Electrodynamics of Rotating Systems

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1 Problem

Discuss methods of analysis of the electrodynamics of rotating systems in the practical limit that the velocity v of any point in the system with respect to the (inertial) laboratory frame is small compared to the speed of light c.

In particular, discuss whether the magnetic field $\mathbf{B}'$ observed in the frame of the rotating system is that same as the field $\mathbf{B}$ observed in the lab frame,

$$\mathbf{B}' = \mathbf{B}, \tag{1}$$

or whether a better approximation is,

$$\mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}, \tag{2}$$

in Gaussian units, where $\mathbf{E}$ is the electric field in the lab frame, and $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}'$ is the velocity in the lab frame of an observer at rest at position $\mathbf{x}'$ in the rotating frame (and at time-dependent position $\mathbf{x}$ in the lab frame),[1] whose angular velocity with respect to the lab frame is $\boldsymbol{\omega}$ (about the $\boldsymbol{\omega}$-axis through the origin).

2 Solution

2.1 Use of Special Relativity

In many examples of electrodynamics of rotating systems we are less interested in the fields observed in the rotating frame than those in the lab frame. Yet, it may be that knowledge of some aspects of the fields in the moving frame is helpful in reaching an understanding of the fields in the lab frame. In this case, it is often convenient to characterize the fields near some point in the moving frame via the use of Lorentz transformations between the lab frame and the comoving, inertial (nonrotating) frame in which that point in the rotating system is instantaneously at rest. For reviews of this approach, see [1, 2].

If the rotating system includes linear, isotropic media with (relative) permittivity ϵ and/or (relative) permeability μ that differ from unity, the electrodynamic description should include the "auxiliary" fields $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ and $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, where $\mathbf{P}$ and $\mathbf{M}$ are the electric and magnetic polarization densities. Solutions to Maxwell's equations for $\mathbf{B}$, $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{H}$ in terms of the free charge density ρ and the conduction current $\mathbf{J}$ can only be obtained using

[1] As the velocity in the rotating frame of an observer at rest in that frame is zero, we do not write the nonzero vector $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}'$ as $\mathbf{v}'$ when considering it in the rotating frame. Rather, $\mathbf{v}'$ will denote the velocity of a moving observer/point in the rotating frame.

knowledge of the constitutive equations that relate $\mathbf{D}$ and $\mathbf{H}$ to $\mathbf{E}$ and $\mathbf{B}$. These relations are simple in an inertial rest frame of the medium,

$$\mathbf{D} = \epsilon \mathbf{E}, \qquad \mathbf{B} = \mu \mathbf{H} \qquad \text{(inertial rest frame)}, \tag{3}$$

and in general take more complicated forms in frames in which the medium is in motion, and in non inertial frames even if the medium is at rest there. For example, the constitutive equations in an inertial frame in which the medium has velocity $\mathbf{v}$ with $v \ll c$ can be written as,

$$\mathbf{D} = \epsilon \mathbf{E} + (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{H}, \quad \mathbf{B} = \mu \mathbf{H} + (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{E}, \qquad \left(\begin{array}{c} \text{medium with velocity } \mathbf{v} \\ \text{w.r.t. an inertial frame} \end{array} \right) \tag{4}$$

as first noted by Minkowski [3] (who gave the relations for any $v < c$).

The lab-frame constitutive equations (3) can also be expressed in terms of the polarizations $\mathbf{P}$ and $\mathbf{M}$ as,

$$\mathbf{P} = \frac{\epsilon - 1}{4\pi}\mathbf{E} + \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \mathbf{B}, \quad \mathbf{M} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} + (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{E}. \tag{5}$$

In a rotating system with nontrivial electrical media, such as the Wilson-Wilson experiment [4, 5], the use of a comoving inertial frame leads us to understand that the relations (4)-(5) are valid in the lab frame in the small region where the medium has velocity $\mathbf{v}$.[2]

2.2 Electrodynamics in a Rotating Frame

If a description is desired of the electrodynamics of a rotating system according to an observer at rest in that system, the methods of general relativity should be used.

One of the first analyses of electrodynamics in a rotating frame using general relativity was made by Schiff [6]. See also [7]-[40]. Just as mechanical analyses in rotating frames include "fictitious" forces, electrodynamic analysis in rotating frames include "fictitious" charges and currents.[3]

Sections 2.2.1-2 present a "naive" derivation, due to Modesitt [20], of Schiff's results [6] for media with $\epsilon = 1 = \mu$. Section 2.2.3 presents a possible alternative description for rotating electrodynamics that appears more compatible with special relativity, but which is found in sec. 2.2.4 to be in contradiction with a thought experiment at order v/c. Section 2.2.5 summarizes a description of the electrodynamics of the nontrivial, rotating electrical media. A long Appendix presents a "covariant" discussion of electrodynamics in a rotating frame when ϵ and μ differ from unity.

[2] We shall find in Appendix A.5 that the constitutive equations have a slightly different form than eq. (3) in the rotating frame.

[3] The adjective "fictitious" is misleading in that to an observer in the rotating frame the "fictitious" forces and charge and current densities appear to be very "real".

2.2.1 Fields Observed in a Rotating Frame

We desire the fields $\mathbf{E}'$ and $\mathbf{B}'$ seen by an observer at rest at $\mathbf{x}'$ in a frame that rotates with angular velocity $\boldsymbol{\omega}$ with respect to the lab frame, where $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ is parallel to the angular velocity $\boldsymbol{\omega}$ of the rotating system in the lab frame. We restrict our attention to points $\mathbf{x}'$ in the rotating frame such that their velocity $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$ with respect to the lab frame is small compared to the speed of light. Then, we can largely ignore issues of whether rods and clocks in the rotating frame measure the same lengths and time intervals as would similar rods and clocks in the lab frame, and whether the speed of light is the same in both frames. That is, we ignore Ehrenfest's paradox [41].

We also restrict ourselves to the case that all media have unit (relative) permittivity ϵ and unit (relative) permeability μ.

The (cylindrical) coordinates in the rotating frame are related to those in the lab frame by,

$$r' = r, \qquad \phi' = \phi - \omega t, \qquad z' = z, \qquad t' = t. \tag{6}$$

This transformation preserves volume, so conservation of electric charge implies that the transformation of charge and current density is,

$$\rho' = \rho, \qquad \mathbf{J}' = \mathbf{J} - \rho\,\mathbf{v}, \tag{7}$$

where $\mathbf{v}$ is the velocity of the observer in the rotating frame with respect to the lab frame. Force $\mathbf{F}$ is also invariant under the transformation (6). Thus, a charge q at rest in the rotating frame experiences the Lorentz force,

$$\mathbf{F}' = q\mathbf{E}' = \mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right), \tag{8}$$

and hence the transformation of the electric field is,

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}. \tag{9}$$

The force density on current $\mathbf{J}$ in a neutral conductor in the lab frame is $\mathbf{J}/c \times \mathbf{B}$. For this to equal the force density $\mathbf{J}'/c \times \mathbf{B}'$ in the rotating frame, we find, using (7), that,

$$\mathbf{B}' = \mathbf{B}, \tag{10}$$

which clearly holds on the axis of rotation.[4]

[4]We shall see in sec. 2.2.2 that the transformations (7) and (9)-(10) lead to Schiff's forms for Maxwell's equations in the rotating frame [6]. As remarked in sec. 4 of [42], these transformations do not leave the form of Maxwell's equations for $\mathbf{B}$ and $\mathbf{E}$ invariant under a Galilean transformation with $\mathbf{v} =$ constant. If the transformations (9)-(10) are supplemented by $\mathbf{D}' = \mathbf{D}$ and $\mathbf{H}' = \mathbf{H} - \mathbf{v}/c \times \mathbf{D}$ and the transformations (7) are taken to apply to the free charge density and the conduction current, then Maxwell's equations for $\mathbf{B}$ and $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{H}$ are invariant under a Galilean transformation with velocity $\mathbf{v}$. Using this generalization for the velocity $\mathbf{v}$ of a rotating observer does not, however, lead to the generalization of Schiff's analysis reported in [33] as it implies that $\boldsymbol{\nabla}' \cdot \mathbf{D}' = 4\pi\rho'$; nor would it lead to an adequate explanation of the Wilson-Wilson experiment [4] (where we must find that $\mathbf{D} = \epsilon\mathbf{E} + (\epsilon\mu - 1)\mathbf{v}/c \times \mathbf{H}$ in the lab frame [5]) by supposing that the constitutive equations for a linear, isotropic medium are $\mathbf{D}' = \epsilon\mathbf{E}'$ and $\mathbf{B}' = \mu\mathbf{H}'$ in the rotating frame.

2.2.2 Maxwell's Equation in a Rotating Frame

According to the transformation (6), intervals of distance, area and volume are measured to be the same in both frames, so that,

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}'. \tag{11}$$

Hence, the lab-frame Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{B}$ transforms to,

$$\boldsymbol{\nabla}' \cdot \mathbf{B}' = 0, \tag{12}$$

recalling eq. (10). Similarly, the lab-frame Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$ transforms to,

$$\boldsymbol{\nabla}' \cdot \mathbf{E}' = 4\pi\rho' + \boldsymbol{\nabla}' \cdot \left(\frac{\mathbf{v}}{c} \times \mathbf{B}'\right) = 4\pi\rho' + \mathbf{B}' \cdot \boldsymbol{\nabla}' \times \frac{\mathbf{v}}{c} - \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}' \times \mathbf{B}' = 4\pi\rho' + \frac{2\boldsymbol{\omega} \cdot \mathbf{B}'}{c} - \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}' \times \mathbf{B}', \tag{13}$$

using eqs. (7)-(10) and recalling that $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}'$, so $\boldsymbol{\nabla}' \times \mathbf{v} = \boldsymbol{\omega}(\boldsymbol{\nabla}' \cdot \mathbf{x}') - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla}')\mathbf{x}' = 2\boldsymbol{\omega}$.

More care is required in dealing with transformations of time derivatives to the rotating frame. We recall that the total lab-frame time derivative of a lab-frame vector field $\mathbf{A}$ according to an observer with velocity $\mathbf{v}$ in the lab is given by the convective derivative,

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}. \tag{14}$$

Similarly, if the observer has velocity $\mathbf{v}'$ in the rotating frame, the total time derivative in that frame is,

$$\frac{d\mathbf{A}}{dt'} = \frac{\partial \mathbf{A}}{\partial t'} + (\mathbf{v}' \cdot \boldsymbol{\nabla}')\mathbf{A}. \tag{15}$$

These two total time derivative are related by,

$$\frac{d\mathbf{A}}{dt} = \frac{d\mathbf{A}}{dt'} + \boldsymbol{\omega} \times \mathbf{A}. \tag{16}$$

For an observer at rest in the rotating frame, $\mathbf{v}' = 0$, while $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$ in the lab frame, so that,

$$\frac{d\mathbf{A}}{dt'} = \frac{\partial \mathbf{A}}{\partial t'}, \tag{17}$$

and eqs. (14) and (16)-(17) can be combined to give,[5]

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\partial \mathbf{A}}{\partial t'} + \boldsymbol{\omega} \times \mathbf{A} - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{A}. \tag{18}$$

A useful vector facts is that,

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \boldsymbol{\nabla} \cdot (\boldsymbol{\omega} \times \mathbf{x}) = -\boldsymbol{\omega} \cdot \boldsymbol{\nabla} \times \mathbf{x} = 0. \tag{19}$$

In rectangular coordinates, but not *in cylindrical or spherical coordinates, we can also write,*

$$\boldsymbol{\omega} \times \mathbf{A} = \boldsymbol{\omega} \times (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{x} = (\mathbf{A} \cdot \boldsymbol{\nabla})(\boldsymbol{\omega} \times \mathbf{x}) = (\mathbf{A} \cdot \boldsymbol{\nabla})\mathbf{v}, \tag{20}$$

[5] A subtle issue here is that the vector $\mathbf{A}$ which appears in the term $\partial\mathbf{A}/\partial t'$ is not the transform $\mathbf{A}'$ of lab-frame vector $\mathbf{A}$ to the rotating frame; rather it is still the lab-frame vector $\mathbf{A}$.

and hence (in rectangular coordinates only),

$$\begin{aligned}\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{A}) &= \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{A})-\mathbf{A}(\boldsymbol{\nabla}\cdot\mathbf{v})+(\mathbf{A}\cdot\boldsymbol{\nabla})\mathbf{v}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}\\ &= \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{A})+\boldsymbol{\omega}\times\mathbf{A}-(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{A}.\end{aligned} \tag{21}$$

Using eq. (21) in (18), we have (in rectangular coordinates only),

$$\frac{\partial\mathbf{A}}{\partial t}+\mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{A})=\frac{\partial\mathbf{A}}{\partial t'}+\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{A}). \tag{22}$$

A special case is that $\mathbf{A}=\mathbf{v}=\boldsymbol{\omega}\times\mathbf{x}$ *(for which* $\partial\mathbf{v}/\partial t'=0=\partial\mathbf{v}/\partial t$*).*

Thus, the partial time derivatives of the magnetic field $\mathbf{B}$ are related by,

$$\frac{\partial\mathbf{B}}{\partial t}=\frac{\partial\mathbf{B}}{\partial t'}+\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B})=\frac{\partial\mathbf{B}'}{\partial t'}+\boldsymbol{\nabla}'\times(\mathbf{v}\times\mathbf{B}'). \tag{23}$$

Faraday's law can now be transformed as,

$$\boldsymbol{\nabla}\times\mathbf{E}=\boldsymbol{\nabla}'\times\left(\mathbf{E}'-\frac{\mathbf{v}}{c}\times\mathbf{B}'\right)=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=-\frac{1}{c}\frac{\partial\mathbf{B}'}{\partial t'}-\boldsymbol{\nabla}'\times\left(\frac{\mathbf{v}}{c}\times\mathbf{B}'\right), \tag{24}$$

which simplifies to,

$$\boldsymbol{\nabla}'\times\mathbf{E}'=-\frac{1}{c}\frac{\partial\mathbf{B}'}{\partial t'}. \tag{25}$$

