and energy. For motion that begins at time  $t = 0$ , the left side of the pipe is at  $x(t)$  and the height of the water in the right vertical segment of the pipe is  $h(t)$  above the horizontal segment (until  $h$  goes to zero, after which  $h$  is no longer meaningful). The water that has exited the pipe has a complicated pattern whose details we can omit by consideration only of the momentum and energy of the pipe + water therein at time  $t$ .

## 2.1 Momentum

The pipe has mass  $m$  and cross sectional area  $A$ , and the water has density  $\rho$ . The horizontal velocity of the pipe is  $dx/dt = \dot{x}$ , and the velocity of the water inside the pipe relative to it is  $dh/dt = \dot{h}$ . The mass of the pipe + water inside it is  $m_{\text{in}} = m + m_{\text{water,in}}$ , where  $m_{\text{water,in}} = \rho A(H + L + h)$ , and  $\dot{m}_{\text{in}} = \dot{m}_{\text{water,in}} = \rho A \dot{h}$ , so long as  $h \geq 0$ .

Then, the horizontal momentum of the pipe + water therein at time  $t$  is,

$$p_{\text{in},x} = [m + \rho A(h + H)]\dot{x} + \rho AL(\dot{x} + \dot{h}) = [m + \rho A(h + H + L)]\dot{x} + \rho AL\dot{h}, \quad (1)$$

since the water in the horizontal section of pipe has mass  $\rho AL$  and horizontal velocity  $\dot{x} + \dot{h}$ . The horizontal momentum of the pipe + water still therein at time  $t + dt$  has changed by amount,

$$dp_{\text{in},x} = [m + \rho A(h + H + L)]\ddot{x} dt + \rho A\dot{h} dt \dot{x} + \rho AL\ddot{h} dt, \quad (2)$$

to order  $dt$ , while the horizontal momentum of the water newly outside the pipe is,

$$dp_{\text{out},x} = -\rho A\dot{h} dt \dot{x}, \quad (3)$$

since the water that exits the pipe has horizontal velocity  $\dot{x}$  in the lab frame, and the volume of newly exited water is  $-A\dot{h} dt$ .

If the pipe rolls without friction, horizontal momentum is conserved,  $dp_{\text{in},x} + dp_{\text{out},x} = 0$ , such that,<sup>2</sup>

$$[m + \rho A(h + H + L)]\ddot{x} = m_{\text{in}}\ddot{x} = -\rho AL\ddot{h}. \quad (5)$$

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<sup>2</sup>Alternatively [1], we could analyze the horizontal position  $x_{\text{cm}}$  of the system which must be fixed if the pipe rolls/slides without friction.

We take the left end of the pipe to start from rest at  $x = 0$  at time  $t = 0$ , such that at time  $t$ ,

$$m_{\text{total}}x_{\text{cm}} = m(x + \tilde{x}_p) + \rho AHx + \rho AL(x + L/2) + \rho Ah(x + L) + \int_0^t (-\rho A\dot{h} dt')X(t, t'), \quad (4)$$

where  $\tilde{x}_p$  is the distance of the center of mass of the pipe from its left end, and  $X(t, t') = x(t') + \dot{x}(t')(t - t')$  is the present position of the element of water of mass  $-\rho A\dot{h}(t') dt'$  that left the pipe at time  $t'$  with zero relative horizontal velocity, and hence with lab-frame horizontal velocity  $\dot{x}(t')$ . Taking the time derivative of this we find eq. (7), and taking the time derivative of eq. (7) we find eq. (5).

In an  $F = ma$  interpretation, the horizontal acceleration  $\ddot{x}$  of the pipe + water inside is due to the reaction force associated with the horizontal acceleration  $-\dot{h}$  relative to the pipe of the mass  $\rho AL$  of water in the horizontal segment of the pipe. Alternatively, in a view that emphasizes the pipe, we could write eq. (5) as,

$$m\ddot{x} = -m_{\text{water,in}}\ddot{x} - \dot{m}_{\text{water,in}}\dot{x} - \rho AL\ddot{h} + \dot{m}_{\text{water,in}}\dot{x}, \quad (6)$$

where  $-m_{\text{water,in}}\ddot{x}$  is the inertial force of the water in the pipe back on the walls of the pipe that are accelerating the water;  $-\dot{m}_{\text{water,in}}\dot{x}$  is a correction to the inertial force of the water because the amount of water inside the pipe is changing;  $-\rho AL\ddot{h}$  is the force of the water on the left bend of the pipe where the horizontal velocity water relative to the pipe is reduced from  $\dot{h}$  to zero; and  $\dot{m}_{\text{water,in}}\dot{x}$  is the reaction force of the water that leaves the pipe back on the pipe. Because the water leaves the pipe with zero relative horizontal velocity, the reaction force of the water leaving the pipe exactly cancels the correction to the inertial force of the water left in the pipe.

