

The Rolling Motion of a Disk on a Horizontal Plane

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Abstract

Recent interest in the old problem of the motion of a coin spinning on a tabletop has focused on mechanisms of dissipation of energy as the angle α of the coin to the table decreases, while the angular velocity Ω of the point of contact increases. Following a review of the general equations of motion of a thin disk rolling without slipping on a horizontal surface, we present results of a simple experiment on the time dependence of the motion that indicate the dominant dissipative power loss to be proportional to the Ω^2 up to and including the last observable cycle.

1 Introduction

This classic problem has been treated by many authors, perhaps in greatest detail but very succinctly by Routh in Art. 244 of [1]. About such problems, Lamb [2] has said, *It is not that the phenomena, though familiar and often interesting, are held to be specially important, but it was regarded rather as a point of honour to shew how the mathematical formulation could be effected, even if the solution should prove to be impracticable, or difficult of interpretation.* Typically, the role of friction was little discussed other than in relation to “rising” [1, 2, 3, 4, 5, 6]. The present paper is motivated by recent discussions [7, 8, 9] of friction for small angles of inclination of a spinning disk to the horizontal supporting surface.

The issues of rolling motion of a disk are introduced in sec. 2 in the larger context of non-rigid-body motion and rolling motion on curved surfaces, using the science toy “Euler’s Disk” as an example. In our analysis of the motion of a rigid disk on a horizontal plane we adopt a vectorial approach as advocated by Milne [10]. The equations of motion assuming rolling without slipping are deduced in sec. 3, steady motion is discussed in secs. 4 and 5, and oscillation about steady motion is considered in sec. 7. The case of zero friction is discussed in secs. 8 and 9, and effects of dynamic friction are discussed in secs. 6, 10 and 11. Section 12 presents a brief summary of the various aspects of the motions discussed in secs. 3-11.

2 The Tangent Toy “Euler’s Disk”

An excellent science toy that illustrates the topic of this article is “Euler’s Disk”, distributed by Tangent Toy Co. [11]. Besides the disk itself, a base is included that appears to be the

key to the superior performance exhibited by this toy. The surface of the base is a thin, curved layer of glass, glued to a plastic backing. The base rests on three support points to minimize rocking. As the disk rolls on the base, the latter is noticeably deformed. If the same disk is rolled on a smooth, hard surface such as a granite surface plate, the motion dies out more quickly, and rattling sounds are more prominent. It appears that a small amount of flexibility in the base is important in damping the perturbations of the rolling motion if long spin times are to be achieved.

Thus, high-performance rolling motion is not strictly a rigid-body phenomenon. However, we do not pursue the theme of elasticity further in this paper.

The concave shape of the Tangent Toy base helps center the rolling motion of the disk, and speeds up the reduction of an initially nonzero radius b to the desirable value of zero.

An analysis of the motion of a disk rolling on a curved surface is more complex than that of rolling on a horizontal plane [12]. For rolling near the bottom of the sphere, the results are very similar to those for rolling on a plane. A possibly nonintuitive result is that a disk can roll stably on the inside of the upper hemisphere of a fixed sphere, as demonstrated in the motorcycle riding act “The Globe of Death” [13].

3 The Equations of Motion for Rolling Without Slipping

In addition to the $\hat{\mathbf{z}}$ axis which is vertically upwards, we introduce a right-handed coordinate triad of unit vectors $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ related to the geometry of the disk, as shown in Fig. 1. Axis $\hat{\mathbf{1}}$ lies along the symmetry axis of the disk. Axis $\hat{\mathbf{3}}$ is directed from the center of the disk to the point of contact with the horizontal plane, and makes angle α to that plane. The vector from the center of the disk to the point of contact is then,

$$\mathbf{a} = a \hat{\mathbf{3}}. \quad (1)$$

Axis $\hat{\mathbf{2}} = \hat{\mathbf{3}} \times \hat{\mathbf{1}}$ lies in the plane of the disk, and also in the horizontal plane. The sense of axis $\hat{\mathbf{1}}$ is chosen so that the component ω_1 of the angular velocity vector $\boldsymbol{\omega}$ of the disk about this axis is positive. Consequently, axis $\hat{\mathbf{2}}$ points in the direction of the velocity of the point of contact. (For the special case where the point of contact does not move, $\omega_1 = 0$ and analysis is unaffected by the choice of direction of axis $\hat{\mathbf{1}}$.)

3.1 Kinematics

Before discussing the dynamics of the problem, a considerable amount can be deduced from kinematics. The total angular velocity $\boldsymbol{\omega}$ can be thought of as composed of two parts,

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{123} + \omega_{\text{rel}} \hat{\mathbf{1}}, \quad (2)$$

where $\boldsymbol{\omega}_{123}$ is the angular velocity of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$, and $\omega_{\text{rel}} \hat{\mathbf{1}}$ is the angular velocity of the disk relative to that triad; the relative angular velocity can only have a component along $\hat{\mathbf{1}}$ by definition. The angular velocity of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ has component $\dot{\alpha}$ about

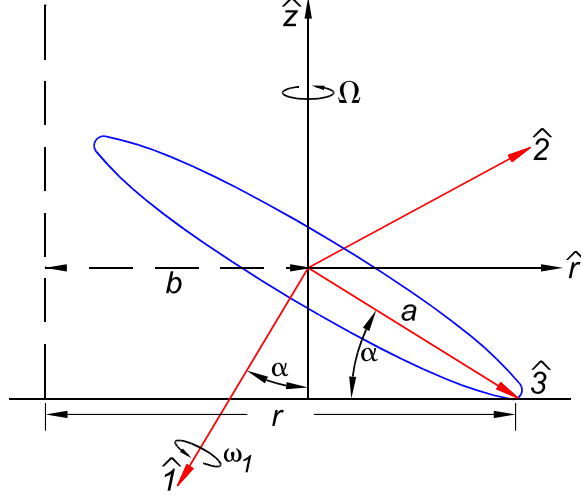


Figure 1: A disk of radius a rolls without slipping on a horizontal plane. The symmetry axis of the disk is called axis $\hat{\mathbf{1}}$, and makes angle α to the $\hat{\mathbf{z}}$ axis, which is vertically upwards, with $0 \leq \alpha \leq \pi$. The line from the center of the disk to the point of contact with the plane is called axis $\hat{\mathbf{3}}$, which makes angle α to the horizontal. The horizontal axis $\hat{\mathbf{2}}$ is defined by $\hat{\mathbf{2}} = \hat{\mathbf{3}} \times \hat{\mathbf{1}}$, and the horizontal axis $\hat{\mathbf{r}}$ is defined by $\hat{\mathbf{r}} = \hat{\mathbf{2}} \times \hat{\mathbf{z}}$. The angular velocity of the disk about axis $\hat{\mathbf{1}}$ is called ω_1 , and the angular velocity of the axes $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ about the vertical is called Ω . The motion of the point of contact is instantaneously in a circle of radius r , in the $\hat{\mathbf{2}}$ -direction. The distance from the axis of this motion to the center of mass of the disk is labeled b .