The partial time derivative of the electric field $\mathbf{E}=\mathbf{E}'-\mathbf{v}/c\times\mathbf{B}'$, where $\boldsymbol{\nabla}\cdot\mathbf{E}=4\pi\rho$, is, according to eq. (22),

$$\frac{\partial\mathbf{E}}{\partial t}+4\pi\rho\,\mathbf{v}=\frac{\partial\mathbf{E}}{\partial t'}+\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{E})=\frac{\partial\mathbf{E}'}{\partial t'}-\frac{\mathbf{v}}{c}\times\frac{\partial\mathbf{B}'}{\partial t'}+\boldsymbol{\nabla}'\times\left[\mathbf{v}\times\left(\mathbf{E}'-\frac{\mathbf{v}}{c}\times\mathbf{B}'\right)\right]. \tag{26}$$

Using this, and recalling from eq. (7) that $\mathbf{J}=\mathbf{J}'+\rho\,\mathbf{v}$, the fourth Maxwell equation transforms as,

$$\boldsymbol{\nabla}\times\mathbf{B}=\boldsymbol{\nabla}'\times\mathbf{B}'=\frac{4\pi}{c}\mathbf{J}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}=\frac{4\pi}{c}\mathbf{J}'+\frac{1}{c}\frac{\partial\mathbf{E}'}{\partial t'}-\frac{\mathbf{v}}{c^2}\times\frac{\partial\mathbf{B}'}{\partial t'}+\boldsymbol{\nabla}'\times\left[\frac{\mathbf{v}}{c}\times\left(\mathbf{E}'-\frac{\mathbf{v}}{c}\times\mathbf{B}'\right)\right]. \tag{27}$$

Maxwell's equations (12)-(13), (25) and (27) in the rotating frame are,

$$\boldsymbol{\nabla}'\cdot\mathbf{E}'=4\pi\rho'+\frac{2\boldsymbol{\omega}\cdot\mathbf{B}'}{c}-\frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}'\times\mathbf{B}', \tag{28}$$

$$\boldsymbol{\nabla}'\times\mathbf{E}'=-\frac{1}{c}\frac{\partial\mathbf{B}'}{\partial t'}, \tag{29}$$

$$\boldsymbol{\nabla}'\cdot\mathbf{B}'=0, \tag{30}$$

$$\boldsymbol{\nabla}'\times\mathbf{B}'=\frac{4\pi}{c}\mathbf{J}'+\frac{1}{c}\frac{\partial\mathbf{E}'}{\partial t'}-\frac{\mathbf{v}}{c^2}\times\frac{\partial\mathbf{B}'}{\partial t'}+\boldsymbol{\nabla}'\times\left[\frac{\mathbf{v}}{c}\times\left(\mathbf{E}'-\frac{\mathbf{v}}{c}\times\mathbf{B}'\right)\right], \tag{31}$$

which are the forms given by Schiff [6].

In practical rotating systems, terms in v^2/c^2 are negligible, so eq. (31) simplifies slightly to,

$$\boldsymbol{\nabla}'\times\left(\mathbf{B}'-\frac{\mathbf{v}}{c}\times\mathbf{E}'\right)\approx\frac{4\pi}{c}\mathbf{J}'+\frac{1}{c}\frac{\partial\mathbf{E}'}{\partial t'}-\frac{\mathbf{v}}{c^2}\times\frac{\partial\mathbf{B}'}{\partial t'}. \tag{32}$$

Using this in eq. (28) we have, to order v/c,

$$\boldsymbol{\nabla}'\cdot\mathbf{E}'\approx4\pi\left(\rho'-\frac{\mathbf{v}\cdot\mathbf{J}'}{c^2}\right)+\frac{2\boldsymbol{\omega}\cdot\mathbf{B}'}{c}-\frac{\mathbf{v}}{c^2}\cdot\frac{\partial\mathbf{E}'}{\partial t'}. \tag{33}$$

2.2.3 Another Description of Electrodynamics in the Rotating Frame

The forms (32)-(33) suggest that it might be better to regard the transformations of the charge and current densities and of the fields from the lab frame to the rotating frame as,

$$\rho^\star = \rho - \frac{\mathbf{v}\cdot\mathbf{J}}{c^2}\,, \qquad \mathbf{J}^\star = \mathbf{J} - \rho\,\mathbf{v}, \tag{34}$$

and,

$$\mathbf{E}^\star = \mathbf{E} + \frac{\mathbf{v}}{c}\times\mathbf{B}, \qquad \mathbf{B}^\star = \mathbf{B} - \frac{\mathbf{v}}{c}\times\mathbf{E}, \tag{35}$$

which we recognize as the low-velocity forms of the Lorentz transformations from the lab frame to an inertial frame with velocity $\mathbf{v}$ with respect to the lab frame.[6,7] Then, to order v/c,

$$\rho^\star = \rho' - \frac{\mathbf{v}\cdot\mathbf{J}'}{c^2}\,, \qquad \mathbf{J}^\star = \mathbf{J}', \qquad \mathbf{E}^\star = \mathbf{E}' \qquad \mathbf{B}^\star = \mathbf{B}' - \frac{\mathbf{v}}{c}\times\mathbf{E}'. \tag{36}$$

If we also write the time and space derivatives in the rotating frame as $\boldsymbol{\nabla}^\star$ and $\partial/\partial t^\star$, then Maxwell's equations (29)-(30) and (32)-(33) in the rotating frame can be written to order v/c as,

$$\boldsymbol{\nabla}^\star\cdot\mathbf{E}^\star = 4\pi\rho^\star + \frac{2\boldsymbol{\omega}\cdot\mathbf{B}^\star}{c} - \frac{\mathbf{v}}{c^2}\cdot\frac{\partial\mathbf{E}^\star}{\partial t^\star}\,, \tag{37}$$

$$\boldsymbol{\nabla}^\star\times\mathbf{E}^\star = -\frac{1}{c}\frac{\partial}{\partial t^\star}\left(\mathbf{B}^\star + \frac{\mathbf{v}}{c}\times\mathbf{E}^\star\right), \tag{38}$$

$$\boldsymbol{\nabla}^\star\cdot\mathbf{B}^\star = \boldsymbol{\nabla}^\star\cdot\frac{\mathbf{v}}{c}\times\mathbf{E}^\star = \frac{2\boldsymbol{\omega}\cdot\mathbf{E}^\star}{c} - \frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}^\star\times\mathbf{E}^\star = \frac{2\boldsymbol{\omega}\cdot\mathbf{E}^\star}{c} + \frac{\mathbf{v}}{c^2}\cdot\frac{\partial\mathbf{B}^\star}{\partial t^\star}\,, \tag{39}$$

$$\boldsymbol{\nabla}^\star\times\mathbf{B}^\star = \frac{4\pi}{c}\mathbf{J}^\star + \frac{1}{c}\frac{\partial\mathbf{E}^\star}{\partial t^\star} - \frac{\mathbf{v}}{c^2}\times\frac{\partial\mathbf{B}^\star}{\partial t^\star}\,. \tag{40}$$

These forms have been advocated in [15, 40].[8]

2.2.4 Does a Rotating Observer Find $\mathbf{B}' = \mathbf{B}$ or $\mathbf{B} - \mathbf{v}/c\times\mathbf{E}$?

We use a thought experiment to show that the relation $\mathbf{B}' = \mathbf{B}$ for the magnetic field according to an observer in a rotating frame is more consistent than $\mathbf{B}' = \mathbf{B} - \mathbf{v}/c\times\mathbf{E}$.

A parallel-plate capacitor with circular plates is at rest in the lab, and charged up to create electric field $\mathbf{E} \approx 4\pi\sigma\,\hat{\mathbf{z}}$ inside the capacitor, where $\mp\sigma$ is the surface-charge density

[6] The transformations (34)-(35) do not correspond to any Galilean limit of electrodynamics, as discussed in [42]. However, (as also noted in [42]) electromagnetic waves do not exist in any Galilean approximation, so the forms (34)-(35), which preserve the invariance of $\mathbf{E}^2 - \mathbf{B}^2$ to order v/c, might be preferred on this basis.

[7] The Lorentz force is invariant under a low-velocity Lorentz transformation: $q(\mathbf{E}^\star + \mathbf{v}_q^\star/c\times\mathbf{B}^\star) = q(\mathbf{E} + \mathbf{v}_q/c\times\mathbf{B})$ using eq. (35) and $\mathbf{v}_q^\star = \mathbf{v}_q - \mathbf{v}$.

[8] The forms described in this section are examples of the results of making assumptions as to the forms the equations of electrodynamics in a rotating frame, rather than supposing that there is a transformation between the equations in an inertial frame and in a rotating frame (as pursued in Appendix A below). Different assumptions will, of course, lead to different forms. A comparison of the results of three sets of assumptions is given in [22].

on the plates, and $\hat{\mathbf{z}}$ is along the symmetry axis of the capacitor. There is no magnetic field in the lab.

A (dielectric) rotating platform is placed between the plates of the capacitor and rotates with angular velocity $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ about the symmetry axis of the capacitor. An observer at rest on the rotating platform should find magnetic field $\mathbf{B}' = 0$ according to Schiff's transformation (10),[9] but the magnetic field inside the capacitor should be $\mathbf{B}^\star = -\mathbf{v}/c \times \mathbf{E} = \omega E r\,\hat{\mathbf{r}}/c$ (in cylindrical coordinates) according to eq. (36). In both prescriptions the electric field in the rotating frame is the same as that in the lab, $\mathbf{E} = \mathbf{E}' = \mathbf{E}^\star$.

Consider an electron of charge e that is released from rest in the lab on the upper plate of the capacitor. The electron accelerates straight downwards in the lab.

In the rotating frame, the electron has uniform circular motion in the plane of the plates, as well as downwards acceleration.

This motion is consistent with Schiff's view that there is no magnetic field in the rotating frame.

In view of sec. 2.2.3, the downward velocity, $v_z^\star$ of the electron combines with the radial magnetic field $\mathbf{B}^\star$ to give an azimuthal component of the Lorentz force, $F_\phi = e v_z^\star B^\star/c = e v_z^\star \omega r E/c^2$. While $\omega r \ll c$, we do not require that $v_z^\star \ll c$, since the voltage between the plates could be, say, 1 million volts, so that $v_z^\star \approx c$ as the electron nears the lower plate. Then, F_ϕ can be of order $eE(\omega r/c)$, which is not negligible. As a result, the electron's azimuthal velocity would not maintain the expected constant value of $-\omega r$.

Thus, there is an inconsistency at order v/c in the description according to sec. 2.2.3, which argues that Schiff's formulation is the more "realistic", despite the appearance of "fictitious" currents.

We shall see in the Appendix that Schiff's forms correspond to use of the covariant electromagnetic field tensors, while the forms $\mathbf{B}^\star = \mathbf{B} - \mathbf{v}/c \times \mathbf{E}$ (together with $\mathbf{E}^\star = \mathbf{E}$) are components of the contravariant field tensor in the rotating frame, for which the form of the Lorentz force is altered slightly from its familiar form. Consistent dynamics can be achieved with these forms in the rotating frame by introduction of a "fictitious" term in the Lorentz force law, as discussed in sec. A.4.

2.2.5 Summary of Electrodynamics in a Rotating Frame

For reference, we reproduce the principles of electrodynamics in the frame of a slowly rotating medium where ϵ and μ differ from unity.[10] These relations are derived in the Appendix, noting that the electromagnetic fields in the rotating frame summarized below are the covariant fields, the Lorentz force vector is the covariant force vector, but the velocity vector $\mathbf{v}'$ is the contravariant velocity and the charge and current densities ρ' and $\mathbf{J}'$ are the contravariant components of the 4-current. Extreme care is required when making a "covariant" analysis in the rotating frame, because, for example, a particle at rest in the rotating frame is to be

[9] The capacitor plates rotate with angular velocity $-\boldsymbol{\omega}$ with respect to the rotating frame, where the rotating charge density $\rho' = \rho$ leads to a surface current $\mathbf{J}' = -\rho'\,\mathbf{v}$ according to Schiff. However, there are also "fictitious" currents in Schiff's prescription, given by $\boldsymbol{\nabla}' \times (\mathbf{v} \times \mathbf{E}')/4\pi = (\boldsymbol{\nabla}' \cdot \mathbf{E}')\,\mathbf{v}/4\pi = \rho'\,\mathbf{v} = -\mathbf{J}'$, so the total current is zero and $\mathbf{B}' = 0$.

[10] This case is discussed most thoroughly by Ridgely [33, 35], but primarily for the interesting limit of steady charge and current distributions.

described as having a nonzero covariant 3-velocity (sec. A.3.1), and use of the contravariant derivative operator does not lead to meaningful physics relations (sec. A.3.4).

The (cylindrical) coordinate transformation is,

$$r' = r, \qquad \phi' = \phi - \omega t, \qquad z' = z, \qquad t' = t, \tag{41}$$

where quantities in observed in the rotating frame are labeled with a $'$. The transformations of free charge and current density are,

$$\rho'_{\rm free} = \rho_{\rm free}, \qquad \mathbf{J}'_{\rm free} = \mathbf{J}_{\rm free} - \rho_{\rm free}\,\mathbf{v}, \tag{42}$$

where $\mathbf{v}$ (with $v = \omega r \ll c$) is the velocity with respect to the lab frame of the observer in the rotating frame at distance $r' = r$ from its axis. The transformations of the electromagnetic fields are,

$$\mathbf{B}' = \mathbf{B}, \qquad \mathbf{D}' = \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, \qquad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad \mathbf{H}' = \mathbf{H}. \tag{43}$$

The transformations of the electric and magnetic polarizations are,

$$\mathbf{P}' = \mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{M}, \qquad \mathbf{M}' = \mathbf{M}, \tag{44}$$

if we regard these polarizations as defined by $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{B}' = \mathbf{H}' + 4\pi\mathbf{M}'$.