A formal time integration of eq. (5) yields (using an integration by parts), for the system starting from rest at  $t = 0$ ,

$$m_{\text{in}}\dot{x} - \int_0^t \dot{m}_{\text{in}}(t')\dot{x}(t') dt' = -\rho LA\dot{h}, \quad (7)$$

$$\dot{x} = -\frac{\rho AL}{m_{\text{in}}}\dot{h} + \frac{\rho A}{m_{\text{in}}} \int_0^t \dot{h}(t')\dot{x}(t') dt', \quad (8)$$

recalling that  $\dot{m}_{\text{in}} = \rho A\dot{h}$ . At time  $t = 0^+$ , when the water starts to drain out of the pipe, the integral in eq. (8) is negligible and we have that,

$$\dot{x}(0^+) = -\frac{\rho AL}{m_{\text{in}}}\dot{h}(0^+) > 0, \quad (9)$$

since  $\dot{h}$  is always negative as the water drains. That is, the pipe initially moves to the right, as the water drains out of the left end of the pipe, such that the center of mass of the system stays at rest.

As time increases, the first term on the right of eq. (8) remains positive, while the second term becomes increasingly negative, such that the sign of  $\dot{x}$  can reverse. Indeed, we noted at the beginning of sec. 2 that the sign of  $\dot{x}$  must eventually reverse, as the momentum of the system would eventually be nonzero and positive if  $\dot{x}$  is always positive. However, it can still be that the height  $h$  of the water in the right vertical pipe segment goes to zero before  $\dot{x}$  changes sign.

## 2.2 Energy

The kinetic energy of the pipe + water therein at time  $t$  in the lab frame is,

$$T_{\text{in}} = [m + \rho A(h + H)]\frac{\dot{x}^2}{2} + \rho A(h + H)\frac{\dot{h}^2}{2} + \rho AL\frac{(\dot{x} + \dot{h})^2}{2}. \quad (10)$$

The kinetic energy of the pipe + water still therein at time  $t + dt$  has changed by amount,

$$dT_{\text{in}} = [m + \rho A(h + H)]\dot{x}\ddot{x} dt + \rho A(h + H)\dot{h}\ddot{h} dt + \rho A\dot{h} dt \frac{\dot{x}^2 + \dot{h}^2}{2} + \rho AL(\dot{x} + \dot{h})(\ddot{x} + \ddot{h}) dt, \quad (11)$$

to order  $dt$ , while the kinetic energy of the water newly outside the pipe is,

$$dT_{\text{out}} = -\rho A\dot{h} dt \frac{\dot{x}^2 + \dot{h}^2}{2}. \quad (12)$$

Assuming that no dissipative forces are present, the change in kinetic energy equals to work  $W$  done on the system by gravity as mass element  $-\rho A\dot{h}$  is transferred from height  $h$  to height  $-H$ ,<sup>3</sup>

$$dT_{\text{in}} + dT_{\text{out}} = W = -\rho A\dot{h}g(h + H) = d\text{Potential Energy}. \quad (13)$$

That is,

$$[m + \rho A(h + H + L)]\dot{x}\ddot{x} + \rho A(h + H + L)\dot{h}\ddot{h} + \rho AL(\dot{x}\ddot{h} + \dot{h}\ddot{x}) + \rho A(h + H)g\dot{h} = 0. \quad (14)$$

The first term can be eliminated using the momentum equation (5), leading to,<sup>4,5</sup>

$$\ddot{x} = -\frac{h + H + L}{L}\ddot{h} - \frac{h + H}{L}g \quad (20)$$

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<sup>3</sup>Equivalently, mass of water  $\rho A(h + H)$  drops by distance  $-\dot{h} dt$ , and the work done by gravity is again  $\rho A(h + H)g(-\dot{h} dt)$ .