the horizontal axis $\dot{\hat{\mathbf{2}}}$ (where the dot indicates differentiation with respect to time), and is defined to have component Ω about the vertical axis $\hat{\mathbf{z}}$. Since axis $\hat{\mathbf{2}}$ is always horizontal, ω_{123} has no component along the (horizontal) axis $\hat{\mathbf{2}} \times \hat{\mathbf{z}} \equiv \hat{\mathbf{r}}$. Hence, the angular velocity of the triad $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ can be written,

$$\omega_{123} = \Omega \hat{\mathbf{z}} + \dot{\alpha} \hat{\mathbf{2}} = -\Omega \cos \alpha \hat{\mathbf{1}} + \dot{\alpha} \hat{\mathbf{2}} - \Omega \sin \alpha \hat{\mathbf{3}}, \quad (3)$$

noting that,

$$\hat{\mathbf{z}} = -\cos \alpha \hat{\mathbf{1}} - \sin \alpha \hat{\mathbf{3}}, \quad (4)$$

as can be seen from Fig. 1. The time rates of change of the axes are therefore,

$$\frac{d\hat{\mathbf{1}}}{dt} = \omega_{123} \times \hat{\mathbf{1}} = -\Omega \sin \alpha \hat{\mathbf{2}} - \dot{\alpha} \hat{\mathbf{3}}, \quad (5)$$

$$\frac{d\hat{\mathbf{2}}}{dt} = \omega_{123} \times \hat{\mathbf{2}} = \Omega \sin \alpha \hat{\mathbf{1}} - \Omega \cos \alpha \hat{\mathbf{3}} = -\Omega \hat{\mathbf{r}}, \quad (6)$$

$$\frac{d\hat{\mathbf{3}}}{dt} = \omega_{123} \times \hat{\mathbf{3}} = \dot{\alpha} \hat{\mathbf{1}} + \Omega \cos \alpha \hat{\mathbf{2}}, \quad (7)$$

where the rotating horizontal axis $\hat{\mathbf{r}}$ is related by,

$$\hat{\mathbf{r}} = \hat{\mathbf{2}} \times \hat{\mathbf{z}} = -\sin \alpha \hat{\mathbf{1}} + \cos \alpha \hat{\mathbf{3}}, \quad (8)$$

whose time rate of change is,

$$\frac{d\hat{\mathbf{r}}}{dt} = \Omega \hat{\mathbf{2}}. \quad (9)$$

Combining eqs. (2) and (3), we write the total angular velocity vector as,

$$\boldsymbol{\omega} = \omega_1 \hat{\mathbf{1}} + \dot{\alpha} \hat{\mathbf{2}} - \Omega \sin \alpha \hat{\mathbf{3}}, \quad (10)$$

where,

$$\omega_1 = \omega_{\text{rel}} - \Omega \cos \alpha. \quad (11)$$

The (nonholonomic) constraint that the disk rolls without slipping relates the velocity of the center of mass to the angular velocity vector $\boldsymbol{\omega}$ of the disk. In particular, the instantaneous velocity of the point contact of the disk with the horizontal plane is zero,

$$\mathbf{v}_{\text{contact}} = \mathbf{v}_{\text{cm}} + \boldsymbol{\omega} \times \mathbf{a} = 0. \quad (12)$$

Hence,

$$\mathbf{v}_{\text{cm}} = \frac{d\mathbf{r}_{\text{cm}}}{dt} = a \hat{\mathbf{3}} \times \boldsymbol{\omega} = -a \dot{\alpha} \hat{\mathbf{1}} + a \omega_1 \hat{\mathbf{2}}, \quad (13)$$

using eqs. (1) and (10). Then, using eqs. (5) and (6),¹

$$\frac{d\mathbf{v}_{\text{cm}}}{dt} = \frac{d^2\mathbf{r}_{\text{cm}}}{dt^2} = a[(\omega_1 \Omega \sin \alpha - \ddot{\alpha}) \hat{\mathbf{1}} + (\dot{\omega}_1 + \dot{\alpha} \Omega \sin \alpha) \hat{\mathbf{2}} + (\dot{\alpha}^2 - \omega_1 \Omega \cos \alpha) \hat{\mathbf{3}}], \quad (14)$$

$$\frac{d^2\mathbf{r}_{\text{cm}}}{dt^2} \times \hat{\mathbf{3}} = a[(\dot{\omega}_1 + \dot{\alpha} \Omega \sin \alpha) \hat{\mathbf{1}} + (\ddot{\alpha} - \omega_1 \Omega \sin \alpha) \hat{\mathbf{2}}]. \quad (15)$$

Additional kinematic relations can be deduced by noting that the point of contact between the disk and the horizontal plane can always be considered as moving instantaneously in a circle whose radius vector we define as $\mathbf{r} = r \hat{\mathbf{r}}$ with $r \geq 0$, as shown in Fig. 1, and whose center is defined to have position $x_A \hat{\mathbf{x}} + y_A \hat{\mathbf{y}}$ where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are fixed horizontal unit vectors in the lab frame. Then, the center of mass of the disk has position,

$$\mathbf{r}_{\text{cm}} = x_A \hat{\mathbf{x}} + y_A \hat{\mathbf{y}} + r \hat{\mathbf{r}} - a \hat{\mathbf{3}}, \quad (16)$$

and,

$$\frac{d\mathbf{r}_{\text{cm}}}{dt} = \dot{x}_A \hat{\mathbf{x}} + \dot{y}_A \hat{\mathbf{y}} + \dot{r} \hat{\mathbf{r}} - a \dot{\alpha} \hat{\mathbf{1}} + (r - a \cos \alpha) \Omega \hat{\mathbf{2}}. \quad (17)$$

In the special case of steady motion, $\dot{x}_A = \dot{y}_A = \dot{r} = \dot{\alpha} = 0$, eqs. (13) and (17) combine to give,

$$\omega_1 = \frac{b}{a} \Omega, \quad (18)$$

where,

$$b = r - a \cos \alpha \quad (19)$$

¹Aug. 16, 2023. A counterintuitive feature of eq. (14) is that when ω_1 is increasing we might expect the component of the acceleration of the c.m. along the forward ($\hat{\mathbf{2}}$) direction to be positive, but if the disk is falling/leaning to the left with $\dot{\alpha} < 0$ while the trajectory of the disk curves to the left ($\Omega > 0$), then the acceleration of the c.m. along the forward direction can be negative. This is a consequence of the rolling constraint (12).

is the horizontal distance from the axis of the circular motion to the center of mass of the disk. Thus, for steady motion the “spin” angular velocity ω_1 is related to the “precession” angular velocity Ω according to eq. (18). While ω_1 is defined to be nonnegative, length b can be negative if Ω is negative as well.

Except for axis $\hat{\mathbf{1}}$, the rotating axes $(\hat{\mathbf{1}}, \hat{\mathbf{2}}, \hat{\mathbf{3}})$ are not body axes, but the inertia tensor \mathbf{I} (with components I_{ij}) is diagonal with respect to them in view of the symmetry of the disk. We write,

$$I_{11} = 2kma^2, \quad I_{22} = kma^2 = I_{33}, \quad (20)$$

which holds for any thin, circularly symmetric disk according to the perpendicular axis theorem; $k = 1/2$ for a disk with mass m concentrated at the rim, $k = 1/4$ for a uniform disk, *etc.* The angular momentum \mathbf{L}_{cm} of the disk with respect to its center of mass can now be written as,

$$\mathbf{L}_{\text{cm}} = \mathbf{I} \cdot \boldsymbol{\omega} = kma^2(2\omega_1 \hat{\mathbf{1}} + \dot{\alpha} \hat{\mathbf{2}} - \Omega \sin \alpha \hat{\mathbf{3}}), \quad (21)$$

and,

$$\begin{aligned} & \frac{d\mathbf{L}_{\text{cm}}}{dt} \\ &= kma^2[2\dot{\omega}_1 \hat{\mathbf{1}} - (\Omega^2 \sin \alpha \cos \alpha + 2\omega_1 \Omega \sin \alpha - \ddot{\alpha}) \hat{\mathbf{2}} - (\dot{\Omega} \sin \alpha + 2\dot{\alpha} \Omega \cos \alpha + 2\omega_1 \dot{\alpha}) \hat{\mathbf{3}}]. \end{aligned} \quad (22)$$

3.2 Dynamics if No Air/Rolling Friction

Turning at last to the dynamics of the rolling disk, we suppose that the only forces on it are $-mg\hat{\mathbf{z}}$ due to gravity and \mathbf{F} at the point of contact with the horizontal plane.² For now, we ignore rolling friction and friction due to the air surrounding the disk.