The lab-frame bound charge and current densities $\rho_{\rm bound} = -\boldsymbol{\nabla}\cdot\mathbf{P}$ and $\mathbf{J}_{\rm bound} = \partial\mathbf{P}/\partial t + c\,\boldsymbol{\nabla} \times \mathbf{M}$ transform to,[11]

$$\rho'_{\rm bound} = -\boldsymbol{\nabla}'\cdot\mathbf{P}' - \frac{2\boldsymbol{\omega}\cdot\mathbf{M}'}{c} + \frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}'\times\mathbf{M}', \tag{45}$$

$$\mathbf{J}'_{\rm bound} = \frac{\partial\mathbf{P}'}{\partial t'} + c\,\boldsymbol{\nabla}'\times\mathbf{M}' + \mathbf{v}(\boldsymbol{\nabla}'\cdot\mathbf{P}') + \frac{\mathbf{v}}{c}\times\frac{\partial\mathbf{M}'}{\partial t'} + (\mathbf{P}'\cdot\boldsymbol{\nabla})\mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{P}', \tag{46}$$

Force $\mathbf{F}$ is invariant under the transformation (41). In particular, a charge q with velocity $\mathbf{v}_q$ in the lab frame experiences a Lorentz force in the rotating frame given by,

$$\mathbf{F}' = q\left(\mathbf{E}' + \frac{\mathbf{v}'_q}{c}\times\mathbf{B}'\right) = q\left(\mathbf{E} + \frac{\mathbf{v}_q}{c}\times\mathbf{B}\right) = \mathbf{F}, \tag{47}$$

where $\mathbf{v}'_q = \mathbf{v}_q - \mathbf{v}$. Similarly, the Lorentz force density $\mathbf{f}'$ on charge and current densities in the rotating frame is,

$$\mathbf{f}' = \rho'\mathbf{E}' + \frac{\mathbf{J}'}{c}\times\mathbf{B}' = (\rho'_{\rm free} + \rho'_{\rm bound})\,\mathbf{E}' + \frac{\mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm bound}}{c}\times\mathbf{B}'. \tag{48}$$

Maxwell's equations in the rotating frame can be written as,

$$\boldsymbol{\nabla}'\cdot\mathbf{B}' = 0, \tag{49}$$

[11]The term $-2\boldsymbol{\omega}\cdot\mathbf{M}'/c$ is in effect a charge density $-\boldsymbol{\nabla}'\cdot\mathbf{P}'_{\rm mag}$ associated with an electric polarization density $\mathbf{P}'_{\rm mag}$ that appears along with magnetization in a rotating frame [43].

$$\nabla' \cdot \mathbf{D}' = 4\pi\rho'_{\rm free,total} = 4\pi\left(\rho'_{\rm free} + \rho'_{\rm other}\right), \tag{50}$$

$$\nabla' \times \mathbf{E}' + \frac{\partial \mathbf{B}'}{\partial ct'} = 0, \tag{51}$$

$$\nabla' \times \mathbf{H}' - \frac{\partial \mathbf{D}'}{\partial ct'} = \frac{4\pi}{c}\mathbf{J}'_{\rm free,total} = \frac{4\pi}{c}\left(\mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm other}\right), \tag{52}$$

where $\rho'_{\rm free} = \rho_{\rm free}$ and $\mathbf{J}'_{\rm free} = \mathbf{J}_{\rm free} - \rho_{\rm free}\mathbf{v}$ are the free charge and current densities, and the "other" charge and current densities that appear to an observer in the rotating frame are,

$$\rho'_{\rm other} = -\frac{\mathbf{v}\cdot\mathbf{J}'_{\rm free}}{c^2} + \frac{\boldsymbol{\omega}\cdot\mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial \mathbf{D}'}{\partial ct'}, \tag{53}$$

$$\mathbf{J}'_{\rm other} = \rho'_{\rm free}\mathbf{v} + \left(\frac{\mathbf{D}'}{4\pi}\cdot\nabla\right)\mathbf{v} - (\mathbf{v}\cdot\nabla)\frac{\mathbf{D}'}{4\pi} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial \mathbf{H}'}{\partial t'}. \tag{54}$$

The "other" charge and current distributions are sometimes called "fictitious" [6], but we find this term misleading.[12] For an example with an "other" charge density $\boldsymbol{\omega}\cdot\mathbf{H}'/2\pi c$ in the rotating frame, see Appendix B below, (and also [45]).

Maxwell's equations can also be expressed only in terms of the fields $\mathbf{E}'$ and $\mathbf{B}'$ and charge and current densities associated with free charges as well as with electric and magnetic polarization,

$$\nabla' \cdot \mathbf{E}' = 4\pi\rho'_{\rm total}, \tag{55}$$

and,

$$\nabla' \times \mathbf{B}' - \frac{\partial \mathbf{E}'}{\partial ct'} = \frac{4\pi}{c}\mathbf{J}'_{\rm total}, \tag{56}$$

where,

$$\begin{aligned}
\rho'_{\rm total} &= \rho'_{\rm free} - \frac{\mathbf{v}}{c^2}\cdot\mathbf{J}'_{\rm free} - \nabla'\cdot\mathbf{P}' + \frac{\boldsymbol{\omega}\cdot\mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial \mathbf{D}'}{\partial ct'} \\
&= \rho'_{\rm free,total} - \nabla'\cdot\mathbf{P}' \\
&= \rho'_{\rm free} + \rho'_{\rm bound} + \rho'_{\rm more}, \qquad (57)\\
\rho'_{\rm more} &= -\frac{\mathbf{v}}{c^2}\cdot\left(\mathbf{J}'_{\rm free} + \frac{\partial \mathbf{P}'}{\partial t'} + c\nabla'\times\mathbf{M}'\right) + \frac{\boldsymbol{\omega}\cdot\mathbf{B}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial \mathbf{E}'}{\partial ct'}, \qquad (58)\\
\mathbf{J}'_{\rm total} &= \mathbf{J}'_{\rm free} + \frac{\partial \mathbf{P}'}{\partial t'} + c\nabla'\times\mathbf{M}' + \rho'_{\rm free}\mathbf{v} + \left(\frac{\mathbf{D}'}{4\pi}\cdot\nabla\right)\mathbf{v} - (\mathbf{v}\cdot\nabla)\frac{\mathbf{D}'}{4\pi} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial \mathbf{H}'}{\partial t'} \\
&= \mathbf{J}'_{\rm free,total} + \frac{\partial \mathbf{P}'}{\partial t'} + c\nabla'\times\mathbf{M}' \\
&= \mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm bound} + \mathbf{J}'_{\rm more}, \qquad (59)\\
\mathbf{J}'_{\rm more} &= \mathbf{v}\left(\rho'_{\rm free} - \nabla'\cdot\mathbf{P}' - \frac{2\boldsymbol{\omega}\cdot\mathbf{M}'}{c} + \frac{\mathbf{v}}{c}\cdot\nabla'\times\mathbf{M}'\right) \\
&\quad + \left(\frac{\mathbf{E}'}{4\pi}\cdot\nabla\right)\mathbf{v} - (\mathbf{v}\cdot\nabla)\frac{\mathbf{e}'}{4\pi} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial \mathbf{B}'}{\partial t'}. \qquad (60)
\end{aligned}$$

[12] According to Einstein [44]: "Fields which can be transformed into each other by such transformations [as eq. (41)] describe the same real situation." In particular, the "fictitious" sources (53)-(54) are an aspect of a description by an observer in the rotating frame of the same physical situation as observed in the lab frame.

The contribution of the polarization densities to the source terms in Maxwell's equations in much more complex in the rotating frame than in the lab frame.[13] Because of the "other" source terms that depend on the fields in the rotating frame, Maxwell's equations cannot be solved directly in this frame. Rather, an iterative approach is required in general.

The constitutive equations for a linear, isotropic medium at rest in the rotating frame are,[14]

$$\mathbf{D}' = \epsilon \mathbf{E}', \qquad \mathbf{B}' = \mu \mathbf{H}' - (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{E}', \tag{61}$$

in the rotating frame, and,

$$\mathbf{D} = \epsilon \mathbf{E} + (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{H}, \qquad \mathbf{B} = \mu \mathbf{H} - (\epsilon\mu - 1)\frac{\mathbf{v}}{c} \times \mathbf{E}, \tag{62}$$

in the lab frame, where ϵ and μ are the permittivity and permeability of the medium when at rest in an inertial frame. The lab-frame constitutive equations (62) are the same as for a nonrotating medium that moves with constant velocity $\mathbf{v}$ with respect to the lab frame.

We can also write the constitutive equations (61) for a linear, isotropic medium in terms of the fields $\mathbf{B}'$, $\mathbf{E}'$, $\mathbf{P}'$ and $\mathbf{M}'$ by noting that $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{H}' = \mathbf{B}' - 4\pi\mathbf{M}'$, so that,

$$\begin{aligned}
\mathbf{P}' &= \frac{\epsilon - 1}{4\pi}\mathbf{E}', \\
\mathbf{M}' &= \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{E}'}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c} \times \mathbf{P}'.
\end{aligned} \tag{63}$$

Similarly, the constitutive equations (62) in the lab frame can be written to order v/c as,

$$\begin{aligned}
\mathbf{P} &= \frac{\epsilon - 1}{4\pi}\mathbf{E} + \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{B}}{4\pi} \qquad = \frac{\epsilon - 1}{4\pi}\mathbf{E} + \frac{\epsilon\mu - 1}{\mu - 1}\frac{\mathbf{v}}{c} \times \mathbf{M}, \\
\mathbf{M} &= \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} - \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{E}}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c} \times \mathbf{P}.
\end{aligned} \tag{64}$$

Ohm's law for the conduction current $\mathbf{J}_C$ has the same form for a medium with velocity $\mathbf{u}'$ relative to the rotating frame as it does for a medium with velocity $\mathbf{u}$ relative to the lab frame,

$$\mathbf{J}'_C = \sigma\left(\mathbf{E}' + \frac{\mathbf{u}'}{c} \times \mathbf{B}'\right) = \sigma\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right) = \mathbf{J}_C, \tag{65}$$

where σ is the electric conductivity of a medium at rest in an inertial frame.

A Appendix: Electrodynamics Using Covariant and Contravariant Vectors and Tensors

In the preceding we have used only vector notation when discussing electrodynamics, even though it is more proper to consider the vector fields $\mathbf{B}$ and $\mathbf{E}$ as part of a field tensor

[13]Neglect of this complexity can lead to apparent paradoxes involving rotating magnetic media [13, 14, 31].

[14]Different constitutive equations hold for a rotating permanent magnet [46] or for a rotating electret.

F. Furthermore, one should distinguish between covariant and contravariant vectors and tensors.

In an inertial frame, the distinction between covariant and contravariant vectors is rather trivial, so that one can neglect this distinction and then speak of the field vectors $\mathbf{B}$ and $\mathbf{E}$ without confusion. However, the metric tensor is not diagonal in a rotating frame, so that greater care should be made in distinguishing covariant and contravariant vectors and tensors [30, 35].

The summary of electrodynamics in a rotating frame given in sec. 2.2.5 does not mention covariant and contravariant components, and simply uses a $'$ to indicate quantities observed in the rotating frame. As shown in this Appendix, it is consistent to make such a summary if one obeys the following conventions,

1. All components of electromagnetic field vectors $\mathbf{B}'$, $\mathbf{D}'$, $\mathbf{E}'$, $\mathbf{H}'$, $\mathbf{P}'$ and $\mathbf{M}'$ that appear in sec.2.2.5 are components of their respective covariant tensors.

2. The gradient operator $\boldsymbol{\nabla}'$ is part of the covariant 4-derivative operator.

3. The velocity $\mathbf{u}'$, and the charge and current densities ρ', $\rho'_{\rm free}$, $\mathbf{J}'$ and $\mathbf{J}'_{\rm free}$ are contravariant components of their respective contravariant 4-vectors.[15]

4. The bound charge and current densities $\rho'_{\rm bound}$ and $\mathbf{J}'_{\rm bound}$ are representations of their contravariant 4-vectors in terms of covariant components of the relevant electromagnetic fields.

These conventions are relatively straightforward except for item 4, which is needed so that charge conservation, $\boldsymbol{\nabla}'\cdot\mathbf{J}'+\partial\rho'/\partial t' = 0$, holds for bound as well was for free charge, while observing the convention that electromagnetic fields are represented by covariant components. For examples exhibiting charge conservation in rotating media, see [5, 46].

A.1 Notation

We will indicate a contravariant 4-vector or 4-tensor by indices that are Greek superscripts, and covariant 4-vector or 4-tensor by indices that are Greek subscripts.[16] The components of contravariant 4-vectors and tensors (other than the position 4-vector x^α) will have a $\tilde{}$ above the symbol. Quantities measured in the rotating frame will be designated with a $'$. As before, we only consider velocities small compared to the speed of light. The vector $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$ is the velocity in the (inertial) lab frame of a point at position $\mathbf{x}$ that is at rest in the rotating frame, whose angular velocity is $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ with respect to the lab frame.

[15] In this Appendix, components of contravariant vectors and tensors will include a $\tilde{}$ above the symbol.

[16] The notion of covariant and contravariant vectors and tensors arises from the character of their transformations from one frame to another. We defer discussion of details of these transformations until sec. A.3.

The metric tensor for rectangular coordinates in an inertial frame is written as,

$$g_{\alpha\beta} = g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{66}$$

where the Greek indices run from 0 to 3 (and we ignore the effect of gravity on electrodynamics).

The contravariant position vector x^α has components in the lab frame $(x^0, x^1, x^2, x^3) = (ct, \tilde{x}, \tilde{y}, \tilde{z}) = (ct, \tilde{\mathbf{x}})$ $[= (ct, x, y, z)]$, so the corresponding covariant position vector in the has lab-frame components $x_a = g_{\alpha\beta} x^\beta = (ct, -\mathbf{x})$.