<sup>4</sup>If  $L = 0$ , the system is simply a vertical pipe with vertically falling water, and eq. (20) reduces to  $\ddot{h} = -g$  as expected.

<sup>5</sup>We could also work in the accelerated frame of the pipe, if we take into account the apparent horizontal “coordinate” force  $-M\ddot{x}$  on any mass  $M$  in the accelerated frame. The kinetic energy of the pipe + water therein at time  $t$  in the accelerated frame is,

$$T_{\text{in}}^* = +\rho A(h + H)\frac{\dot{h}^2}{2} + \rho AL\frac{\dot{h}^2}{2}. \quad (15)$$

The kinetic energy of the pipe + water still therein at time  $t + dt$  has changed by amount,

$$dT_{\text{in}}^* = \rho A(h + H)\dot{h}\ddot{h} dt + \rho A\dot{h} dt \frac{\dot{h}^2}{2} + \rho AL\dot{h}\ddot{h} dt, \quad (16)$$

to order  $dt$ , while the kinetic energy of the water newly outside the pipe is,

$$dT_{\text{out}}^* = -\rho A\dot{h} dt \frac{\dot{h}^2}{2}. \quad (17)$$

Assuming that no dissipative forces are present, the change in kinetic energy equals the work  $W$  done by gravity as mass element  $-\rho A\dot{h}$  is transferred from height  $h$  to height  $-H$ , plus the work done by the coordinate force  $-M\ddot{x}$  on the mass of water  $M = \rho AL$  that moves horizontal distance  $\dot{h} dt$ ,

$$dT_{\text{in}}^* + dT_{\text{out}}^* = W^* = -\rho A\dot{h}g(h + H) - \rho AL\ddot{x}\dot{h} dt. \quad (18)$$

That is,

$$\rho A(h + H + L)\dot{h}\ddot{h} + \rho AL\dot{h}\ddot{x} + \rho A(h + H)g\dot{h} = 0, \quad (19)$$

which becomes eq. (20) after dividing by  $\rho AL\dot{h}$ . This equation can be regarded as an example of the so-called extended Bernoulli equation, eq. (12) of [3], for nonsteady fluid flow in possibly accelerating frames.

A formal integration of eq. (20), for a system that starts from rest at  $t = 0$ , is,

$$\dot{x} = -\frac{h + H + L}{L}\dot{h} + \frac{1}{L}\int_0^t \dot{h}^2(t') dt - \frac{H}{L}gt - \frac{g}{L}\int_0^t h(t') dt'. \quad (21)$$

At a small time  $\epsilon$  after the motion begins,  $\dot{h}$  and  $\dot{x}$  are still small, so the first integral in eq. (21) can be neglected, and,

$$\dot{x}(t = \epsilon) \approx -\dot{h} - \frac{h + H}{L}(\dot{h} + \epsilon g). \quad (22)$$

However, we cannot make a crisp inference from this as the small value of  $\dot{h}$  at early times is not yet determined.

Instead, we combine the two equations (5) and (20) for  $\ddot{x}$ , obtaining (for  $h > 0$ ),

$$\ddot{h} = -\frac{m_{\text{in}}\rho A(h + H)}{m_{\text{in}}^2 - (\rho AL)^2}g, \quad (23)$$

which is negative so long as any water remains in the right vertical segment of the tube. Then, eq. (5) tells us that  $\ddot{x}$  is positive when  $h > 0$ , and since  $\dot{x}(0^+) > 0$ , as in eq. (9), the velocity  $\dot{x}$  of the pipe remains positive for  $h > 0$ .

We noted at the beginning of sec. 2 that the velocity of the pipe must change sign eventually, but we now understand that this does not happen until after the water has completely drained from the right vertical pipe segment.

Once this has happened, we can consider that the water separates from the right “elbow” in the pipe, and the water no longer exerts a force to the right on the pipe. The now rapidly moving water continues to impact on the left “elbow” of the pipe, which decelerates the pipe and changes the sign of  $\dot{x}$  before the water drains past that “elbow”.

That is, the most dramatic behavior in this problem occurs only after the height  $h$  has gone to zero. In principle we can deduce the equations of motion after this event, but as we don’t know the water speed when  $h$  goes to 0, we cannot complete the solution analytically.