The equation of motion for the position \mathbf{r}_{cm} of the center of mass of the disk is then,

$$m \frac{d^2 \mathbf{r}_{\text{cm}}}{dt^2} = \mathbf{F} - mg \hat{\mathbf{z}}. \quad (23)$$

The torque equation of motion for the angular momentum \mathbf{L}_{cm} about the center of mass is,

$$\frac{d\mathbf{L}_{\text{cm}}}{dt} = \boldsymbol{\tau}_{\text{cm}} = \mathbf{a} \times \mathbf{F}. \quad (24)$$

We eliminate the unknown force \mathbf{F} in eq. (24) via eqs. (1) and (23) to find,

$$\frac{1}{ma} \frac{d\mathbf{L}_{\text{cm}}}{dt} + \frac{d^2 \mathbf{r}_{\text{cm}}}{dt^2} \times \hat{\mathbf{3}} = g \hat{\mathbf{3}} \times \hat{\mathbf{z}} = -g \cos \alpha \hat{\mathbf{2}}. \quad (25)$$

Then, using eqs. (15) and (22) in eq. (25), the $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$ components of the equation of motion are,

$$(2k+1)\dot{\omega}_1 + \dot{\alpha}\Omega \sin \alpha = 0, \quad (26)$$

$$k\Omega^2 \sin \alpha \cos \alpha + (2k+1)\omega_1 \Omega \sin \alpha - (k+1)\ddot{\alpha} = \frac{g}{a} \cos \alpha, \quad (27)$$

$$\dot{\Omega} \sin \alpha + 2\dot{\alpha}\Omega \cos \alpha + 2\omega_1 \dot{\alpha} = 0. \quad (28)$$

²The horizontal component of \mathbf{F} is due to static friction, with magnitude less than μN , where μ is the coefficient of static friction and N is the vertical (normal) force of contact between the disk and the horizontal plane.

These equations agree with eqs. (1)-(3) of Art. 244 of Routh [1], noting that his A , C , θ , $\psi' = \dot{\psi}$ and ω_3 are expressed as ka^2 , $2ka^2$, α , Ω and $-\omega_1$, respectively, in our notation.³

Besides the coordinates (x_A, y_A) of the center of motion, we can readily identify only one other constant of the motion, the total energy,

$$\begin{aligned} E &= T + V = \frac{1}{2}mv_{\text{cm}}^2 + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I} \cdot \boldsymbol{\omega} + mgz \\ &= \frac{ma^2}{2} \left[(2k+1)\omega_1^2 + (k+1)\dot{\alpha}^2 + k\Omega^2 \sin^2 \alpha + \frac{2g}{a} \sin \alpha \right]. \end{aligned} \quad (29)$$

The time derivative of the energy is consistent with the equations of motion (26)-(28), but does not provide any independent information.

4 Steady Motion

For steady motion, $\dot{\alpha} = \ddot{\alpha} = \dot{\Omega} = \dot{\omega}_1 = 0$, and we define $\alpha_{\text{steady}} = \alpha_0$, $\Omega_{\text{steady}} = \Omega_0$ and $\omega_{1,\text{steady}} = \omega_{10}$. The equations of motion (26) and (28) are now trivially satisfied, and eq. (27) becomes,

$$k\Omega_0^2 \sin \alpha_0 \cos \alpha_0 + (2k+1)\omega_{10}\Omega_0 \sin \alpha_0 = \frac{g}{a} \cos \alpha_0, \quad (30)$$

A special case of steady motion is $\alpha_0 = \pi/2$, corresponding to the plane of the disk being vertical. In this case, eq. (30) requires that $\omega_{10}\Omega_0 = 0$. If $\Omega_0 = 0$, the disk rolls along a straight line and ω_{10} is the rolling angular velocity. If $\omega_{10} = 0$, the disk spins in place about the vertical axis with angular velocity Ω_0 .

For $\alpha_0 \neq \pi/2$, the angular velocity $\Omega_0 \hat{\mathbf{z}}$ of the axes about the vertical must be nonzero. We can then replace ω_{10} by the radius b of the horizontal circular motion of the center of mass using eqs. (18)-(19),

$$\omega_{10} = \frac{b}{a}\Omega_0 = \Omega_0 \left(\frac{r}{a} - \cos \alpha_0 \right). \quad (31)$$

Inserting this in (30), we find,

$$\Omega_0^2 = \frac{g \cot \alpha_0}{ka \cos \alpha_0 + (2k+1)b} = \frac{g \cot \alpha_0}{(2k+1)r - (k+1)a \cos \alpha_0}. \quad (32)$$

For $\pi/2 < \alpha_0 < \pi$ the denominator of eq. (32) is positive, since r is positive by definition, but the numerator is negative. Hence, Ω_0 is imaginary, and steady motion is not possible in this quadrant of angle α_0 .

³Aug. 16, 2023. Equation (26) indicates that ω_1 , and hence also the horizontal velocity $a\omega_1$, increases when angle α decreases (and the disk leans farther away from the vertical), provided the path of the point of contact is curved and Ω is nonzero [14]. This behavior is somewhat counterintuitive as it might seem that for the horizontal velocity to increase there must be a component, \mathbf{F}_{\parallel} , of the contact force in the direction of the horizontal velocity, and the torque $a\hat{\mathbf{z}} \times \mathbf{F}_{\parallel}$ points in the $-\hat{\mathbf{1}}$ direction which would imply that ω_1 decreases not increases. However, when the trajectory of the disk is curved, additional terms enter the $\hat{\mathbf{1}}$ -component of the equation of motion, leading to eq. (26). We can consider that the increase in the velocity of the disk as it leans more is associated with conversion of some of the gravitational potential energy, liberated as the c.m. of disk “falls”, into kinetic energy of the c.m. motion (as well as into rotational kinetic energy).

For $0 < \alpha_0 < \pi/2$, Ω_0 is real and steady motion is possible so long as,

$$b > -\frac{ak \cos \alpha_0}{2k + 1}. \quad (33)$$

In addition to the commonly observed case of $b > 0$, steady motion is possible with small negative values of b

A famous special case is when $b = 0$, and the center of mass of the disk is at rest. Here, eq. (32) becomes,

$$\Omega_0^2 = \frac{g}{ak \sin \alpha_0}, \quad (34)$$

and $\omega_{10} = 0$ according to eq. (31), so that,

$$\omega_{\text{rel}} = \Omega_0 \cos \alpha_0, \quad (35)$$

recalling eq. (11). Also, the total angular velocity becomes simply $\boldsymbol{\omega} = -\Omega_0 \sin \alpha_0 \hat{\mathbf{3}}$ according to eq. (10), so the instantaneous axis of rotation is axis $\mathbf{3}$ which contains the center of mass and the point of contact, both of which are instantaneously at rest.