A particle with velocity $\mathbf{u}$ ($u \ll c$) in the lab frame has 4-velocity,

$$u^\alpha = (c, \tilde{\mathbf{u}}) = (c, \mathbf{u}) \qquad \text{and} \qquad u_\alpha = g_{\alpha\beta} u^\beta = (c, -\mathbf{u}). \tag{67}$$

Lab-frame charge density ρ and current density $\mathbf{J}$ are described by the 4-vectors,

$$J^\alpha = (c\tilde{\rho}, \tilde{\mathbf{J}}) = (c\rho, \mathbf{J}) \qquad \text{and} \qquad J_\alpha = (c\rho, -\mathbf{J}). \tag{68}$$

The lab-frame electromagnetic fields $\mathbf{B}$ and $\mathbf{E}$ are described by the 4-tensors,

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -\tilde{E}_x & -\tilde{E}_y & -\tilde{E}_z \\ \tilde{E}_x & 0 & -\tilde{B}_z & \tilde{B}_y \\ \tilde{E}_y & \tilde{B}_z & 0 & -\tilde{B}_x \\ \tilde{E}_z & -\tilde{B}_y & \tilde{B}_x & 0 \end{pmatrix} \qquad \text{and} \qquad F_{\alpha\beta} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \tag{69}$$

where $\tilde{\mathbf{B}} = \mathbf{B}$ and $\tilde{\mathbf{E}} = \mathbf{E}$, using $F^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta}$. The lab-frame electromagnetic fields $\tilde{\mathbf{D}} = \mathbf{D}$ and $\tilde{\mathbf{H}} = \mathbf{H}$ are described by the 4-tensors,

$$H^{\alpha\beta} = \begin{pmatrix} 0 & -\tilde{D}_x & -\tilde{D}_y & -\tilde{D}_z \\ \tilde{D}_x & 0 & -\tilde{H}_z & \tilde{H}_y \\ \tilde{D}_y & \tilde{H}_z & 0 & -\tilde{H}_x \\ \tilde{D}_z & -\tilde{H}_y & \tilde{H}_x & 0 \end{pmatrix} \qquad \text{and} \qquad H_{\alpha\beta} = \begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & -H_z & H_y \\ -D_y & H_z & 0 & -H_x \\ -D_z & -H_y & H_x & 0 \end{pmatrix}. \tag{70}$$

The lab-frame electric and magnetic polarization densities $\tilde{\mathbf{P}} = \mathbf{P}$ and $\tilde{\mathbf{M}} = \mathbf{M}$ are described by the 4-tensors,

$$M^{\alpha\beta} = \begin{pmatrix} 0 & \tilde{P}_x & \tilde{P}_y & \tilde{P}_z \\ -\tilde{P}_x & 0 & -\tilde{M}_z & \tilde{M}_y \\ -\tilde{P}_y & \tilde{M}_z & 0 & -\tilde{M}_x \\ -\tilde{P}_z & -\tilde{M}_y & \tilde{M}_x & 0 \end{pmatrix} \qquad \text{and} \qquad M_{\alpha\beta} = \begin{pmatrix} 0 & -P_x & -P_y & -P_z \\ P_x & 0 & -M_z & M_y \\ P_y & M_z & 0 & -M_x \\ P_z & -M_y & M_x & 0 \end{pmatrix}, \tag{71}$$

such that,

$$H^{\alpha\beta} = F^{\alpha\beta} - 4\pi M^{\alpha\beta} \qquad \text{and} \qquad H_{\alpha\beta} = F_{\alpha\beta} - 4\pi M_{\alpha\beta}. \tag{72}$$

A.2 Electrodynamics in an Inertial Frame

Electric charge is conserved, which can be expressed in terms of charge density and current density as,

$$\partial^\alpha J_\alpha = \tilde{\boldsymbol{\nabla}} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 = \partial_\alpha J^\alpha = \boldsymbol{\nabla} \cdot \tilde{\mathbf{J}} + \frac{\partial \tilde{\rho}}{\partial t}, \tag{73}$$

where $\tilde{\boldsymbol{\nabla}} = \boldsymbol{\nabla}$ and,

$$\partial^\alpha = \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial ct}, -\tilde{\boldsymbol{\nabla}}\right) \qquad \text{and} \qquad \partial_\alpha = \frac{\partial}{\partial x_\alpha} = \left(\frac{\partial}{\partial ct}, \boldsymbol{\nabla}\right). \tag{74}$$

The Lorentz force $\mathbf{f}$ on a charge q that moves with velocity $\mathbf{u}$ in the lab frame is described by the 4-vectors,

$$f^\alpha = (\tilde{\mathbf{f}} \cdot \mathbf{u}/c, \tilde{\mathbf{f}}) = qF^{\alpha\beta} u_\beta = q(\tilde{\mathbf{E}} \cdot \mathbf{u}/c, \tilde{\mathbf{E}} + \mathbf{u}/c \times \tilde{\mathbf{B}}), \tag{75}$$

$$f_\alpha = (\mathbf{f} \cdot \tilde{\mathbf{u}}/c, -\mathbf{f}) = qF_{\alpha\beta} u^\beta = q(\mathbf{E} \cdot \tilde{\mathbf{u}}/c, -\mathbf{E} - \tilde{\mathbf{u}}/c \times \mathbf{B}). \tag{76}$$

In inertial frames we can ignore the fact that the Lorentz force vector is composed of a mix of covariant and contravariant vectors.

The two Maxwell's equations for $\mathbf{B}$ and $\mathbf{E}$ that involve the total charge density ρ and the total current density $\mathbf{J}$,

$$\boldsymbol{\nabla} \cdot \tilde{\mathbf{E}} = 4\pi\tilde{\rho} = 4\pi\rho = \tilde{\boldsymbol{\nabla}} \cdot \mathbf{E}, \qquad \boldsymbol{\nabla} \times \tilde{\mathbf{B}} - \frac{\partial \tilde{\mathbf{E}}}{\partial ct} = \frac{4\pi}{c}\tilde{\mathbf{J}} = \frac{4\pi}{c}\mathbf{J} = \tilde{\boldsymbol{\nabla}} \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial ct}, \tag{77}$$

can be written as,[17]

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta \qquad \text{or} \qquad \partial^\alpha F_{\alpha\beta} = \frac{4\pi}{c} J_\beta, \tag{79}$$

while the other two Maxwell's equations,

$$\boldsymbol{\nabla} \cdot \tilde{\mathbf{B}} = \tilde{\boldsymbol{\nabla}} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \times \tilde{\mathbf{E}} + \frac{\partial \tilde{\mathbf{B}}}{\partial ct} = \tilde{\boldsymbol{\nabla}} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial ct} = 0, \tag{80}$$

can be written as

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \qquad \text{or} \qquad \partial^\alpha \mathcal{F}_{\alpha\beta} = 0, \tag{81}$$

where the dual of an antisymmetric, covariant tensor $F_{\alpha\beta}$ is the contravariant tensor $F^{\alpha\beta}$ defined by,

$$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} \tag{82}$$

[17]The field tensors can be derived from 4-potentials according to

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha, \qquad A^\alpha = (\tilde{V}, \tilde{\mathbf{A}}) = (V, \mathbf{A}), \qquad F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \qquad A_\alpha = (V, -\mathbf{A}). \tag{78}$$

when the components of tensor $F_{\alpha\beta}$ are defined as in eq. (69), and where,

$$\epsilon^{\alpha\beta\gamma\delta} = -\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ is an even permutation of 0123,} \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ is an odd permutation of 0123,} \\ 0 & \text{if any two indices are equal.} \end{cases} \tag{83}$$

Similarly, we define,

$$\mathcal{F}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta} = \begin{pmatrix} 0 & \tilde{B}_x & \tilde{B}_y & \tilde{B}_z \\ -\tilde{B}_x & 0 & \tilde{E}_z & -\tilde{E}_y \\ -\tilde{B}_y & -\tilde{E}_z & 0 & \tilde{E}_x \\ -\tilde{B}_z & \tilde{E}_y & -\tilde{E}_x & 0 \end{pmatrix}. \tag{84}$$

The components of the dual tensor $\mathcal{F}^{\alpha\beta}$ can be obtained from those of $F^{\alpha\beta}$ by the **duality transformation**,

$$\tilde{\mathbf{E}} \to \mathbf{B}, \qquad \tilde{\mathbf{B}} \to -\mathbf{E} \qquad (F^{\alpha\beta} \to \mathcal{F}^{\alpha\beta}), \tag{85}$$

while those of the dual tensor $\mathcal{F}_{\alpha\beta}$ can be obtained from those of $F_{\alpha\beta}$ by the transformation,

$$\mathbf{E} \to \tilde{\mathbf{B}}, \qquad \mathbf{B} \to -\tilde{\mathbf{E}} \qquad (F_{\alpha\beta} \to \mathcal{F}_{\alpha\beta}). \tag{86}$$

If ρ_{free} and $\mathbf{J}_{\text{free}}$ represent only the free charge density and the conduction current, then the Maxwell equations,

$$\boldsymbol{\nabla} \cdot \tilde{\mathbf{D}} = 4\pi\tilde{\rho}_{\text{free}} = 4\pi\rho_{\text{free}} = \tilde{\boldsymbol{\nabla}} \cdot \mathbf{D}, \qquad \boldsymbol{\nabla} \times \tilde{\mathbf{H}} - \frac{\partial\tilde{\mathbf{D}}}{\partial ct} = \frac{4\pi}{c}\tilde{\mathbf{J}} = \frac{4\pi}{c}\mathbf{J} = \tilde{\boldsymbol{\nabla}} \times \mathbf{H} - \frac{\partial\mathbf{D}}{\partial ct}, \tag{87}$$

can be written as,

$$\partial_\alpha H^{\alpha\beta} = \frac{4\pi}{c}J^\beta_{\text{free}} \qquad \text{or} \qquad \partial^\alpha H_{\alpha\beta} = \frac{4\pi}{c}J_{\text{free}\beta}. \tag{88}$$

It is not obvious from the preceding how the alternative forms of Maxwell's equations (81) and (88) should be paired. However, once we consider transformations to noninertial frames it becomes clear that the pairing must be,

$$\partial^\alpha \mathcal{F}_{\alpha\beta} = 0 \qquad \text{and} \qquad \partial^\alpha H_{\alpha\beta} = \frac{4\pi}{c}J_{\text{free}\beta}, \tag{89}$$

or,

$$\partial_\alpha \mathcal{F}^{\alpha\beta} = 0 \qquad \text{and} \qquad \partial_\alpha H^{\alpha\beta} = \frac{4\pi}{c}J^\beta_{\text{free}}. \tag{90}$$

This has the implication that when Maxwell's equations are expressed in terms of covariant and contravariant 3-vectors $\mathbf{B}$, $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{H}$ and $\tilde{\mathbf{B}}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ they have the mixed forms,

$$\tilde{\boldsymbol{\nabla}} \cdot \mathbf{D} = 4\pi\rho_{\text{free}}, \qquad \tilde{\boldsymbol{\nabla}} \times \tilde{\mathbf{E}} + \frac{\partial\tilde{\mathbf{B}}}{\partial ct} = 0, \qquad \tilde{\boldsymbol{\nabla}} \cdot \tilde{\mathbf{B}} = 0, \qquad \tilde{\boldsymbol{\nabla}} \times \mathbf{H} - \frac{\partial\mathbf{D}}{\partial ct} = \frac{4\pi}{c}\mathbf{J}_{\text{free}}, \tag{91}$$

or,

$$\boldsymbol{\nabla} \cdot \tilde{\mathbf{D}} = 4\pi \tilde{\rho}_{\rm free}, \qquad \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial ct} = 0, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad \boldsymbol{\nabla} \times \tilde{\mathbf{H}} - \frac{\partial \tilde{\mathbf{D}}}{\partial ct} = \frac{4\pi}{c} \tilde{\mathbf{J}}_{\rm free}. \quad (92)$$

Since there is no difference between covariant and contravariant components in inertial frames, the mixing seen in eqs. (91)-(92) goes unnoticed there.

The relation between the total and free charge and current densities can be written as,

$$\tilde{\rho} = \tilde{\rho}_{\rm free} - \boldsymbol{\nabla} \cdot \tilde{\mathbf{P}}, \qquad \tilde{\mathbf{J}} = \tilde{\mathbf{J}}_{\rm free} + \frac{\partial \tilde{\mathbf{P}}}{\partial t} + c\, \boldsymbol{\nabla} \times \tilde{\mathbf{M}},$$

$$\text{or} \qquad \rho = \rho_{\rm free} - \tilde{\boldsymbol{\nabla}} \cdot \mathbf{P}, \qquad \mathbf{J} = \mathbf{J}_{\rm free} + \frac{\partial \mathbf{P}}{\partial t} + c\, \tilde{\boldsymbol{\nabla}} \times \mathbf{M}. \quad (93)$$

or in 4-vector form as,

$$J^\beta = J^\beta_{\rm free} + c\, \partial_\alpha M^{\alpha\beta} \qquad \text{or} \qquad J_\beta = J_{{\rm free}\beta} + c\, \partial^\alpha M_{\alpha\beta}, \quad (94)$$

The constitutive equations,

$$\mathbf{D} = \tilde{\mathbf{D}} = \epsilon \tilde{\mathbf{E}} = \epsilon \mathbf{E} \qquad \text{and} \qquad \mathbf{B} = \tilde{\mathbf{B}} = \mu \tilde{\mathbf{H}} = \mu \mathbf{H}, \quad (95)$$

hold only in an inertial rest frame of a linear, isotropic medium. If the medium has velocity $\mathbf{u}$ ($u \ll c$) with respect to the (inertial) lab frame, the constitutive equations in the lab frame are,

$$\tilde{\mathbf{D}} + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{H}} = \epsilon \left(\tilde{\mathbf{E}} + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{B}} \right) \qquad \text{and} \qquad \tilde{\mathbf{B}} - \frac{\mathbf{v}}{c} \times \tilde{\mathbf{E}} = \mu \left(\tilde{\mathbf{H}} - \frac{\mathbf{v}}{c} \times \tilde{\mathbf{D}} \right), \quad (96)$$

$$\text{or} \quad \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H} = \epsilon \left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right) \qquad \text{and} \qquad \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} = \mu \left(\mathbf{H} - \frac{\mathbf{v}}{c} \times \mathbf{D} \right), \quad (97)$$

which can be expressed in tensor form as,

$$H_{\alpha\beta} u^\beta = \epsilon F_{\alpha\beta} u^\beta \qquad \text{and} \qquad \mathcal{F}_{\alpha\beta} u^\beta = \mu\, \mathcal{H}_{\alpha\beta} u^\beta, \quad (98)$$

$$\text{or} \qquad H^{\alpha\beta} u_\beta = \epsilon F^{\alpha\beta} u_\beta \qquad \text{and} \qquad \mathcal{F}^{\alpha\beta} u_\beta = \mu\, \mathcal{H}^{\alpha\beta} u_\beta, \quad (99)$$

using the dual tensors introduced in eqs. (84)-(86), as first noted by Minkowski [3]. The pairings in eqs. (98)-(99) follow those made for Maxwell's equations (89)-(90).