In contrast, the approximations of the leaky-tank-car problem [1] permit a simpler analytic discussion of the time dependence and reversal of the velocity  $\dot{x}$  of the tank car as water drains out from it.

*Oct. 1, 2023. An analysis of the motion after the water has drained from the vertical section of the pipe has been carried out in [10] (where this behavior is called Phase 2), and exhibits a change in the sign of the horizontal velocity.*

## A Appendix: A Lagrangian Analysis

A Lagrangian approach to variable-mass problems has been given in [4, 5], which was applied in Appendix B of [6] to a leaky tank at rest (Torricelli’s problem), and in sec. 2.2 of [7] to a leaky bucket suspended from a spring. Lagrangian analyses of two variants of leaky tank cars have been given in [8]. See also [9, 10].

This method considers the kinetic energy  $T(q_k, \dot{q}_k, t)$  (but not the potential energy) of a system described by coordinates  $q_k$ , and supplements the generalized forces of Lagrange

with additional terms, related to a so-called *control volume* whose velocity is  $\mathbf{w}$ , according to eq. (5.6) of [4] and eq. (1) of [5],

$$\frac{d}{dt} \frac{\partial T_w}{\partial \dot{q}_k} - \frac{\partial T_w}{\partial q_k} + \int \frac{\partial \tilde{T}}{\partial \dot{q}_k} (\mathbf{v} - \mathbf{w}) \cdot d\mathbf{Area} - \int \tilde{T} \frac{\partial (\mathbf{v} - \mathbf{w})}{\partial \dot{q}_k} \cdot d\mathbf{Area} = Q_k, \quad (24)$$

where  $T_w$  is the kinetic energy within the control volume,  $\tilde{T}$  is the kinetic energy per unit volume,  $\mathbf{v}$  is the velocity of the material at a point in the system, and the generalized forces  $Q_k$  are related to the external forces on the system by,

$$Q_k = \sum_i \mathbf{F}_i^{\text{ext}} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}, \quad (25)$$

supposing the system to consist of particles with mass  $m_i$  at positions  $\mathbf{r}_i$  subject to external forces  $\mathbf{F}_i^{\text{ext}}$ .

In the present example we take the system to be the zig-zag pipe (of mass  $m_p$ ) and the water (of density  $\rho$ ) therein, which system can be characterized by two coordinates,  $x =$  horizontal position of the drain pipe (as in the figure on p. 1), and  $h =$  height of water in the vertical pipe on the right. The cross sectional area  $A$  of the pipe is everywhere the same.

We take the surface of the control volume to be just outside the physical surface of the zig-zag pipe. The velocity of the control volume is then  $\mathbf{w} = \dot{\mathbf{x}}$ . There is no matter of the system on the surface of the control volume, except for the water that is exiting the drain with vertical velocity,

$$\mathbf{V} = \dot{h} \hat{\mathbf{y}}, \quad (26)$$

(approximating the water as incompressible), The horizontal velocity of the exiting water is, of course  $\dot{\mathbf{x}} = \mathbf{w}$ , so  $\mathbf{v} - \mathbf{w} = \dot{\mathbf{x}} + \mathbf{V} - \mathbf{w} = \mathbf{V} = \dot{h} \hat{\mathbf{y}}$ . The kinetic energy within the control volume is, for  $h \geq 0$ , as in eq. (10),

$$T_w = T_{\text{in}} = [m + \rho A(h + H)] \frac{\dot{x}^2}{2} + \rho A(h + H) \frac{\dot{h}^2}{2} + \rho AL \frac{(\dot{x} + \dot{h})^2}{2}. \quad (27)$$

The kinetic energy per unit volume at the drain is,

$$\tilde{T}_{\text{drain}} = \rho \frac{\dot{x}^2 + V^2}{2} = \rho \frac{\dot{x}^2 + \dot{h}^2}{2}. \quad (28)$$

The area vector at the drain is  $d\mathbf{Area} = -A \hat{\mathbf{y}}$ .

The external force on the system is  $-[m + \rho A(h + H + L)]g \hat{\mathbf{y}}$ .<sup>6</sup>

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<sup>6</sup>We recall that Lagrange's method distinguishes between external and constraint forces. In the present example, the upward normal on the rolling pipe is a constraint force, and so is not included in the computation of the generalized force.