5 Shorter Analysis of Steady Motion with $b = 0$

The analysis of a spinning disk (for example, a coin) whose center is at rest can be shortened considerably by noting at the outset that in this case axis $\hat{\mathbf{3}}$ is the instantaneous axis of rotation. Then, the angular velocity is $\boldsymbol{\omega} = \omega \hat{\mathbf{3}}$, and the angular momentum is simply,

$$\mathbf{L} = I_{33}\omega \hat{\mathbf{3}} = kma^2\omega \hat{\mathbf{3}}. \quad (36)$$

Since the center of mass is at rest, the contact force \mathbf{F} is just $mg\hat{\mathbf{z}}$, so the torque about the center of mass is,

$$\boldsymbol{\tau} = a \hat{\mathbf{3}} \times mg \hat{\mathbf{z}} = \frac{d\mathbf{L}}{dt}. \quad (37)$$

We see that the equation of motion for \mathbf{L} has the form,

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\Omega}_0 \times \mathbf{L}, \quad (38)$$

where,

$$\boldsymbol{\Omega}_0 = -\frac{g}{ak\omega} \hat{\mathbf{z}}. \quad (39)$$

Thus, the angular momentum, and the coin precesses about the vertical at rate Ω_0 .

A second relation between $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_0$ is obtained from eqs. (2) and (3) by noting that $\boldsymbol{\omega}_{123} = \boldsymbol{\omega}_0$, so that,

$$\boldsymbol{\omega} = (-\Omega_0 \cos \alpha_0 + \omega_{\text{rel}}) \hat{\mathbf{1}} - \Omega_0 \sin \alpha_0 \hat{\mathbf{3}} = \omega \hat{\mathbf{3}}, \quad (40)$$

using eq. (4). Hence,

$$\omega = -\Omega_0 \sin \alpha_0, \quad (41)$$

and the angular velocity ω_1 about the symmetry axis vanishes, so that,

$$\omega_{\text{rel}} = \Omega_0 \cos \alpha_0. \quad (42)$$

Combining eqs. (39) and (41), we again find that,

$$\Omega_0^2 = \frac{g}{ak \sin \alpha_0}, \quad (43)$$

As α_0 approaches zero, the angular velocity of the point of contact becomes very large, and one hears a high-frequency sound associated with the spinning coin. However, a prominent aspect of what one sees is the rotation of the figure on the face of the coin, whose angular velocity $\Omega_0 - \omega_{\text{rel}} = \Omega_0(1 - \cos \alpha_0)$ approaches zero. The total angular velocity ω also vanishes as $\alpha_0 \rightarrow 0$.

6 Radial Slippage During “Steady” Motion with $b \ll a$

The behavior of an actual rolling disk involves dissipation of energy, such that the angle α decreases with time and the distance b of the center of the disk from the center of the motion goes to zero. To get a sense of how this might occur, we consider the “steady” motion when α_0 is small and $b \ll a$. Then, the contact force \mathbf{F} during steady motion at a small angle α_0 is obtained from eqs. (6), (13), (23), (31) and (34) as,

$$\mathbf{F} = mg \hat{\mathbf{z}} - b\Omega_0^2 \hat{\mathbf{r}} \approx mg \hat{\mathbf{z}} - \frac{b}{ak \sin \alpha_0} mg \hat{\mathbf{r}}. \quad (44)$$

The horizontal component of force \mathbf{F} is due to static friction at the point of contact. The coefficient μ of static friction must therefore satisfy,

$$\mu \geq \frac{|b|}{ak \sin \alpha_0}, \quad (45)$$

otherwise the disk will slip in the direction opposite to the radius vector \mathbf{b} . Since coefficient μ is typically one or less, slippage will occur whenever $ak \sin \alpha_0 \lesssim |b|$. As the disk loses energy and angle α decreases, the slippage will reduce $|b|$ as well. The trajectory of the center of the disk will be a kind of inward spiral leading toward $b = 0$ for small α .

If distance b is negative, it must obey $|b| < ak \cos \alpha_0 / (2k + 1)$ according to eq. (33). In this case, eq. (45) becomes,

$$\mu \geq \frac{\cot \alpha_0}{2k + 1}, \quad (46)$$

which could be satisfied for a uniform disk only for $\alpha_0 \gtrsim \pi/3$. Motion with negative b is likely to be observed only briefly before large radial slippage when α_0 is large reduces b to zero.

Once b is zero, the contact force is purely vertical, according to eq. (44). Surprisingly, the condition of rolling without slipping can be maintained in this special case without any friction at the point of contact. Hence, an analysis of the motion with $b = 0$ could be made

with the assumption of zero friction, as discussed in secs. 8-9. In practice, there will always be some friction, aspects of which are further discussed in secs. 10-11. From the argument here, we see that if $b = 0$, there is no frictional force to oppose the radial slippage that accompanies a change in angle α .

7 Small Oscillations about Steady Motion

We now consider oscillations at angular frequency ϖ about steady motion, assuming that the disk rolls without slipping. We suppose that α , Ω and ω_1 have the form,

$$\alpha = \alpha_0 + \epsilon \cos \varpi t, \quad (47)$$

$$\Omega = \Omega_0 + \delta \cos \varpi t, \quad (48)$$

$$\omega_1 = \omega_{10} + \gamma \cos \varpi t, \quad (49)$$

where ϵ , δ and γ are small constants. Inserting these in the equation of motion (28) and equating terms of first order of smallness, we find that,

$$\delta = -\frac{2\epsilon}{\sin \alpha_0}(\Omega_0 \cos \alpha_0 + \omega_{10}). \quad (50)$$

From this as well as from eq. (47), we see that $\epsilon/\sin \alpha_0 \ll 1$ for small oscillations. Similarly, eq. (26) leads to,

$$\gamma = -\epsilon \frac{\Omega_0 \sin \alpha_0}{2k+1}, \quad (51)$$

and eq. (27) leads to,

$$\begin{aligned} \epsilon \varpi^2(k+1) = & -(2k+1)(\epsilon \omega_{10} \Omega_0 \cos \alpha_0 + \gamma \Omega_0 \sin \alpha_0 + \delta \omega_{10} \sin \alpha_0) + \epsilon k \Omega_0^2(1 - 2 \cos^2 \alpha_0) \\ & - 2\delta k \Omega_0 \sin \alpha_0 \cos \alpha_0 - \epsilon \frac{g}{a} \sin \alpha_0. \end{aligned} \quad (52)$$

Combining eqs. (50)-(52), we obtain,

$$\begin{aligned} \varpi^2(k+1) = & \Omega_0^2(k(1 + 2 \cos^2 \alpha_0) + \sin^2 \alpha_0) + (6k+1)\omega_{10}\Omega_0 \cos \alpha_0 \\ & + 2(2k+1)\omega_{10}^2 - \frac{g}{a} \sin \alpha_0, \end{aligned} \quad (53)$$

which agrees with Ex. 3, Art. 244 of Routh [1], noting that our k , Ω_0 , and ω_{10} are his k^2 , μ , and $-n$.

For the special case of a wheel rolling in a straight line, $\alpha_0 = \pi/2$, $\Omega_0 = 0$, and,

$$\varpi^2(k+1) = 2(2k+1)\omega_{10}^2 - \frac{g}{a}. \quad (54)$$

This motion is stable only if,⁴

⁴Aug. 16, 2023. Note that stable oscillations about rolling in a straight line ($\Omega_0 = 0$), do not involve oscillation in Ω or ω_1 as $\delta = 0 = \gamma$ in this case according to eqs. (50)-(51), so the motion of the contact point remains a straight line, with ω_1 (and hence the velocity of the contact point) remaining constant.

$$\omega_{10}^2 > \frac{g}{2(2k+1)a}. \quad (55)$$

Another special case is that of a disk spinning about a vertical diameter, for which $\alpha_0 = \pi/2$ and ω_{10} and b are zero. Then, eq. (53) indicates that the spinning is stable only for,

$$|\Omega_0| > \sqrt{\frac{g}{a(k+1)}}, \quad (56)$$

which has been called the condition for “sleeping”. Otherwise, angle α decreases when perturbed, and the motion of the disc becomes that of the more general case. Further discussion of this special case is given in the following section.