In the present case of a medium that is at rest in a rotating frame, the constitutive equations (96)-(97) hold in the lab frame, since these relations summarize physical effects in a very small region about the point of observation, for which a Lorentz transformation between the lab frame and the comoving local inertial frame of a point at rest in the rotating frame provides an adequate description. A remaining issue is the form of the constitutive equations in the noninertial rotating frame. This technical issue is not relevant to an analysis of quantities in the lab frame, as observed, for example, in the Wilson-Wilson experiment [4, 5].

A third constitutive equation is Ohm's law for the conduction current $\mathbf{J}_C$, which has the form,

$$\mathbf{J}_C = \sigma \mathbf{E} \quad (100)$$

in the rest frame of a medium with DC conductivity σ. We consider only the case of conduction currents in a neutral medium, so that their is no net charge density associated with the conduction current.[18] Then, it is consistent to write Ohm's law in a form similar to that of the Lorentz force law (76),

$$J_C^\alpha = (\tilde{\mathbf{J}}_C \cdot \mathbf{u}/c, \tilde{\mathbf{J}}_C) = \sigma F^{\alpha\beta} u_\beta = \sigma(\tilde{\mathbf{E}} \cdot \mathbf{u}/c, \tilde{\mathbf{E}} + \mathbf{u}/c \times \tilde{\mathbf{B}}), \tag{101}$$

$$J_{C\alpha} = (\mathbf{J}_C \cdot \tilde{\mathbf{u}}/c, -\mathbf{J}_C) = \sigma F_{\alpha\beta} u^\beta = \sigma(\mathbf{E} \cdot \tilde{\mathbf{u}}/c, -\mathbf{E} - \tilde{\mathbf{u}}/c \times \mathbf{B}), \tag{102}$$

where $\mathbf{u}$ is the velocity of the conductor relative to the lab frame. Thus, the conduction current is,

$$\tilde{\mathbf{J}}_C = \sigma\left(\tilde{\mathbf{E}} + \frac{\mathbf{u}}{c} \times \tilde{\mathbf{B}}\right) = \sigma\left(\mathbf{E} + \frac{\tilde{\mathbf{u}}}{c} \times \mathbf{B}\right) = \mathbf{J}_C \tag{103}$$

in a moving conductor. There is little physical significance to the time components $\sigma\tilde{\mathbf{E}} \cdot \mathbf{u}/c$ and $\sigma\mathbf{E} \cdot \tilde{\mathbf{u}}/c$ in the case of a moving conductor. For a more general discussion, see [48].

A.3 Transformations from Lab to Rotating Frame

The (cylindrical) coordinates in the rotating frame (denoted with a $'$) are related to those in the lab frame by,

$$t' = t, \qquad r' = r, \qquad \phi' = \phi - \omega t, \qquad z' = z, \tag{104}$$

so the contravariant, rectangular coordinate transformation is,

$$x'^0 = x^0, \quad x'^1 = x^1 \cos\frac{\omega}{c}x^0 + x^2 \sin\frac{\omega}{c}x^0, \quad x'^2 = -x^1 \sin\frac{\omega}{c}x^0 + x^2 \cos\frac{\omega}{c}x^0, \quad x'^3 = x^3, \tag{105}$$

whose inverse is,

$$x^0 = x'^0, \quad x^1 = x'^1 \cos\frac{\omega}{c}x'^0 - x'^2 \sin\frac{\omega}{c}x'^0, \quad x^2 = x'^1 \sin\frac{\omega}{c}x'^0 + x'^2 \cos\frac{\omega}{c}x'^0, \quad x^3 = x'^3. \tag{106}$$

A contravariant 4-vector A^α transforms according to,

$$A'^\alpha = \frac{\partial x'^\alpha}{\partial x^\beta} A^\beta, \tag{107}$$

while a covariant 4-vector A_α transforms according to,

$$A'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} A_\beta. \tag{108}$$

When writing $\partial x'^\alpha / \partial x^\beta$ and $\partial x^\beta / \partial x'^\alpha$ as matrices (and A^α and A_α as column vectors), indices α and β label the rows and columns, respectively.

[18] See prob. 11.16 of [47] for the case when convection currents are present.

A.3.1 Covariant Vectors and Tensors in the Rotating Frame

To transform covariant vectors and tensors from the lab frame to the rotating frame we use,

$$\frac{\partial x^\beta}{\partial x'^\alpha} = \begin{pmatrix} 1 & \frac{v_x}{c} & \frac{v_y}{c} & 0 \\ 0 & \cos\frac{\omega}{c}x^0 & -\sin\frac{\omega}{c}x^0 & 0 \\ 0 & \sin\frac{\omega}{c}x^0 & \cos\frac{\omega}{c}x^0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{v_x}{c} & \frac{v_y}{c} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{\omega}{c}x^0 & -\sin\frac{\omega}{c}x^0 & 0 \\ 0 & \sin\frac{\omega}{c}x^0 & \cos\frac{\omega}{c}x^0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{109}$$

where $\mathbf{v} = (v_x, v_y, 0) = (-\omega y, \omega x, 0)$ is the velocity in the lab frame of the point $\mathbf{x}'$ in the rotating frame. The transformation (109) is composed of a rotation about the z axis and another transformation that mixes space and time components. When describing 3-vectors, a rotation of the coordinate axes can be said to change the components of that vector, but not the direction and magnitude of that vector with respect to the "fixed stars". If we accept this view, then it suffices to describe the transformation from the lab frame to the rotating frame by only the first of the two transformations in the last form of eq. (109),

$$\left.\frac{\partial x^\beta}{\partial x'^\alpha}\right|_{\text{NR}} = \begin{pmatrix} 1 & \frac{v_x}{c} & \frac{v_y}{c} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{110}$$

where the subscript NR, means that a nonrotating basis for 3-vectors is used in the rotating frame.[19]

Using the form (110) in eq. (108), we find that the covariant 4-velocity $u_\alpha = (c, -\mathbf{u})$ transforms to,

$$u'_\alpha = (c', -\mathbf{u}') = (c(1 - \mathbf{u}\cdot\mathbf{v}/c^2), -\mathbf{u}) = (c, -\mathbf{u}), \tag{111}$$

after ignoring the second-order term $\mathbf{u}\cdot\mathbf{v}/c^2$. The covariant speed of light in the rotating frame remains c, but the covariant velocity $\mathbf{u}' = \mathbf{u}$ is unaffected by the transformation to the rotating frame. This alerts us to the need for care in interpreting vector and tensor quantities in the rotating frame. In particular, a particle at rest in the rotating frame has lab-frame velocity $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{x}$, so **the covariant velocity $\mathbf{u}'$ "observed" in the rotating frame of a particle at rest in that frame is nonzero!**[20]

Similarly, the covariant 4-current density $J_\alpha = (c\rho, -\mathbf{J})$ transforms to,

$$J'_\alpha = (c\rho', -\mathbf{J}') = (c(\rho - \mathbf{J}\cdot\mathbf{v}/c^2), -\mathbf{J}). \tag{112}$$

[19]The transformation (110) is also NonRelativistic in that terms of order v^2/c^2 have been omitted.

[20]This author has no idea how a measurement could be made in the rotating frame of a particle at rest in that frame so as to assign a nonzero velocity to that particle. Rather, it seems that the covariant velocity $\mathbf{u}' = \mathbf{u} = (\mathbf{u}-\mathbf{v}) + \mathbf{v}$ in the rotating frame has a "fictitious" component equal to the velocity $\mathbf{v}$ of the point of observation relative to the lab frame.

Thus, the covariant current density $\mathbf{J}' = \mathbf{J}$ is unaffected by the transformation to the rotating frame, while the covariant charge density $\rho' = \rho - \mathbf{J}\cdot\mathbf{v}/c^2$ follows that of a low-velocity Lorentz transformation.[21]

The transformation of a covariant tensor $F_{\alpha\beta}$ from the lab frame to the rotating frame (with nonrotating basis) has the form,

$$F'_{\alpha\beta} = \left.\frac{\partial x^\gamma}{\partial x'^\alpha}\right|_{\text{NR}} \left.\frac{\partial x^\delta}{\partial x'^\beta}\right|_{\text{NR}} F_{\gamma\delta}. \tag{113}$$

Applying this to the covariant electromagnetic tensors (69), (70) and (71) we find the covariant field vectors in the rotating frame to be,

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \qquad \mathbf{B}' = \mathbf{B}, \tag{114}$$

$$\mathbf{D}' = \mathbf{D} + \frac{\mathbf{v}}{c} \times \mathbf{H}, \qquad \mathbf{H}' = \mathbf{H}, \tag{115}$$

and,

$$\mathbf{P}' = \mathbf{P} - \frac{\mathbf{v}}{c} \times \mathbf{M}, \qquad \mathbf{M}' = \mathbf{M}. \tag{116}$$

A.3.2 Contravariant Vectors and Tensors in the Rotating Frame

We now have two methods to obtain the forms of contravariant vectors and tensors in the rotating frame. We can use eq. (107) to go directly from the lab frame to the rotating frame, once we have found the version of $\partial x'^\alpha/\partial x^\beta$ that applies to a nonrotating basis in the rotating frame. Or, we can use the metric tensor g' in the rotating frame (once we have identified this) to go from covariant to contravariant vectors and tensors in that frame according to,

$$A'^\alpha = g'^{\alpha\beta} A'_\beta, \qquad F'^{\alpha\beta} = g'^{\alpha\gamma} g'^{\beta\delta} F'_{\gamma\delta}. \tag{117}$$

For these two approaches to be consistent, we need that,

$$A'^\alpha = g'^{\alpha\beta} A'_\beta = g'^{\alpha\gamma} \left.\frac{\partial x^\beta}{\partial x'^\gamma}\right|_{\text{NR}} A_\beta = g'^{\alpha\gamma} \left.\frac{\partial x^\delta}{\partial x'^\gamma}\right|_{\text{NR}} g_{\delta\beta} A^\beta = \left.\frac{\partial x'^\alpha}{\partial x^\beta}\right|_{\text{NR}} A^\beta, \tag{118}$$

i.e., that,

$$\left.\frac{\partial x'^\alpha}{\partial x^\beta}\right|_{\text{NR}} = g'^{\alpha\gamma} \left.\frac{\partial x^\delta}{\partial x'^\gamma}\right|_{\text{NR}} g_{\delta\beta}. \tag{119}$$

Recalling eq. (105), we have that,

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{v'_x}{c} & \cos\frac{\omega}{c}x^0 & \sin\frac{\omega}{c}x^0 & 0 \\ \frac{v'_y}{c} & -\sin\frac{\omega}{c}x^0 & \cos\frac{\omega}{c}x^0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{v'_x}{c} & 1 & 0 & 0 \\ \frac{v'_y}{c} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\frac{\omega}{c}x^0 & \sin\frac{\omega}{c}x^0 & 0 \\ 0 & -\sin\frac{\omega}{c}x^0 & \cos\frac{\omega}{c}x^0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{120}$$

[21]This behavior is called the magnetic Galilean transformation in [42].

where $\mathbf{v}' = (v'_x, v'_y, 0) = (\omega y', -\omega x', 0)$ is the velocity in the rotating frame of the point in the lab frame whose present coordinates are $\mathbf{x}'$ in the rotating frame. As before, we omit the rotation so that vectors in the rotating frame are defined with respect to a nonrotating basis. Then velocity $\mathbf{v}' = -\mathbf{v}$, where as before $\mathbf{v}$ is the velocity of point $\mathbf{x}'$ with respect to the lab frame. Hence,

$$\left.\frac{\partial x'^\alpha}{\partial x^\beta}\right|_{\rm NR} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{v_x}{c} & 1 & 0 & 0 \\ -\frac{v_y}{c} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{121}$$

To identify the metric tensor g' we recall eqs. (104)-(105) to write the invariant interval as,

$$\begin{aligned} ds^2 &= d(x^0)^2 - d(x^1)^2 - d(x^2)^2 - d(x^3)^2 \\ &\approx d(x'^0)^2 - d(x'^1)^2 - d(x'^2)^2 - d(x'^3)^2 - 2\frac{\omega x'^2}{c} d(x'^0)d(x'^1) - 2\frac{\omega x'^1}{c} d(x'^0)d(x'^2), \end{aligned} \tag{122}$$

where we ignore a term of order $\omega^2 r^2/c^2$.[22] Thus, the (symmetric) metric tensor in the rotating frame is,

$$g'_{\alpha\beta} = g'^{\alpha\beta} \approx \begin{pmatrix} 1 & -\frac{v_x}{c} & -\frac{v_y}{c} & 0 \\ -\frac{v_x}{c} & -1 & 0 & 0 \\ -\frac{v_y}{c} & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{123}$$

where we again write $(\omega x'^2, -\omega x'^1)$ as $(-v_x, -v_y)$.[23]

As a check, we verify that,

$$\begin{aligned} g'^{\alpha\gamma} \left.\frac{\partial x^\delta}{\partial x'^\gamma}\right|_{\rm NR} g_{\delta\beta} &= \begin{pmatrix} 1 & -\frac{v_x}{c} & -\frac{v_y}{c} & 0 \\ -\frac{v_x}{c} & -1 & 0 & 0 \\ -\frac{v_y}{c} & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & \frac{v_x}{c} & \frac{v_y}{c} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{v_x}{c} & 1 & 0 & 0 \\ -\frac{v_y}{c} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \left.\frac{\partial x'^\alpha}{\partial x^\beta}\right|_{\rm NR}, \end{aligned} \tag{124}$$

[22]By considering only radii where the velocity ωr is small compared to c we avoid difficulties such as Ehrenfest's paradox. For the latter, see [41].