## A.1 Equation of Motion for Coordinate $x$

The generalized force  $Q_x$  is,

$$Q_x = - \sum_i m_{\text{in},i} g \hat{\mathbf{y}} \cdot \frac{\partial \mathbf{r}_i}{\partial x} = - \sum_i m_{\text{in},i} g \hat{\mathbf{y}} \cdot \hat{\mathbf{x}} = 0. \quad (29)$$

From the kinetic energy (27) we have,

$$\frac{d}{dt} \frac{\partial T_w}{\partial \dot{x}} - \frac{\partial T_w}{\partial x} = [m + \rho A(h + H)]\ddot{x} + \rho A \dot{h} \dot{x} + \rho AL(\ddot{x} + \ddot{h}). \quad (30)$$

From eq. (28) we have,

$$\frac{\partial \tilde{T}_{\text{drain}}}{\partial \dot{x}} = \rho \dot{x}. \quad (31)$$

Then,

$$\int_{\text{drain}} \frac{\partial \tilde{T}_{\text{drain}}}{\partial \dot{x}} (\mathbf{v} - \mathbf{w}) \cdot d\mathbf{Area} = \rho \dot{x} (\dot{h} \hat{\mathbf{y}}) \cdot (-A \hat{\mathbf{y}}) = -\rho A \dot{h} \dot{x}, \quad (32)$$

and since  $\partial(\mathbf{v} - \mathbf{w})/\partial \dot{x} = \partial \mathbf{V}/\partial \dot{x} = 0$ , we have that,

$$\int_{\text{drain}} \tilde{T}_{\text{drain}} \frac{\partial(\mathbf{v} - \mathbf{w})}{\partial \dot{x}} \cdot d\mathbf{Area} = 0. \quad (33)$$

Altogether, the equation of motion for coordinate  $x$ , according to eq. (24), is,

$$[m + \rho A(h + H + L)]\ddot{x} + \rho AL\ddot{h} = 0, \quad (34)$$

as found in eq. (5) above.

## A.2 Equation of Motion for Coordinate $h$

The generalized force  $Q_h$  is (delicately),

$$Q_h = - \sum_i [m + \rho A(h + H + L)]_i g \hat{\mathbf{y}} \cdot \frac{\partial \mathbf{r}_i}{\partial h} = - \sum_i \rho A(h_i + H_i) g \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = -\rho A(h + H) g, \quad (35)$$

in that only for water in the vertical sections of the pipe are the  $y$ -coordinates of particles related to the water level  $h$ . From the kinetic energy (27) we have,

$$\frac{d}{dt} \frac{\partial T_w}{\partial \dot{h}} - \frac{\partial T_w}{\partial h} = \rho A(h + H)\ddot{h} + \rho AL(\ddot{x} + \ddot{h}) + \rho A \dot{h}^2 - \rho A \frac{\dot{x}^2 + \dot{h}^2}{2}. \quad (36)$$

From eq. (28) we have,

$$\frac{\partial \tilde{T}_{\text{drain}}}{\partial \dot{h}} = \rho \dot{h}. \quad (37)$$

Then,

$$\int_{\text{drain}} \frac{\partial \tilde{T}_{\text{drain}}}{\partial \dot{h}} (\mathbf{v} - \mathbf{w}) \cdot d\mathbf{Area} = \rho \dot{h} (\dot{h} \hat{\mathbf{y}}) \cdot (-A \hat{\mathbf{y}}) = -\rho A \dot{h}^2, \quad (38)$$

and since  $\partial \mathbf{V} / \partial \dot{h} = \hat{\mathbf{y}}$ , we have that,

$$\int_{\text{drain}} \tilde{T}_{\text{drain}} \frac{\partial (\mathbf{v} - \mathbf{w})}{\partial \dot{h}} \cdot d\mathbf{Area} = \rho \frac{\dot{x}^2 + \dot{h}^2}{2} (\hat{\mathbf{y}}) \cdot (-A \hat{\mathbf{y}}) = -\rho A \frac{\dot{x}^2 + \dot{h}^2}{2}. \quad (39)$$

Altogether, the equation of motion for coordinate  $h$  according to eq. (24) is,