Returning to the general analysis of eq. (53), we eliminate ω_{10} using eq. (31) and replace the term $(g/a) \sin \alpha_0$ via eq. (32) to find,

$$\begin{aligned} \frac{\varpi^2}{\Omega_0^2}(k+1) &= 3k \cos^2 \alpha_0 + \sin^2 \alpha_0 + \frac{b}{a} \left((6k+1) \cos \alpha_0 - (2k+1) \frac{\sin^2 \alpha_0}{\cos \alpha_0} \right) \\ &+ 2 \frac{b^2}{a^2} (2k+1). \end{aligned} \quad (57)$$

The term in eq. (57) in large parentheses is negative for $\alpha_0 > \tan^{-1} \sqrt{(6k+1)/(2k+1)}$, which is about 60° for a uniform disk. Hence, for positive b the motion is unstable for large α_0 , and the disk will appear fall over quickly into a rolling motion with $\alpha_0 \lesssim 60^\circ$, after which α_0 will decrease more slowly due to the radial slippage discussed in sec. 5, until b becomes very small. The subsequent motion at small α_0 is considered further in sec. 11.

The motion with negative b is always stable against small oscillations, but the radial slippage is large as noted in sec. 5.

For motion with $b \ll a$, such as for a spinning coin whose center is nearly fixed, the frequency of small oscillation is given by,

$$\frac{\varpi}{\Omega_0} = \sqrt{\frac{3k \cos^2 \alpha_0 + \sin^2 \alpha_0}{k+1}}. \quad (58)$$

For small angles this becomes,

$$\frac{\varpi}{\Omega_0} \approx \sqrt{\frac{3k}{k+1}}. \quad (59)$$

For a uniform disk with $k = 1/4$, the frequency ϖ of small oscillation approaches $\sqrt{3/5}\Omega_0 = 0.77\Omega_0$, while for a hoop with $k = 1/2$, $\varpi \rightarrow \Omega_0$ as $\alpha_0 \rightarrow 0$.

The effect of this small oscillation of a spinning coin is to produce a kind of rattling sound during which the frequency sounds a bit “wrong”. This may be particularly noticeable if a surface imperfection suddenly excites the oscillation to a somewhat larger amplitude.

The radial slippage of the point of contact discussed in sec. 5 will be enhanced by the rattling, which requires a larger peak frictional force to maintain slip-free motion.

As angle α_0 approaches zero, the slippage keeps the radius b of order $a \sin \alpha_0$. For small α_0 , $b \approx \alpha_0 a$ and eq. (57) gives the frequency of small oscillation as,

$$\varpi \approx \Omega_0 \sqrt{\frac{3k + (6k + 1)\alpha_0}{k + 1}}. \quad (60)$$

For a uniform disk, $k = 1/4$, and eq. (60) gives,

$$\varpi \approx \Omega_0 \sqrt{\frac{3 + 10\alpha_0}{5}}. \quad (61)$$

When $\alpha_0 \approx 0.2$ rad, the oscillation and rotation frequencies are nearly identical, at which time a very low frequency beat can be discerned in the nutations of the disk. Once α_0 drops below about 0.1 rad, the low-frequency nutation disappears and the disk settles into a motion in which the center of mass hardly appears to move, and the rotation frequency $\Omega_0 \approx \sqrt{g/ak\alpha_0}$ grows very large.

For a hoop ($k = 1/2$), the low-frequency beat will be prominent for angles α near zero.

8 Disk Spinning About a Vertical Diameter

When a disc is spinning about a vertical diameter the condition of contact with the horizontal surface is not obviously that of rolling without slipping, which requires nonzero static friction. Olsson [15] has suggested that there is zero friction between the disk and the surface in this case.

If there is no friction, all forces on the disc are vertical. Then, the center of mass moves only vertically, and there is no vertical torque component about the center of mass, so the vertical component L_z of angular momentum is constant.

The equations of motion in the absence of friction can be found by the method of sec. 3, writing the position of the center of mass as,

$$\mathbf{r}_{\text{cm}} = a \sin \alpha \hat{\mathbf{z}}. \quad (62)$$

Using this in eq. (25), the $\hat{\mathbf{1}}$, $\hat{\mathbf{2}}$ and $\hat{\mathbf{3}}$ components of the equation of motion are,

$$\dot{\omega}_1 = 0, \quad (63)$$

$$(k\Omega^2 + \dot{\alpha}^2) \sin \alpha \cos \alpha + 2k\omega_1 \Omega \sin \alpha - (k + \cos^2 \alpha) \ddot{\alpha} = \frac{g}{a} \cos \alpha, \quad (64)$$

$$\dot{\Omega} \sin \alpha + 2\dot{\alpha} \Omega \cos \alpha + 2\omega_1 \dot{\alpha} = 0. \quad (65)$$

According to eq. (63), the angular velocity ω_1 about the symmetry axis of the disk is constant. Then, eq. (65) can be multiplied by $kma^2 \sin \alpha$ and integrated to give,

$$kma^2(\Omega \sin^2 \alpha - 2\omega_1 \cos \alpha) = L_z = \text{constant}, \quad (66)$$

recalling eq. (21).

In the case of motion of a disk with no friction we find five constants of the motion, x_{cm} , y_{cm} , ω_1 , L_z and the total energy E , in contrast to the case of rolling without slipping in which the only (known) constants of the motion are the energy E and the coordinates (x_A, y_A) of the center of motion.

For spinning about a vertical diameter, $\alpha = \pi/2$ and $\omega_1 = 0$. For small perturbations about this motion we write $\alpha = \pi/2 - \epsilon$, and for small ϵ eq. (64) becomes,

$$\ddot{\epsilon} + \left(\Omega^2 - \frac{g}{ak} \right) \epsilon = 0. \quad (67)$$

Hence, in the case of no friction, spinning about a vertical diameter is stable for,

$$|\Omega| > \sqrt{\frac{g}{ak}}. \quad (68)$$

For a uniform disk with $k = 1/4$, this stability condition is that $|\Omega| > 2\sqrt{g/a}$.

In contrast, the stability condition (56) for a uniform disk that rolls without slipping is that $|\Omega| > 2\sqrt{g/5a} \approx 0.9\sqrt{g/a}$.

As the stability conditions (56) and (68) differ by more than a factor of two for a uniform disk, there is hope of distinguishing between them experimentally.

We conducted several tests in which a U.S. quarter dollar was spun initially about a vertical diameter on a vinyl floor, on a sheet of glossy paper, and on the glass surface of the base of the Tangent Toy Euler's Disk. (The Euler's Disk is so thick that when spun about a vertical diameter it comes to rest without falling over.)

We found it essentially impossible to spin a coin such that there is no motion of its center of mass. Rather, the center of mass moves slowly in a spiral before the coin falls over into the "steady" motion with small b described in sec. 5. A centripetal force is required for such spiral motion, and so friction cannot be entirely neglected. The occasional observation of "rising", as discussed further in sec. 10, is additional evidence for the role of friction in nearly vertical spinning.

Analysis of frames taken with a digital video camera [16] at 30 frames per second with exposure time 1/8000 s did not reveal a sharp transition from spinning of a coin nearly vertically about a diameter to the settling motion of sec. 5, but in our judgment the transition point for Ω in several data sets was in the range $1.5\text{-}3\sqrt{g/a}$. This suggests that during the spinning about a nearly vertical diameter friction plays only a small role, as advocated by Olsson [15].