[23]The determinant of g' is unity, which excuses some glibness in the definition of the co- and contravariant derivative operators in eq. (74).

on neglect of terms of order v^2/c^2.

Using the form (121) in eq. (107), we find that the contravariant 4-velocity $u^\alpha = (c, \tilde{\mathbf{u}}) = (c, \mathbf{u})$ transforms to,

$$u'^\alpha = (\tilde{c}', \tilde{\mathbf{u}}') = (c, \tilde{\mathbf{u}} - \mathbf{v}) = (c, \mathbf{u} - \mathbf{v}) = g'^{\alpha\beta} u'_\beta. \tag{125}$$

The contravariant speed of light remains c, and the contravariant velocity $\tilde{\mathbf{u}}' = \mathbf{u} - \mathbf{v}$ behaves as expected for a low-velocity Lorentz transformation to the rotating frame. Thus, we might feel that the contravariant velocity is of greater physical significance than the covariant velocity $\mathbf{u}' = \mathbf{u}$ in the rotating frame.

Similarly, the contravariant 4-current density $J^\alpha = (c\tilde{\rho}, \tilde{\mathbf{J}}) = (c\rho, \mathbf{J})$ transforms to,

$$J'^\alpha = (c\tilde{\rho}', \tilde{\mathbf{J}}') = (c\tilde{\rho}, \tilde{\mathbf{J}} - \tilde{\rho}\mathbf{v}) = (c\rho, \mathbf{J} - \rho\mathbf{v}) = g'^{\alpha\beta} J'_\beta. \tag{126}$$

The the contravariant charge density $\tilde{\rho}' = \rho$ is unaffected by the transformation to the rotating frame, while the contravariant current density $\tilde{\mathbf{J}}' = \mathbf{J} - \rho\mathbf{v}$ follows that of a low-velocity Lorentz transformation.[24]

The transformation of a contravariant tensor $F^{\alpha\beta}$ from the lab frame to the rotating frame (with nonrotating basis) has the form,

$$F'^{\alpha\beta} = \left.\frac{\partial x'^\alpha}{\partial x^\gamma}\right|_{\text{NR}} \left.\frac{\partial x'^\beta}{\partial x^\delta}\right|_{\text{NR}} F^{\gamma\delta}. \tag{127}$$

Applying this to the contravariant electromagnetic tensors (69), (70) and (71) we find the contravariant field vectors in the rotating frame to be,

$$\tilde{\mathbf{E}}' = \tilde{\mathbf{E}} = \mathbf{E}, \qquad \tilde{\mathbf{B}}' = \tilde{\mathbf{B}} - \frac{\mathbf{v}}{c} \times \tilde{\mathbf{E}} = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}, \tag{128}$$

$$\tilde{\mathbf{D}}' = \tilde{\mathbf{D}} = \mathbf{D}, \qquad \tilde{\mathbf{H}}' = \tilde{\mathbf{H}} - \frac{\mathbf{v}}{c} \times \tilde{\mathbf{D}} = \mathbf{H} - \frac{\mathbf{v}}{c} \times \mathbf{D}, \tag{129}$$

and,

$$\tilde{\mathbf{P}}' = \tilde{\mathbf{P}} = \mathbf{P}, \qquad \tilde{\mathbf{M}}' = \tilde{\mathbf{M}} + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{P}} = \mathbf{M} + \frac{\mathbf{v}}{c} \times \mathbf{P}. \tag{130}$$

A.3.3 Consistency of the Transformations of the Field Tensors F and $\mathcal{F}$

When we apply the covariant transformation (113) to the covariant dual field tensor $\mathcal{F}_{\alpha\beta}$ defined in eq. (84), whose components involve the contravariant 3-vectors $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{E}}$, we obtain,

$$\tilde{\mathbf{B}}' = \tilde{\mathbf{B}} - \frac{\mathbf{v}}{c} \times \tilde{\mathbf{E}} = \mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}, \qquad \tilde{\mathbf{E}}' = \tilde{\mathbf{E}} = \mathbf{E}, \tag{131}$$

which is consistent with the results (128) of the contravariant transformation (127) of the contravariant tensor $F^{\alpha\beta}$ of (69). Likewise, the contravariant transformation (127) of the contravariant dual field tensor $\mathcal{F}^{\alpha\beta}$ defined in eq. (82) yields the same results (114),

$$\mathbf{B}' = \mathbf{B}, \qquad \mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{132}$$

[24] This behavior is called the electric Galilean transformation in [42].

as does the covariant transformation (113) of the covariant tensor $F_{\alpha\beta}$ of (69).

Hence, it is consistent that the covariant dual tensor $\mathcal{F}_{\alpha\beta}$ is described in terms of the contravariant fields $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{E}}$ of the contravariant field tensor $F^{\alpha\beta}$, rather than in terms of the covariant fields $\mathbf{B}$ and $\mathbf{E}$ of the covariant field tensor $F_{\alpha\beta}$.

The relations between the covariant and contravariant fields in the rotating frame follow from eqs. (112), (114)-(116) and (126), (128)-(130) as,

$$\mathbf{B}' = \tilde{\mathbf{B}}' + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{E}}', \qquad \mathbf{D}' = \tilde{\mathbf{D}}' + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{H}}', \qquad \mathbf{E}' = \tilde{\mathbf{E}}' + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{B}}', \qquad \mathbf{H}' = \tilde{\mathbf{H}}' + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{D}}',$$

$$\mathbf{P}' = \tilde{\mathbf{P}}' - \frac{\mathbf{v}}{c} \times \tilde{\mathbf{M}}', \qquad \mathbf{M}' = \tilde{\mathbf{M}}' - \frac{\mathbf{v}}{c} \times \tilde{\mathbf{P}}', \qquad \rho' = \tilde{\rho}' - \frac{\tilde{\mathbf{J}}' \cdot \mathbf{v}}{c^2}, \qquad \mathbf{J}' = \tilde{\mathbf{J}}' + \tilde{\rho}'\mathbf{v}, \qquad (133)$$

and,

$$\tilde{\mathbf{B}}' = \mathbf{B}' - \frac{\mathbf{v}}{c} \times \mathbf{E}', \qquad \tilde{\mathbf{D}}' = \mathbf{D}' - \frac{\mathbf{v}}{c} \times \mathbf{H}', \qquad \tilde{\mathbf{E}}' = \mathbf{E}' - \frac{\mathbf{v}}{c} \times \mathbf{B}', \qquad \tilde{\mathbf{H}}' = \mathbf{H}' - \frac{\mathbf{v}}{c} \times \mathbf{D}',$$

$$\tilde{\mathbf{P}}' = \mathbf{P}' + \frac{\mathbf{v}}{c} \times \mathbf{M}', \qquad \tilde{\mathbf{M}}' = \mathbf{M}' + \frac{\mathbf{v}}{c} \times \mathbf{P}', \qquad \tilde{\rho}' = \rho' + \frac{\mathbf{J}' \cdot \mathbf{v}}{c^2}, \qquad \tilde{\mathbf{J}}' = \mathbf{J}' - \rho'\mathbf{v}, \qquad (134)$$

to order v/c. These relations also follow from raising or lowering indices using the metric tensor, $F'^{\alpha\beta} = g'^{\alpha\gamma} g'^{\beta\delta} F'_{\gamma\delta}$, $F'_{\alpha\beta} = g'_{\alpha\gamma} g'_{\beta\delta} F'^{\gamma\delta}$, *etc.*

A.3.4 The Derivative Operators ∂'_α and ∂'^α in the Rotating Frame

If we regard the lab-frame derivative operators ∂'_α and ∂'^α, defined in eq. (74), as 4-vectors then we expect that their forms in the rotating frame could be written as,

$$\partial'_\alpha = \left(\frac{\partial}{\partial ct'}, \boldsymbol{\nabla}'\right) = \left(\frac{\partial}{\partial ct} + \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}, \boldsymbol{\nabla}\right) \quad \text{and} \quad \partial'^\alpha = \left(\frac{\partial}{\partial ct'}, -\tilde{\boldsymbol{\nabla}}'\right) = \left(\frac{\partial}{\partial ct}, -\boldsymbol{\nabla} - \frac{\mathbf{v}}{c}\frac{\partial}{\partial ct}\right), \qquad (135)$$

recalling the transformations (110) and (121). Thus, we are led to two different meanings of the operator $\partial/\partial t'$. To determine whether either of the operations ∂'_α and ∂'^α is meaningful, we consider whether they imply that charge is conserved in the rotating frame,

$$\partial'_\alpha J'^\alpha = \boldsymbol{\nabla}' \cdot \tilde{\mathbf{J}}' + \frac{\partial \tilde{\rho}'}{\partial t'} = \boldsymbol{\nabla} \cdot \tilde{\mathbf{J}} - \boldsymbol{\nabla} \cdot \tilde{\rho}\mathbf{v} + \frac{\partial \tilde{\rho}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\tilde{\rho} = \boldsymbol{\nabla} \cdot \tilde{\mathbf{J}} + \frac{\partial \tilde{\rho}}{\partial t} = 0, \qquad (136)$$

noting that $\boldsymbol{\nabla} \cdot \tilde{\rho}\mathbf{v} = (\mathbf{v} \cdot \boldsymbol{\nabla})\tilde{\rho}$ since $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$ according to eq. (19). However,

$$\begin{aligned} \partial'^\alpha J'_\alpha &= \tilde{\boldsymbol{\nabla}}' \cdot \mathbf{J}' + \frac{\partial \rho'}{\partial t'} = \boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\mathbf{v}}{c}\frac{\partial \mathbf{J}}{\partial ct} + \frac{\partial \rho}{\partial t} - \frac{\partial \mathbf{J} \cdot \mathbf{v}/c}{ct} = \boldsymbol{\nabla} \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} - \mathbf{J} \cdot \frac{\partial \mathbf{v}/c}{ct} \\ &= -\mathbf{J} \cdot \frac{\partial \mathbf{v}/c}{ct} \neq 0, \end{aligned} \qquad (137)$$

since $\mathbf{v}$ is the lab-frame velocity of a point that is at rest in the rotating frame.

Hence, it appears that we can use the covariant derivative operator ∂'_α, but not the contravariant derivative operator ∂'^α, to obtain meaningful physical results in the rotating frame. This issue has been discussed in greater detail by Crater [30]. A consequence of this is that Maxwell's equations should be expressed in the rotating frame only in terms of the covariant derivative operator (sec. A.6).

A.3.5 The Relation between Free, Bound and Total Charge and Current Densities in the Rotating Frame

While we might expect the relations (94) between the lab-frame free and total[25] charge and current densities would transform to

$$J'^{\beta} = J'^{\beta}_{\rm free} + c\,\partial'_\alpha M'^{\alpha\beta} \qquad \text{or} \qquad J'_\beta = J'_{{\rm free}\beta} + c\,\partial'^\alpha M'_{\alpha\beta}, \tag{138}$$

in the rotating frame, we understand from sec. A.3.4 that only the contravariant current density J'^α will be physical when nonzero electric or magnetic polarization are present. The covariant current density will be nonphysical (when nonzero electric or magnetic polarization are present) due to the bad behavior of the contravariant derivative operator ∂'^α. Hence, it is better to avoid use of the covariant components of polarization in the rotating frame when relating polarization to sources of the electromagnetic fields.

The relation between the contravariant total and free charge and current densities can be written as,

$$\tilde\rho' = \tilde\rho'_{\rm free} + \tilde\rho'_{\rm bound}, \qquad \tilde{\mathbf J}' = \tilde{\mathbf J}'_{\rm free} + \tilde{\mathbf J}'_{\rm bound} \tag{139}$$

where,

$$\tilde\rho'_{\rm bound} = -\boldsymbol\nabla'\cdot\tilde{\mathbf P}', \qquad \tilde{\mathbf J}'_{\rm bound} = \frac{\partial\tilde{\mathbf P}'}{\partial t'} + c\,\boldsymbol\nabla'\times\tilde{\mathbf M}'. \tag{140}$$

We can verify that eq. (139) is consistent with the transformations (126), (130) and (135),

$$\tilde\rho = \tilde\rho' = \tilde\rho'_{\rm free} - \boldsymbol\nabla'\cdot\tilde{\mathbf P}' = \tilde\rho_{\rm free} - \boldsymbol\nabla\cdot\tilde{\mathbf P}, \tag{141}$$

$$\begin{aligned}
\tilde{\mathbf J}' &= \tilde{\mathbf J} - \tilde\rho\mathbf v = \tilde{\mathbf J}_{\rm free} + \frac{\partial\tilde{\mathbf P}}{\partial t} + c\,\boldsymbol\nabla\times\tilde{\mathbf M} - \tilde\rho_{\rm free}\mathbf v + \mathbf v(\boldsymbol\nabla\cdot\tilde{\mathbf P}) \\
&= \tilde{\mathbf J}'_{\rm free} + \frac{\partial\tilde{\mathbf P}'}{\partial t'} + c\,\boldsymbol\nabla'\times\tilde{\mathbf M}' \\
&= \tilde{\mathbf J}_{\rm free} - \tilde\rho_{\rm free}\mathbf v + \frac{\partial\tilde{\mathbf P}}{\partial t} + \mathbf v(\boldsymbol\nabla\cdot\tilde{\mathbf P}) - \boldsymbol\nabla\times(\mathbf v\times\tilde{\mathbf P}) + c\,\boldsymbol\nabla\times\tilde{\mathbf M} + \boldsymbol\nabla\times(\mathbf v\times\tilde{\mathbf P}) \\
&= \tilde{\mathbf J}_{\rm free} + \frac{\partial\tilde{\mathbf P}}{\partial t} + c\,\boldsymbol\nabla\times\tilde{\mathbf M} - \tilde\rho_{\rm free}\mathbf v + \mathbf v(\boldsymbol\nabla\cdot\tilde{\mathbf P}),
\end{aligned} \tag{142}$$

where we have used the (subtle) relation (22) between the time derivatives $\partial\tilde{\mathbf P}'/\partial t' = \partial\tilde{\mathbf P}/\partial t'$ and $\partial\tilde{\mathbf P}/\partial t$.