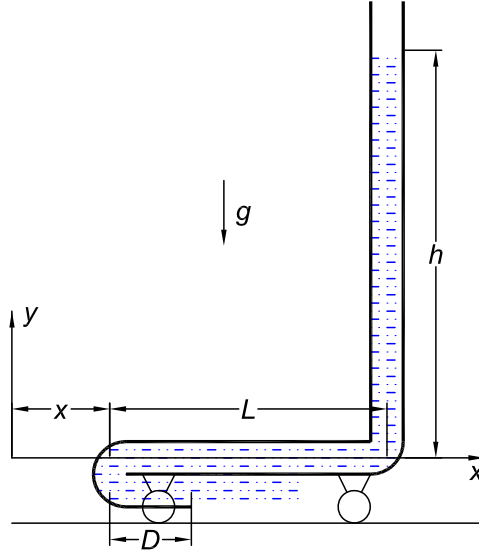
$$\rho A (h + H + L) \ddot{h} + \rho A L \ddot{x} + \rho A \frac{\dot{h}^2 - \dot{x}^2}{2} - \rho A \dot{h}^2 + \rho A \frac{\dot{x}^2 + \dot{h}^2}{2} = -\rho A (h + H) g, \quad (40)$$

$$(h + H + L) \ddot{h} + L \ddot{x} = -(h + H) g, \quad (41)$$

as previously found in eq. (20).

## B Another Variant

In this Appendix, we consider the variant of the rolling pipe sketched below, where the horizontal pipe of length  $D$  is actually at the same height as the horizontal pipe of length  $L$ .



Because the water exits the pipe of length  $D$  with a horizontal velocity to the right, providing a kind of “rocket propulsion,” we might expect that the rolling pipe moves to the left in the lab frame. However, when the system starts from rest, water is taken away from the top of the vertical pipe and appears at the exit of the horizontal pipe of length  $D$ , which moves the center of mass of pipe + water to the left with respect to the pipe. Assuming no friction, the horizontal coordinate of the center of mass of the entire system must remain at rest, so the pipe initially moves to the right.

The water that has left the pipe continue to move to the right, assuming that it does not hit anything, and is eventually to the right of the initial position of the vertical pipe.



Then, the center of mass of the entire system, relative to the pipe, is to the right of its initial location, so now the pipe must be moving to the left in the lab frame. That is, the velocity of the pipe in the lab frame reverses direction at some time, and this time is earlier than that of the reversal of velocity in the example shown on p. 1.

## B.1 Equations of Motion for $h > 0$

We now use the Lagrangian method of Appendix A to deduce the equations of motion for coordinates  $x$  and  $h$  when water is still in the vertical pipe ( $h > 0$ ).

We again take the surface of the control volume to be just outside the physical surface of the zig-zag pipe (of mass  $m$  and cross-sectional area  $A$ ). The velocity of the control volume is then  $\mathbf{w} = \dot{\mathbf{x}}$ , where  $x$  is the coordinate of the left side of the pipe in the lab frame. There is no matter of the system on the surface of the control volume, except for the water that is exiting the drain with horizontal velocity,

$$\mathbf{V} = -\dot{h} \hat{\mathbf{x}}, \quad (42)$$

in the frame of the pipe (approximating the water as incompressible), The horizontal velocity of the exiting water in the lab frame is  $\mathbf{v} = \dot{\mathbf{x}} + \mathbf{V}$ , so  $\mathbf{v} - \mathbf{w} = \mathbf{V} = -\dot{h} \hat{\mathbf{x}}$ . The kinetic energy within the control volume is, for  $h \geq 0$ ,

$$T_w = T_{\text{in}} = m \frac{\dot{x}^2}{2} + \rho A h \frac{\dot{x}^2 + \dot{h}^2}{2} + \rho A L \frac{(\dot{x} + \dot{h})^2}{2} + \rho A D \frac{(\dot{x} - \dot{h})^2}{2}. \quad (43)$$

The kinetic energy per unit volume at the drain is,

$$\tilde{T}_{\text{drain}} = \rho \frac{(\dot{x} - \dot{h})^2}{2}. \quad (44)$$

The area vector at the drain is  $d\mathbf{Area} = A \hat{\mathbf{x}}$ .

The external force on the system is  $-[m + \rho A(h + D + L)]g \hat{\mathbf{y}}$ .