9 Small Oscillations About Steady Motion with No Friction

It is interesting to pursue the consequences of the equations of motion (63)-(65), deduced assuming no friction, when angle α is different from $\pi/2$. For motion that has evolved from $\alpha = \pi/2$ initially, we expect the constant ω_1 to be zero still. Then, eq. (64) indicates that the value of Ω_0 for steady motion at angle α_0 is the same as that of eq. (34). This was

anticipated in sec. 5, where it was noted that for $b = 0$ no friction is required to enforce the condition of rolling without slipping.

We consider small oscillations about steady motion at angle α_0 of the form,

$$\alpha = \alpha_0 + \epsilon \cos \varpi t, \quad (69)$$

$$\Omega = \Omega_0 + \delta \cos \varpi t, \quad (70)$$

where ϵ and δ are small constants. Inserting these in the equation of motion (65) and equating terms of first order of smallness, we find that,

$$\delta = -2\epsilon\Omega_0 \cot \alpha_0, \quad (71)$$

which is the same as eq. (50) with $\omega_1 = 0$, since eqs. (28) and (65) are the same. Similarly, eq. (64) leads to,

$$\epsilon\varpi^2(k + \cos^2 \alpha_0) = \epsilon k\Omega_0^2(1 - 2\cos^2 \alpha_0) - 2\delta k\Omega_0 \sin \alpha_0 \cos \alpha_0 - \epsilon \frac{g}{a} \sin \alpha_0. \quad (72)$$

Combining eqs. (71)-(72), we obtain,

$$\varpi^2(k + \cos^2 \alpha_0) = k\Omega_0^2(1 + 2\cos^2 \alpha_0) - \frac{g}{a} \sin \alpha_0 = 3k\Omega_0^2 \cos^2 \alpha_0, \quad (73)$$

using eq. (34). The ratio of the frequency ϖ of small oscillations to the frequency Ω_0 of rotation about the vertical axis for $\alpha_0 < \pi/2$ is,

$$\frac{\varpi}{\Omega_0} = \sqrt{\frac{3k}{k + \cos^2 \alpha_0}} \cos \alpha_0, \quad (74)$$

which differs somewhat from the result (58) obtained assuming rolling without slipping. For very small α_0 , both eq. (58) and (74) take on the same limiting value (59).

Because of the similarity of the results for small oscillations about steady motion with $b = 0$ for either assumption of no friction or rolling without slipping, it will be hard to distinguish experimentally which condition is the more realistic, but the distinction is of little consequence.

10 “Rising” of a Rotating Disk When Nearly Vertical ($\alpha \approx \pi/2$)

A rotating disk can exhibit “rising” when launched with spin about a nearly vertical diameter, provided there is slippage at the point of contact with the horizontal plane. That is, the plane of the disc may rise first towards the vertical, before eventually falling towards the horizontal.

The rising of tops appears to have been considered by Euler, but rather inconclusively. The present explanation based on sliding friction can be traced to a note by “H.T.” in 1839 [3].

Briefly, we consider motion that is primarily rotation about a nearly vertical diameter. The angular velocity about the vertical is $\Omega > \sqrt{g/ak}$, large enough so that “sleeping” at the vertical is possible in the absence of friction. The needed sliding friction depends on angular velocity component $\omega_1 = b\Omega/a$ being nonzero, which implies that the center of mass moves in a circle of radius $b \ll a$ in the present case. Then, $\omega_1 \ll \Omega$, and the angular momentum (21) is $\mathbf{L} \approx -\Omega \hat{\mathbf{z}}$, which is almost vertically upwards (see Fig. 1). Rising depends on slippage of the disk at the point of contact such that the lowermost point on the disk is not at rest but moves with velocity $-\epsilon a \omega_1 \hat{\mathbf{z}}$, which is opposite to the direction of motion of the center of mass. Corresponding to this slippage, the horizontal surface exerts friction $F_s \hat{\mathbf{z}}$ on the disk, with $F_s > 0$. The related torque, $\boldsymbol{\tau}_s = a \hat{\mathbf{z}} \times F_s \hat{\mathbf{z}} = -a F_s \hat{\mathbf{x}}$, pushes the angular momentum towards the vertical, and the center of mass of the disk rises.

The torque needed for rising exists in principle even for a disk of zero thickness, provided there is sliding friction at the point of contact.

The most dramatic form of rising motion is that of a “tippe” top, which has recently been reviewed by Gray and Nickel [6].

11 Friction at Very Small α

In practice, the motion of a spinning disk appears to cease rather abruptly for a small value of the angle α , corresponding to large precession angular velocity Ω . If the motion continued, the velocity Ωa of the point of contact would eventually exceed the speed of sound.

This suggests that air friction may play a role in the motion at very small α , as has been discussed recently by Moffatt [7, 8, 9].

When the rolling motion ceases, the disk seems to float for a moment, and then settle onto the horizontal surface. It appears that the upward contact force F_z vanished, and the disk lost contact with the surface. From eqs. (13) and (23), we see that for small α , $F_z \approx mg + ma\ddot{\alpha}$. Since the height of the center of mass above the surface is $h \approx a\alpha$ for small α , we recognize that the disk loses contact with the surface when the center of mass is falling with acceleration g .

Moffatt invited us to relate the power P dissipated by friction to the rate of change dU/dt of total energy of the disk. For a disk moving with $b = 0$ at a small angle $\alpha(t)$,

$$U = \frac{1}{2}m\dot{h}^2 + \frac{1}{2}I_{33}\omega^2 + mgh \approx \frac{1}{2}ma^2\dot{\alpha}^2 + \frac{3}{2}mag\alpha, \quad (75)$$

using eq. (41) and assuming that eq. (43) holds adiabatically. Then,

$$P = \frac{dU}{dt} \approx ma^2\dot{\alpha}\ddot{\alpha} + \frac{3}{2}mag\dot{\alpha} \approx \frac{5}{2}mag\dot{\alpha}, \quad (76)$$

where the second approximation holds when $F_z \approx 0$ and $ma\ddot{\alpha} \approx mg$.

For the dissipation of energy we need a model. First, we consider rolling friction, taken to be the effect of inelastic collisions between the disk and the horizontal surface. For example, suppose the surface has small bumps with average spacing δ and average height $h = \epsilon\delta$. We also suppose that the disk dissipates energy $mgh = mge\delta$ when passing over a bump. The

time taken for the rotating disk to pass over a bump is $\delta/a\Omega$ (at small α), so the rate of dissipation of energy to rolling friction is,

$$P = -\frac{mg\epsilon\delta}{\delta/a\Omega} = -\epsilon mag\Omega. \quad (77)$$

A generalized form of velocity-dependent friction could be written as,

$$P = -\epsilon mag\Omega^\beta, \quad (78)$$

where the drag force varies with (angular) velocity as $\Omega^{\beta-1}$. A rolling frictional force proportional to the velocity of the contact point corresponds to $\beta = 2$; an air drag force proportional to the square of the velocity corresponds to $\beta = 3$. The model of Moffatt [7] emphasizes the viscous shear of the air between the disk and the supporting horizontal surface, and corresponds to $\beta = 4$. A revised version of Moffatt's model reportedly [9] corresponds to $\beta = 2.5$.