Despite the difficulty in using eq. (138) to define covariant components of bound charge and current densities, we will find ourselves desiring expressions for them. Therefore, we use the relations (133)-(134) to convert the contravariant bound charge and current densities (140) to covariant components (to order v/c),

$$\begin{aligned}
\rho'_{\rm bound} &= \tilde\rho'_{\rm bound} - \frac{\mathbf v}{c^2}\cdot\tilde{\mathbf J}'_{\rm bound} = -\boldsymbol\nabla'\cdot\tilde{\mathbf P}' - \frac{\mathbf v}{c^2}\cdot\left(\frac{\partial\tilde{\mathbf P}'}{\partial t'} + c\,\boldsymbol\nabla'\times\tilde{\mathbf M}'\right) \\
&= -\boldsymbol\nabla'\cdot\mathbf P' - \boldsymbol\nabla'\cdot\left(\frac{\mathbf v}{c}\times\mathbf M'\right) - \frac{\mathbf v}{c^2}\cdot\left(\frac{\partial\mathbf P'}{\partial t'} + c\,\boldsymbol\nabla'\times\mathbf M'\right)
\end{aligned} \tag{143}$$

[25] In sec. A.6 we will be led to a more general concept of the total charge and current densities in the rotating frame than that implied by eq. (138).

$$
\begin{aligned}
&= -\boldsymbol{\nabla}' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega} \cdot \mathbf{M}'}{c} - \frac{\mathbf{v}}{c^2} \cdot \frac{\partial \mathbf{P}'}{\partial t'}, \\
\mathbf{J}'_{\text{bound}} &= \tilde{\mathbf{J}}'_{\text{bound}} + \tilde{\rho}'_{\text{bound}} \mathbf{v} = \frac{\partial \tilde{\mathbf{P}}'}{\partial t'} + c\, \boldsymbol{\nabla}' \times \tilde{\mathbf{M}}' - \mathbf{v}(\boldsymbol{\nabla}' \cdot \tilde{\mathbf{P}}') \\
&= \frac{\partial \mathbf{P}'}{\partial t'} + c\, \boldsymbol{\nabla}' \times \mathbf{M}' + \frac{\mathbf{v}}{c} \times \frac{\partial \mathbf{M}'}{\partial t'} + \boldsymbol{\nabla}' \times (\mathbf{v} \times \mathbf{P}') - \mathbf{v}(\boldsymbol{\nabla}' \cdot \mathbf{P}') \\
&= \frac{\partial \mathbf{P}'}{\partial t'} + c\, \boldsymbol{\nabla}' \times \mathbf{M}' + \frac{\mathbf{v}}{c} \times \frac{\partial \mathbf{M}'}{\partial t'} + (\mathbf{P}' \cdot \boldsymbol{\nabla})\mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{P}', \qquad (144)
\end{aligned}
$$

recalling eq. (13). These forms are counterintuitive, but they are consistent with transforming the lab-frame bound charge and current densities $\rho_{\text{bound}} = -\boldsymbol{\nabla} \cdot \mathbf{P}$ and $\mathbf{J}_{\text{bound}} = \partial \mathbf{P}/\partial t + c\boldsymbol{\nabla} \times \mathbf{M}$ to the rotating frame using eq. (116).

For later use we record the relation between the contravariant and covariant charge and current densities in the rotating frame, following eq. (134),

$$
\tilde{\rho}'_{\text{bound}} = \rho'_{\text{bound}} + \frac{\mathbf{v}}{c^2} \cdot \mathbf{J}'_{\text{bound}} = -\boldsymbol{\nabla}' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega} \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}' \times \mathbf{M}', \qquad (145)
$$

$$
\begin{aligned}
\tilde{\mathbf{J}}'_{\text{bound}} &= \mathbf{J}'_{\text{bound}} - \rho'_{\text{bound}} \mathbf{v} \\
&= \frac{\partial \mathbf{P}'}{\partial t'} + c\, \boldsymbol{\nabla}' \times \mathbf{M}' + \mathbf{v}(\boldsymbol{\nabla}' \cdot \mathbf{P}') + \frac{\mathbf{v}}{c} \times \frac{\partial \mathbf{M}'}{\partial t'} + (\mathbf{P}' \cdot \boldsymbol{\nabla})\mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{P}'. \qquad (146)
\end{aligned}
$$

The forms (139) and (143)-(146) reinforce that the contravariant forms of charge and current densities in the rotating frame, as expressed in terms of contravariant field components (139) correspond most closely to our experience in inertial frames. Both the covariant forms (143)-(144) and the contravariant forms (145)-(146) expressed in terms of covariant field components include terms whose significance is not immediately evident. From Newtonian mechanics in rotating frames, we are used to the notion of "fictitious" forces that appear "real" to observers at rest in the rotating frame. The electrodynamics of rotating systems is even more complex in that it leads us to multiple expressions for physical quantities, and these are especially intricate in the case of bound charge and current densities.

The form (145) for the contravariant charge density in terms of covariant components indicates that magnetization in the rotating frame is associated with a bound charge distribution. This gives another perspective on the well-known result (see, for example, sec. 88 of [49], sec. 18-6 of [50] and also [46][26]) that a nonzero bound charge density in the lab frame is associated with moving/rotating magnetization.

A.4 Lorentz Force and Ohm's Law in the Rotating Frame

Now that we have expressions for both covariant and contravariant vectors and tensors in the rotating frame, we can evaluate the Lorentz force on a charge q there.

[26]Consider a cylinder that has unit permittivity ϵ and uniform magnetization $\mathbf{M}_0$ parallel to its axis when are rest in the (inertial) lab frame. Then, when the cylinder rotates with angular velocity $\boldsymbol{\omega}$ about its axis in the lab frame, it has radial covariant electric polarization $\mathbf{P} = \mathbf{v}/c \times \mathbf{M}_0 = (\omega M_0 r/c)\,\hat{\mathbf{r}}$ in the lab frame, while in the rotating frame $\mathbf{P}' = 0$ and the axial covariant magnetization is $\mathbf{M}' = \mathbf{M}_0$ [46]. The bound charge density in the bulk of the cylinder is $\rho_{\text{bound}} = -(1/r)d(rP_r)/dr = -2\omega M_0/c$ in the lab frame and $\tilde{\rho}'_{\text{bound}} = -2\boldsymbol{\omega} \cdot \mathbf{M}'/c = -2\omega M_0/c = \rho_{\text{bound}}$ in the rotating frame.

The contravariant Lorentz force in the rotating frame is,

$$f'^{\alpha} = qF'^{\alpha\beta}u'_{\beta} = q(\tilde{\mathbf{E}}' \cdot \mathbf{u}'/c, \tilde{\mathbf{E}}' + \mathbf{u}'/c \times \tilde{\mathbf{B}}') = q(\mathbf{E} \cdot \mathbf{u}/c, \mathbf{E} + \mathbf{u}/c \times \mathbf{B}), \tag{147}$$

plus terms of order uv/c^2. The contravariant 3-vector Lorentz force in the rotating frame is,

$$\tilde{\mathbf{f}}' = q\left(\tilde{\mathbf{E}}' + \frac{\mathbf{u}'}{c} \times \tilde{\mathbf{B}}'\right) = q\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right) = \mathbf{f}, \tag{148}$$

which equals the 3-vector Lorentz force in the lab frame as expected. However, if one works only with quantities in the rotating frame, one must remember to use the counterintuitive covariant 3-velocity $\mathbf{u}' = \mathbf{u}$ together with the contravariant electric and magnetic fields to obtain a valid result. This prescription has the peculiar feature that a particle at rest in the lab frame ($\mathbf{u} = 0$) feels no magnetic force according to an observer in the rotating frame, although the naive description is that such a particle has velocity $-\mathbf{v}$ in the rotating frame.

We can rewrite eq. (148) as,

$$\tilde{\mathbf{f}}' = q\left(\tilde{\mathbf{E}}' + \frac{\tilde{\mathbf{u}}'}{c} \times \tilde{\mathbf{B}}'\right) + q\frac{\mathbf{v}}{c} \times \tilde{\mathbf{B}}', \tag{149}$$

where $\tilde{\mathbf{u}}' = \mathbf{u} - \mathbf{v}$ is "clearly" the velocity of the charge in the rotating frame. Then, the first term in eq. (149) has the form we expect for the Lorentz force law, while the second term can be called a "fictitious", frame-dependent magnetic force.

The covariant Lorentz force in the rotating frame is,

$$f'_{\alpha} = qF'_{\alpha\beta}u'^{\beta} = q(\mathbf{E}' \cdot \tilde{\mathbf{u}}'/c, -\mathbf{E}' - \tilde{\mathbf{u}}'/c \times \mathbf{B}') = q(\mathbf{E} \cdot (\mathbf{u} - \mathbf{v})/c, -\mathbf{E} - \mathbf{u}/c \times \mathbf{B}), \tag{150}$$

plus terms of order uv/c^2. The covariant 3-vector Lorentz force $\mathbf{f}'$ in the rotating frame is,

$$\mathbf{f}' = q\left(\mathbf{E}' + \frac{\tilde{\mathbf{u}}'}{c} \times \mathbf{B}'\right) = q\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right) = \mathbf{f}. \tag{151}$$

Again, the 3-vector Lorentz force in the rotating frame equals that in the lab frame, but the form of the Lorentz force in the rotating frame matches that expected from experience in inertial frames.

Thus, the covariant form (151) of the Lorentz force law in the rotating frame is more appealing than its contravariant form (148).

The covariant Lorentz force (150) on the contravariant charge and current densities in the rotating frame is,

$$f'_{\alpha} = F'_{\alpha\beta}J'^{\beta}/c = (\tilde{\mathbf{J}}' \cdot \mathbf{E}'/c, -\tilde{\rho}'\mathbf{E}' - \tilde{\mathbf{J}}'/c \times \mathbf{B}'). \tag{152}$$

The covariant 3-vector Lorentz force $\mathbf{f}'$ in the rotating frame is,

$$\mathbf{f}' = \tilde{\rho}'\mathbf{E}' + \frac{\tilde{\mathbf{J}}'}{c} \times \mathbf{B}' = (\tilde{\rho}'_{\text{free}} + \tilde{\rho}'_{\text{bound}})\,\mathbf{E}' + \frac{\tilde{\mathbf{J}}'_{\text{free}} + \tilde{\mathbf{J}}'_{\text{bound}}}{c} \times \mathbf{B}', \tag{153}$$

where the contravariant bound charge and current densities $\tilde{\rho}'_{\text{bound}}$ and $\tilde{\mathbf{J}}'_{\text{bound}}$ can be expressed via covariant field components as in eqs. (145)-(146).

In a similar manner, Ohm's law (101)-(102) transforms to the rotating frame as,

$$J'^{\alpha}_{C} = (\tilde{\mathbf{J}}'_C \cdot \mathbf{u}'/c, \tilde{\mathbf{J}}'_C) = \sigma F'^{\alpha\beta} u'_\beta = \sigma(\tilde{\mathbf{E}}' \cdot \mathbf{u}'/c, \tilde{\mathbf{E}}' + \mathbf{u}'/c \times \tilde{\mathbf{B}}'), \tag{154}$$

$$J'_{C\alpha} = (\mathbf{J}'_C \cdot \tilde{\mathbf{u}}'/c, -\mathbf{J}'_C) = \sigma F'_{\alpha\beta} u'^\beta = \sigma(\mathbf{E}' \cdot \tilde{\mathbf{u}}'/c, -\mathbf{E}' - \tilde{\mathbf{u}}'/c \times \mathbf{B}'), \tag{155}$$

Both the covariant and the contravariant conduction-current 3-vectors $\mathbf{J}_C$ and $\tilde{\mathbf{J}}_C$ in the rotating frame are equal to the conduction current $\mathbf{J}_C$ in the lab frame,

$$\tilde{\mathbf{J}}'_C = \sigma\left(\tilde{\mathbf{E}}' + \frac{\mathbf{u}'}{c} \times \tilde{\mathbf{B}}'\right) = \sigma\left(\mathbf{E} + \frac{\mathbf{u}}{c} \times \mathbf{B}\right) = \mathbf{J}_C = \mathbf{J}'_C = \sigma\left(\mathbf{E}' + \frac{\tilde{\mathbf{u}}'}{c} \times \mathbf{B}'\right), \tag{156}$$

However, the awkward form of the covariant 3-velocity $\mathbf{u}'$ may again lead one to prefer use of the covariant version $\mathbf{J}'_C$ of the conduction current in the rotating frame.