### B.1.1 Equation of Motion for Coordinate $x$

The generalized force  $Q_x$  is,

$$Q_x = - \sum_i m_{\text{in},i} g \hat{\mathbf{y}} \cdot \frac{\partial \mathbf{r}_i}{\partial x} = - \sum_i m_{\text{in},i} g \hat{\mathbf{y}} \cdot \hat{\mathbf{x}} = 0. \quad (45)$$

From the kinetic energy (43) we have,

$$\frac{d}{dt} \frac{\partial T_w}{\partial \dot{x}} - \frac{\partial T_w}{\partial x} = [m + \rho A(h + L + D)]\ddot{x} + \rho A \dot{h} \dot{x} + \rho A(L - D)\ddot{h}. \quad (46)$$

From eq. (44) we have,

$$\frac{\partial \tilde{T}_{\text{drain}}}{\partial \dot{x}} = \rho(\dot{x} - \dot{h}). \quad (47)$$

Then,

$$\int_{\text{drain}} \frac{\partial \tilde{T}_{\text{drain}}}{\partial \dot{x}} (\mathbf{v} - \mathbf{w}) \cdot d\mathbf{Area} = \rho(\dot{x} - \dot{h}) \left( -\dot{h} \hat{\mathbf{x}} \right) \cdot (A \hat{\mathbf{x}}) = -\rho A \dot{h} (\dot{x} - \dot{h}), \quad (48)$$

and since  $\partial(\mathbf{v} - \mathbf{w})/\partial \dot{x} = \partial \mathbf{V}/\partial \dot{x} = 0$ , we have that,

$$\int_{\text{drain}} \tilde{T}_{\text{drain}} \frac{\partial(\mathbf{v} - \mathbf{w})}{\partial \dot{x}} \cdot d\mathbf{Area} = 0. \quad (49)$$

Altogether, the equation of motion for coordinate  $x$ , according to eq. (24), is,

$$[m + \rho A(h + L + D)]\ddot{x} + \rho A(L - D)\ddot{h} + \rho A \dot{h}^2 = 0. \quad (50)$$

### B.1.2 Equation of Motion for Coordinate $h$

The generalized force  $Q_h$  is (delicately),

$$Q_h = - \sum_i [m + \rho A(h + L + D)]_i g \hat{\mathbf{y}} \cdot \frac{\partial \mathbf{r}_i}{\partial h} = - \sum_i \rho A h_i g \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = -\rho A h g, \quad (51)$$

in that only for water in the vertical section of the pipe are the  $y$ -coordinates of particles related to the water level  $h$ . From the kinetic energy (43) we have,

$$\frac{d}{dt} \frac{\partial T_w}{\partial \dot{h}} - \frac{\partial T_w}{\partial h} = \rho A(h + L - D)\ddot{h} + \rho A(L + D)\ddot{x} + \rho A \frac{\dot{h}^2 - \dot{x}^2}{2}. \quad (52)$$

From eq. (44) we have,

$$\frac{\partial \tilde{T}_{\text{drain}}}{\partial \dot{h}} = \rho(\dot{h} - \dot{x}). \quad (53)$$

Then,

$$\int_{\text{drain}} \frac{\partial \tilde{T}_{\text{drain}}}{\partial \dot{h}} (\mathbf{v} - \mathbf{w}) \cdot d\mathbf{Area} = \rho(\dot{h} - \dot{x}) \left( -\dot{h} \hat{\mathbf{x}} \right) \cdot (A \hat{\mathbf{x}}) = \rho A \dot{h} (\dot{x} - \dot{h}), \quad (54)$$

and since  $\partial \mathbf{V}/\partial \dot{h} = -\hat{\mathbf{x}}$ , we have that,

$$\int_{\text{drain}} \tilde{T}_{\text{drain}} \frac{\partial(\mathbf{v} - \mathbf{w})}{\partial \dot{h}} \cdot d\mathbf{Area} = \rho \frac{(\dot{x} - \dot{h})^2}{2} (-\hat{\mathbf{x}}) \cdot (A \hat{\mathbf{x}}) = -\rho A \frac{(\dot{x} - \dot{h})^2}{2}. \quad (55)$$