Equating the frictional power loss to the rate of change (76) of the energy of the disk, we find,

$$\dot{\alpha} = -\frac{2\epsilon}{5}\Omega^\beta \approx -\frac{2\epsilon}{5}\left(\frac{g}{ak}\right)^{\beta/2} \frac{1}{\alpha^{\beta/2}}, \quad (79)$$

using $\Omega^2 \approx g/ak\alpha$ from eq. (34), which integrates to give

$$\alpha^{(\beta+2)/2} = \frac{\epsilon(\beta+2)}{5}\left(\frac{g}{ak}\right)^{\beta/2}(t_0 - t), \quad (80)$$

and,

$$\alpha = \left(\frac{\epsilon(\beta+2)}{5}\right)^{2/(\beta+2)} \left(\frac{g}{ak}\right)^{\beta/(\beta+2)} (t_0 - t)^{2/(\beta+2)}. \quad (81)$$

In this model, the angular velocity Ω obeys,

$$\Omega = \left(\frac{5g/\epsilon(\beta+2)ak}{t_0 - t}\right)^{1/(\beta+2)} \equiv \left(\frac{C}{t_0 - t}\right)^{1/(\beta+2)}, \quad (82)$$

which exhibits what is called by Moffatt a “finite-time singularity” at time t_0 [7] for any value of β greater than -2 .

A premise of this analysis is that it will cease to hold when the disk loses contact with the surface, *i.e.*, when $F_z = 0$, at which time $\ddot{\alpha} = -g/a$, or equivalently $d^2(1/\Omega^2)/dt^2 = -k$. Taking the derivative of eq. (82), the maximum angular velocity is,

$$\Omega_{\max} = \left(\frac{k(\beta+2)^2}{2\beta}\right)^{1/2(\beta+1)} C^{1/(\beta+1)}, \quad (83)$$

which occurs at time t_{\max} given by,

$$t_0 - t_{\max} = \left(\frac{2\beta}{k(\beta+2)^2}\right)^{(\beta+2)/2(\beta+1)} C^{-1/(\beta+1)}. \quad (84)$$

In Moffatt's model based on viscous drag of the air between the disc and the surface [7], $\beta = 4$,

$$\alpha = \left(\frac{2\pi\eta a}{m} (t_0 - t) \right)^{1/3}, \quad (85)$$

where $\eta = 1.8 \times 10^{-4}$ g-cm⁻¹-s is the viscosity of air, and,

$$\Omega = \sqrt{\frac{g}{ak}} \left(\frac{m/2\pi\eta a}{t_0 - t} \right)^{1/6}. \quad (86)$$

This model is notable for having no free parameters.

The main distinguishing feature between the various models for friction is the different time dependences (82) for the angular velocity Ω as angle α decreases.

A recent report [8] indicates that the total times of spin of coins in vacuum and in air are similar, which suggests that air drag is not the dominant mechanism of energy dissipation. Such results do not preclude that air drag could be important for disks of better surface quality, and hence lower rolling friction, or that air drag becomes important only during the high-frequency motion as time t approaches t_0 .

To help determine whether any of the above models corresponds to the practical physics, we have performed an experiment using a Tangent Toy Euler's Disk [11]. The spinning disk was illuminated by a flashlight whose beam was reflected off the surface of the disk onto a phototransistor [17] whose output was recorded by a digital oscilloscope [18] at 5 kS/s. The record length of 50,000 samples permitted the last 10 seconds of the spin history of the disk to be recorded, as shown in Figs. 2-4.

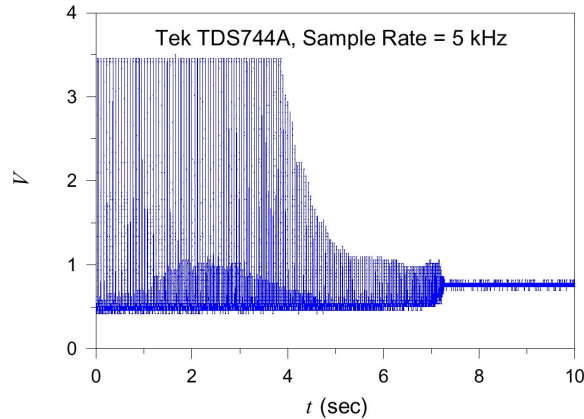


Figure 2: A 10-s record at 5 kS/s of the spinning of a Tangent Toy Euler's Disk [11] as observed by a phototransistor that detected light reflected off the disk.

The analysis of the data shown in Figs. 2-4 consisted of identifying the time t_i of the peak of cycle i of oscillation as the mean of the times of the rising and falling edges of the waveform at roughly one half the peak height. The average angular frequency for each cycle was calculated as $\Omega_i = 2\pi/(t_{i+1} - t_i)$, as shown in Fig. 5, and the rate of change of angular

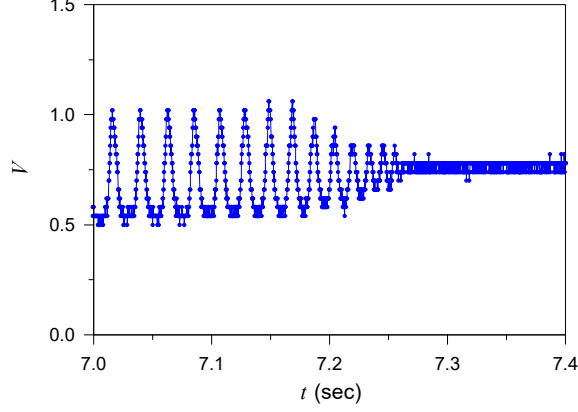


Figure 3: The last 0.25 s of the history of the spinning disk shown in Fig. 2.

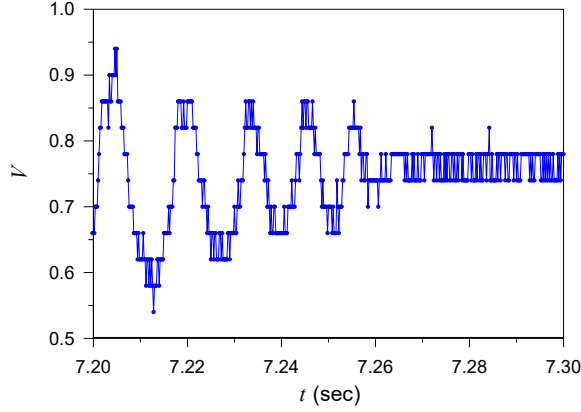


Figure 4: The last 0.06 s of the history of the spinning disk shown in Figs. 2 and 3.

frequency was calculated as $d\Omega_i/dt = 2(\Omega_{i+1} - \Omega_i)/(t_{i+2} - t_i)$. The angular frequency of the last analyzable cycle was $\Omega_{\max} = 680$ Hz.

The data can be conveniently compared to the result (82) in the form,

$$\frac{1}{\Omega} = \left(\frac{t_0 - t}{C} \right)^{1/(\beta+2)} \quad (87)$$

via a log-log plot of $1/\Omega$ vs. $t_0 - t$, given an hypothesis as to t_0 . Inspection of Fig. 4 suggests that t_0 is in the range 7.26-7.28 s for our data sample. Figures 6 and 7 show plots of $1/\Omega$ vs. $t_0 - t$ for $t_0 = 7.26$ and 7.28 s, respectively. The straight lines are not fits to the data, but illustrate the behavior expected according to eq. (82) for various values of parameter β .

A larger value of t_0 has the effect of lowering the apparent value of parameter β for the last few cycles of the motion. Figure 4 suggests that t_0 could hardly be less than 7.26 s, for which case a value of $\beta = 2$ would fit the entire data sample rather well.

In view of the uncertainty in assigning a value to the time t_0 , it is interesting to ask at

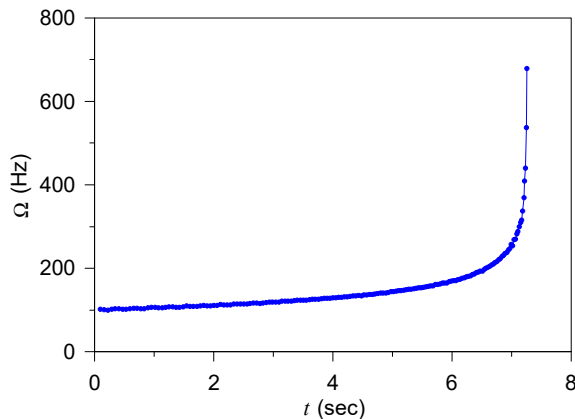


Figure 5: Ω vs. t deduced from the data shown in Figs. 2-4.

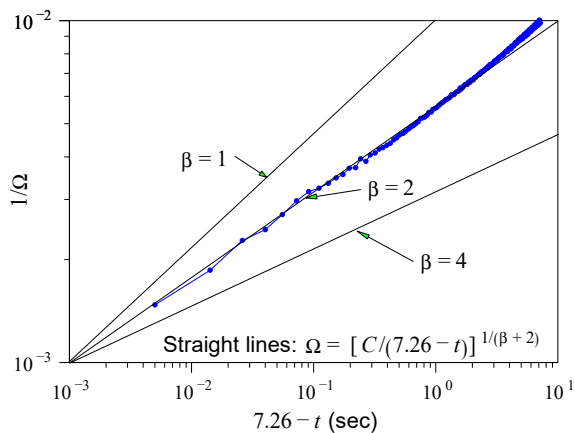


Figure 6: $1/\Omega$ vs. $t_0 - t$ for $t_0 = 7.26$ s, using the data shown in Figs. 2-4. The straight lines illustrate the behavior expected according to eq. (82) for various values of parameter β .

what time t does the time remaining equal exactly one period of the motion, *i.e.*, when does $t_0 - t = 2\pi/\Omega(t)$? For $\beta = 2$, the answer from eq. (82) is when $\Omega = (C/2\pi)^{1/3}$. From Fig. 6 we estimate that $C^{-1/4} = 0.0055$, and hence $\Omega \approx 560$ Hz when the remaining time of the motion is $2\pi/\Omega = 0.011$ s. Recall that the last cycle analyzable in our data sample yielded a value of 680 Hz for Ω . The preceding analysis tells us that time t_0 cannot be more than about 0.01 s after the last observable peak in the data, which suggests that t_0 is closer to 7.26 than to 7.28 s, and that Fig. 6 is the relevant representation of the experiment.

For $\beta = 2$, the spinning disk is predicted by eq. (84) to lose contact with the horizontal surface when $t_0 - t = C^{-1/3} = 0.001$ s for $C^{-1/4} = 0.0055$. The instantaneous angular frequency at that time is predicted by eq. (83) to be $\Omega = C^{1/3} = 1030$ Hz. These values are, of course, beyond those for the last analyzable cycle in the data.

The question as to the value of t_0 can be avoided by noting [19] that the time derivative

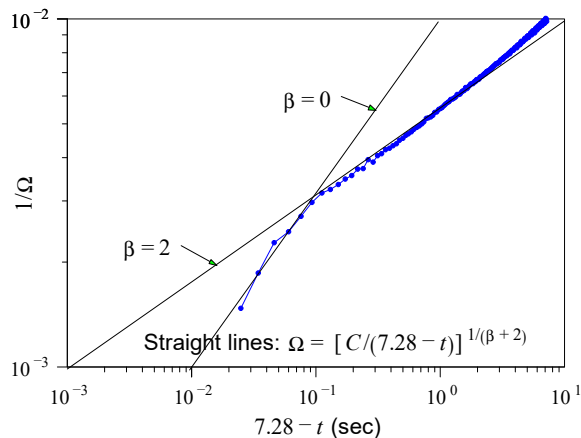


Figure 7: $1/\Omega$ vs. $t_0 - t$ for $t_0 = 7.28$ s, using the data shown in Figs. 2-4.

of eq. (82) yields the relation,

$$\frac{d\Omega}{dt} \propto \Omega^{\beta+3}. \quad (88)$$

However, $d\Omega/dt$ must be calculated from differences of differences of the times of the peaks in the data, so is subject to greater uncertainty than is Ω . Figure 8 shows a log-log plot of $d\Omega/dt$ vs. Ω together with straight lines illustrating the expected behavior for various values of β . Again, $\beta = 2$ is a consistent description of the entire data sample. The value of $\beta = 0$ suggested by Fig. 7 based on $t_0 = 7.28$ s is quite inconsistent with Fig. 8.

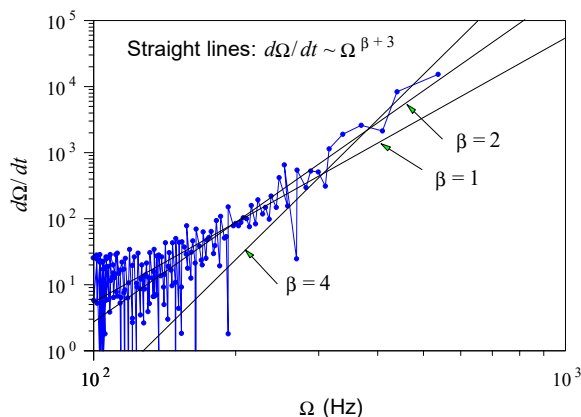


Figure 8: $d\Omega/dt$ vs. Ω for the data shown in Figs. 2-4. The straight lines illustrate the behavior expected according to eq. (88) for various values of parameter β .

The results of our experiment on the time history of the motion of a spinning disk are not definitive, but are rather consistent with the dissipated power being proportional to the square of the velocity of the point of contact, and hence the dissipative force varying linearly

with velocity. Our experiment cannot determine whether during the 0.01 s beyond the last full cycle of the motion an additional dissipative mechanism such as air friction with power loss proportional to the fourth power of the velocity [7] became important.

12 Summary of the Motion of a Disk Spun Initially About a Vertical Diameter

If a uniform disk is given a large initial angular velocity about a vertical diameter, and the initial horizontal velocity of the center of mass is very small, the disk will “sleep” until friction at the point of contact reduces the angular velocity below that of condition (68) (secs. 8). The disk will then appear to fall over rather quickly into a rocking motion with angle $\alpha < 90^\circ$ (sec. 9). After this, the vertical angular velocity Ω will increase ever more rapidly, while angle α decreases (sec. 5), until the disk loses contact with the table at a value of α of a few degrees (sec. 11). The disk then quickly settles on to the horizontal surface. One hears sound at frequency $\Omega/2\pi$, which becomes dramatically higher until the sound abruptly ceases. But if one observes a figure on the face of the disk, this rotates every more slowly and seems almost to have stopped moving before the sounds ceases (sec. 4).

If the initial motion of the disk included a nonzero initial velocity in addition to the spin about a vertical diameter, the center of mass will initially move in a circle (sec. 4). If the initial vertical angular velocity is small, the disc will roll in a large circle, tilting slightly inwards until the rolling angular velocity ω_1 drops below that of condition (55). While in most cases the angle α of the disk will then quickly drop to 60° or so (sec. 6), occasionally α will rise back towards 90° before falling (sec. 9). As the disk rolls and spins, the center of mass traces an inward spiral on average (sec. 5), but nutations about this spiral can be seen, often accompanied by a rattling sound (sec. 6). The nutation is especially prominent for $\alpha \approx 10 - 15^\circ$ at which time a very low beat frequency between that of primary spin and that of the small oscillation can be observed (sec. 6). As α decreases below this, the radius of the circle traced by the center of mass becomes very small, and the subsequent motion is that of a disk without horizontal center of mass motion (secs. 4 and 8).

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