Altogether, the equation of motion for coordinate  $h$  according to eq. (24) is,

$$\rho A(h + L - D)\ddot{h} + \rho A(L + D)\ddot{x} + \rho A \frac{\dot{h}^2 - \dot{x}^2}{2} + \rho A \dot{h} (\dot{x} - \dot{h}) + \rho A \frac{(\dot{x} - \dot{h})^2}{2} = -\rho A h g, \quad (56)$$

$$(h + L - D)\ddot{h} + (L - D)\ddot{x} = -h g, \quad (57)$$

## B.2 Comments on the Motion

As in eq. (7) above, we can make a formal time integration of eq. (50), using an integration by parts, for the system starting from rest at  $t = 0$ ,

$$m_{\text{in}}\dot{x} - \int_0^t \dot{m}_{\text{in}}(t')\dot{x}(t') dt' = -\rho(L - D)A\dot{h} - \rho A \int_0^t \dot{h}^2(t') dt', \quad (58)$$

$$\dot{x} = -\frac{\rho A(L - D)}{m_{\text{in}}}\dot{h} + \frac{\rho A}{m_{\text{in}}} \int_0^t \dot{h}(t')\dot{x}(t') dt' - \frac{\rho A}{m_{\text{in}}} \int_0^t \dot{h}^2(t') dt', \quad (59)$$

with  $m_{\text{in}} = m + \rho A(h + L + D)$  and  $\dot{m}_{\text{in}} = \rho A\dot{h}$ . At time  $t = 0^+$ , when the water starts to drain out of the pipe, the integrals in eq. (59) are negligible and we have that,

$$\dot{x}(0^+) = -\frac{\rho A(L - D)}{m_{\text{in}}}\dot{h}(0^+) > 0, \quad (60)$$

which is positive provided  $L > D$ , since  $\dot{h}$  is always negative as the water drains. That is, the pipe initially moves to the right, as the water drains out of the left end of the pipe, such that the center of mass of the system stays at rest.

As time increases, the integrals in eq. (59) become increasingly negative, such that the sign of  $\dot{x}$  can reverse.

To estimate the time  $t_{\text{reverse}}$ , when  $\dot{x}(t_{\text{reverse}}) = 0$  and the velocity of the rolling pipe reverses, we approximate the relative exit velocity  $V = -\dot{h}$  as  $\sqrt{2gh}$ , which implies that,

$$h(t) \approx \left(\sqrt{h_0} - \sqrt{g/2}t\right)^2 = h_0 \left(1 - \frac{t}{t_e}\right)^2, \quad (61)$$

such that the water has emptied out of the vertical pipe approximately at time  $t_e = \sqrt{2h_0/g}$ . Then, also assuming that  $\dot{x} \ll -\dot{h}$ , eq. (59) tells us that,

$$\begin{aligned} \dot{x} &\approx \frac{\rho A(L - D)}{m_{\text{in}}}\sqrt{2gh} - \frac{2g\rho Ah_0}{m_{\text{in}}} \int_0^t \left(1 - \frac{t'}{t_e}\right)^2 dt' \\ &= \frac{\rho A(L - D)}{m_{\text{in}}}\sqrt{2gh} - \frac{2g\rho Ah_0 t}{m_{\text{in}}} \left(1 - \frac{t}{2t_e} + \frac{t^2}{3t_e^2}\right). \end{aligned} \quad (62)$$

This vanishes when,

$$t_e \frac{L - D}{2h_0} \left(1 - \frac{t}{t_e}\right) = t \left(1 - \frac{t}{2t_e} + \frac{t^2}{3t_e^2}\right) \approx t, \quad (63)$$

such that,

$$t_{\text{reverse}} \approx t_e \frac{L - D}{2h_0(1 + (L - D)/2h_0)} \approx t_e \frac{L - D}{2h_0} \ll t_e, \quad (64)$$

where the last approximation holds when  $L - D \ll h_0$ . That is, the rolling pipe reverses direction long before the water drains completely from the vertical pipe.

In contrast, when the short pipe section of length  $D$  is vertical (with length  $H$  as in sec. 2 above), the reversal of the horizontal velocity of the rolling pipe comes at a time close to  $t_e$  (when the vertical pipe on the right becomes empty). And, if the drain pipe is horizontal and the water exits to the left, the rolling pipe always moves to the right, never reversing direction.

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