A.5 Constitutive Equations in the Rotating Frame

We can now evaluate the constitutive equations (98) for a linear, isotropic medium at rest in the rotating frame, where they have the form,

$$H'^{\alpha\beta} u'_\beta = \epsilon F'^{\alpha\beta} u'_\beta \qquad \text{and} \qquad \mathcal{F}'^{\alpha\beta} u'_\beta = \mu \mathcal{H}'^{\alpha\beta} u'_\beta, \tag{157}$$

$$\text{or} \qquad H'_{\alpha\beta} u'^\beta = \epsilon F'_{\alpha\beta} u'^\beta \qquad \text{and} \qquad \mathcal{F}'_{\alpha\beta} u'^\beta = \mu \mathcal{H}'_{\alpha\beta} u'^\beta. \tag{158}$$

Recalling eqs. (147), (150) and the duality transformation (86), the constitutive relations associated with the use of covariant fields tensors (158) are,

$$\mathbf{D}' + \frac{\tilde{\mathbf{u}}'}{c} \times \mathbf{H}' = \epsilon\left(\mathbf{E}' + \frac{\tilde{\mathbf{u}}'}{c} \times \mathbf{B}'\right), \qquad \tilde{\mathbf{B}}' - \frac{\tilde{\mathbf{u}}'}{c} \times \tilde{\mathbf{E}}' = \mu\left(\tilde{\mathbf{H}}' - \frac{\tilde{\mathbf{u}}'}{c} \times \tilde{\mathbf{D}}'\right), \tag{159}$$

and those associated with use of contravariant field tensors (157) are,

$$\tilde{\mathbf{D}}' + \frac{\mathbf{u}'}{c} \times \tilde{\mathbf{H}}' = \epsilon\left(\tilde{\mathbf{E}}' + \frac{\mathbf{u}'}{c} \times \tilde{\mathbf{B}}'\right), \qquad \mathbf{B}' - \frac{\mathbf{u}'}{c} \times \mathbf{E}' = \mu\left(\mathbf{H}' - \frac{\mathbf{u}'}{c} \times \mathbf{D}'\right), \tag{160}$$

where $\mathbf{u}' = \mathbf{v}$ and $\tilde{\mathbf{u}}' = 0$ are the co- and contravariant 3-velocities of a point that is at rest in the rotating frame. The counterintuitive result that $\mathbf{u}' = \mathbf{v}$ again leads to possibly surprising forms of the constitutive equations in the rotating frame,

$$\mathbf{D}' = \epsilon\mathbf{E}', \qquad \tilde{\mathbf{B}}' = \mu\tilde{\mathbf{H}}', \tag{161}$$

from the use of covariant tensors, and,

$$\tilde{\mathbf{D}}' + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{H}}' = \epsilon\left(\tilde{\mathbf{E}}' + \frac{\mathbf{v}}{c} \times \tilde{\mathbf{B}}'\right), \qquad \mathbf{B}' - \frac{\mathbf{v}}{c} \times \mathbf{E}' = \mu\left(\mathbf{H}' - \frac{\mathbf{v}}{c} \times \mathbf{D}'\right), \tag{162}$$

from the use of contravariant tensors.[27,28]

[27]Equation (161) does not include the form $\mathbf{B}' = \mu\mathbf{H}'$ as naively expected, because the components of the covariant dual field tensors involve contravariant, not covariant, fields $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{E}}$, as confirmed in sec. A.3.3.

[28]Equations (161)-(162) are not independent in view of eqs. (133)-(134).

If we use the relations (114)-(115) and (128)-(128) between the co- and contravariant electromagnetic field in the lab and rotating frames, we quickly recover the constitutive equations (96) in the lab frame, to order v/c.

We can also write the constitutive equations (161)-(162) in terms of the covariant fields $\mathbf{B}'$, $\mathbf{E}'$, $\mathbf{P}'$ and $\mathbf{M}'$ by noting that $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$ and $\mathbf{H}' = \mathbf{B}' - 4\pi\mathbf{M}'$, so that to order v/c,

$$\begin{aligned}
\mathbf{P}' &= \frac{\epsilon - 1}{4\pi}\mathbf{E}', \\
\mathbf{M}' &= \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} + \left(\frac{1}{\mu} - \epsilon\right)\frac{\mathbf{v}}{c} \times \frac{\mathbf{E}'}{4\pi} = \left(1 - \frac{1}{\mu}\right)\frac{\mathbf{B}'}{4\pi} - \frac{\epsilon\mu - 1}{\mu(\epsilon - 1)}\frac{\mathbf{v}}{c} \times \mathbf{P}'. \qquad (163)
\end{aligned}$$

Similarly, writing the constitutive equation in terms of the contravariant components $\tilde{\mathbf{B}}'$, $\tilde{\mathbf{E}}'$, $\tilde{\mathbf{P}}'$ and $\tilde{\mathbf{M}}'$ we find,

$$\begin{aligned}
\tilde{\mathbf{P}}' &= \frac{\epsilon - 1}{4\pi}\tilde{\mathbf{E}}' + \left(\epsilon - \frac{1}{\mu}\right)\frac{\mathbf{v}}{c} \times \frac{\tilde{\mathbf{B}}'}{4\pi} = \frac{\epsilon - 1}{4\pi}\tilde{\mathbf{E}}' + \frac{\epsilon\mu - 1}{\mu - 1}\frac{\mathbf{v}}{c} \times \tilde{\mathbf{M}}', \\
\tilde{\mathbf{M}}' &= \left(1 - \frac{1}{\mu}\right)\frac{\tilde{\mathbf{B}}'}{4\pi}. \qquad (164)
\end{aligned}$$

Compare eqs. (163)-(164) with the lab-frame relations (5).

A.6 Maxwell's Equations in the Rotating Frame

A covariant transcription of the lab-frame Maxwell's equations (81) and (88) into the rotating frame is,

$$\partial'^{\alpha}\mathcal{F}'_{\alpha\beta} = 0 \qquad \text{and} \qquad \partial'^{\alpha}H'_{\alpha\beta} = \frac{4\pi}{c}J'_{\text{free}\beta}, \qquad (165)$$

or,

$$\partial'_{\alpha}\mathcal{F}'^{\alpha\beta} = 0 \qquad \text{and} \qquad \partial'_{\alpha}H'^{\alpha\beta} = \frac{4\pi}{c}J'^{\beta}_{\text{free}}. \qquad (166)$$

However, as discussed in sec. A.3.4 and by Crater [30], the use of the contravariant derivative operator ∂'^{α} does not lead to meaningful physical results in the rotating frame. So, we restrict our discussion of Maxwell's equations in the rotating frame to the form (166), which can be written in terms of the electric and magnetic field vectors in the rotating frame as,

$$\boldsymbol{\nabla}'\cdot\tilde{\mathbf{D}}' = 4\pi\tilde{\rho}'_{\text{free}}, \qquad \boldsymbol{\nabla}'\times\mathbf{E}' + \frac{\partial\mathbf{B}'}{\partial ct'} = 0, \qquad \boldsymbol{\nabla}'\cdot\mathbf{B}' = 0, \qquad \boldsymbol{\nabla}'\times\tilde{\mathbf{H}}' - \frac{\partial\tilde{\mathbf{D}}'}{\partial ct'} = \frac{4\pi}{c}\tilde{\mathbf{J}}'_{\text{free}}, \quad (167)$$

where $\tilde{\rho}' = \rho$ and $\tilde{\mathbf{J}}' = \mathbf{J} - \rho\mathbf{v}$. We can use the relations (134) to obtain the first and fourth Maxwell equations of eq. (167) in terms of the covariant field vectors in the rotating frame as,

$$\boldsymbol{\nabla}'\cdot\mathbf{D}' = 4\pi\left(\tilde{\rho}'_{\text{free}} + \boldsymbol{\nabla}'\cdot\frac{\mathbf{v}}{c}\times\frac{\mathbf{H}'}{4\pi}\right) = 4\pi\left(\tilde{\rho}'_{\text{free}} + \frac{\boldsymbol{\omega}\cdot\mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{c}\cdot\boldsymbol{\nabla}'\times\frac{\mathbf{H}'}{4\pi}\right), \qquad (168)$$

recalling eq. (13), and,

$$\boldsymbol{\nabla}'\times\mathbf{H}' - \frac{\partial\mathbf{D}'}{\partial ct'} = \frac{4\pi}{c}\left[\tilde{\mathbf{J}}'_{\text{free}} + \boldsymbol{\nabla}'\times\left(\mathbf{v}\times\frac{\mathbf{D}'}{4\pi}\right) - \frac{\partial}{\partial t'}\left(\frac{\mathbf{v}}{c}\times\frac{\mathbf{H}'}{4\pi}\right)\right]. \qquad (169)$$

We insert eqs. (168)-(169) into each other and keep terms only to order v/c to find,

$$\boldsymbol{\nabla}' \cdot \mathbf{D}' = 4\pi \left(\tilde{\rho}'_{\rm free} - \frac{\mathbf{v} \cdot \tilde{\mathbf{J}}'_{\rm free}}{c^2} + \frac{\boldsymbol{\omega} \cdot \mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{D}'}{\partial ct'} \right), \qquad (170)$$

and,

$$\boldsymbol{\nabla}' \times \mathbf{H}' - \frac{\partial \mathbf{D}'}{\partial ct'} = \frac{4\pi}{c} \left(\tilde{\mathbf{J}}'_{\rm free} + \tilde{\rho}'_{\rm free}\mathbf{v} + \left(\frac{\mathbf{D}'}{4\pi} \cdot \boldsymbol{\nabla} \right) \mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla}) \frac{\mathbf{D}'}{4\pi} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{H}'}{\partial t'} \right), \qquad (171)$$

using $\partial \mathbf{v}/\partial t' = 0$. We note the appearance in eqs. (170)-(171) of the covariant charge and current densities,

$$\rho'_{\rm free} = \tilde{\rho}'_{\rm free} - \frac{\mathbf{v} \cdot \tilde{\mathbf{J}}'_{\rm free}}{c^2} = \rho_{\rm free} - \frac{\mathbf{v} \cdot \mathbf{J}_{\rm free}}{c^2}, \qquad (172)$$

$$\mathbf{J}'_{\rm free} = \tilde{\mathbf{J}}'_{\rm free} + \tilde{\rho}'_{\rm free}\mathbf{v} = \mathbf{J}_{\rm free}. \qquad (173)$$

This is formally satisfactory, but counterintuitive, as it requires us to use a free current density in the rotating frame that is equal to the free current density in the lab frame. We therefore propose to use a different characterization of the source terms,

$$\boldsymbol{\nabla}' \cdot \mathbf{D}' = 4\pi \tilde{\rho}'_{\rm free,total}, \qquad \boldsymbol{\nabla}' \times \mathbf{H}' - \frac{\partial \mathbf{D}'}{\partial ct'} = \frac{4\pi}{c} \tilde{\mathbf{J}}'_{\rm free,total}, \qquad (174)$$

where,

$$\tilde{\rho}'_{\rm free,total} = \tilde{\rho}'_{\rm free} + \rho'_{\rm other}, \qquad (175)$$

$$\tilde{\rho}'_{\rm free} = \rho_{\rm free}, \qquad (176)$$

$$\rho'_{\rm other} = -\frac{\mathbf{v} \cdot \tilde{\mathbf{J}}'_{\rm free}}{c^2} + \frac{\boldsymbol{\omega} \cdot \mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{D}'}{\partial ct'}, \qquad (177)$$

$$\tilde{\mathbf{J}}'_{\rm free,total} = \tilde{\mathbf{J}}'_{\rm free} + \mathbf{J}'_{\rm other}, \qquad (178)$$

$$\tilde{\mathbf{J}}'_{\rm free} = \mathbf{J}_{\rm free} - \rho_{\rm free}\mathbf{v}, \qquad (179)$$

$$\mathbf{J}'_{\rm other} = \tilde{\rho}'_{\rm free}\mathbf{v} + \left(\frac{\mathbf{D}'}{4\pi} \cdot \boldsymbol{\nabla} \right) \mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla}) \frac{\mathbf{D}'}{4\pi} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{H}'}{\partial t'}. \qquad (180)$$

However, $\rho'_{\rm other}$ and $\mathbf{J}'_{\rm other}$ are not components of a covariant nor of a contravariant 4-vector. These terms are the "fictitious" charge and current densities first identified by Schiff [6], which can be nonzero in vacuum as well as inside conductors and inside polarizable media.

We have adopted the attitude that the charge and current densities $\tilde{\rho}'_{\rm free}$ and $\tilde{\mathbf{J}}'_{\rm free}$ are "real" to an observer in the rotating frame. However, we see that $\rho'_{\rm other}$ of eq. (177) involves the term $-\mathbf{v} \cdot \tilde{\mathbf{J}}'_{\rm free}/c^2$, and the discussion of secs. 2.2.3-4 indicates that we could regard this term as "fictitious". This alerts us to ambiguities as to the meaning of the terms "real" and "fictitious", and we will not use these terms further.

To express Maxwell's equations in the rotating frame in terms of the fields $\mathbf{E}'$ and $\mathbf{B}$ and the total charge and current densities $\tilde{\rho}'$ and $\tilde{\mathbf{J}}'$, we use $\tilde{\mathbf{D}}' = \tilde{\mathbf{E}}' + 4\pi\tilde{\mathbf{P}}'$ and $\tilde{\mathbf{H}}' = \tilde{\mathbf{B}}' - 4\pi\tilde{\mathbf{M}}'$

in eq. (167) to find,

$$
\begin{aligned}
\boldsymbol{\nabla}' \cdot \tilde{\mathbf{E}}' &= 4\pi(\tilde{\rho}'_{\rm free} - \boldsymbol{\nabla}' \cdot \tilde{\mathbf{P}}') = 4\pi\tilde{\rho}', &(181)\\
\boldsymbol{\nabla}' \times \tilde{\mathbf{B}}' - \frac{\partial \tilde{\mathbf{E}}'}{\partial ct'} &= \frac{4\pi}{c}\left(\tilde{\mathbf{J}}'_{\rm free} + \frac{\partial \tilde{\mathbf{P}}'}{\partial t'} + c\boldsymbol{\nabla}' \times \tilde{\mathbf{M}}'\right) = \frac{4\pi}{c}\tilde{\mathbf{J}}', &(182)
\end{aligned}
$$

recalling eqs. (139)-(140). We use the relations (134) to convert eqs.(181)-(182) to covariant field vectors in the rotating frame as,

$$
\boldsymbol{\nabla}' \cdot \mathbf{E}' = 4\pi\left(\tilde{\rho}' + \boldsymbol{\nabla}' \cdot \frac{\mathbf{v}}{c} \times \frac{\mathbf{B}'}{4\pi}\right) = 4\pi\left(\tilde{\rho}' + \frac{\boldsymbol{\omega} \cdot \mathbf{B}'}{2\pi c} - \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}' \times \frac{\mathbf{B}'}{4\pi}\right), \tag{183}
$$

recalling eq. (13), and,

$$
\boldsymbol{\nabla}' \times \mathbf{B}' - \frac{\partial \mathbf{E}'}{\partial ct'} = \frac{4\pi}{c}\left[\tilde{\mathbf{J}}' + \boldsymbol{\nabla}' \times \left(\mathbf{v} \times \frac{\mathbf{E}'}{4\pi}\right) - \frac{\partial}{\partial t'}\left(\frac{\mathbf{v}}{c} \times \frac{\mathbf{B}'}{4\pi}\right)\right]. \tag{184}
$$

We insert eqs. (183)-(184) into each other and keep terms only to order v/c to find,

$$
\begin{aligned}
\boldsymbol{\nabla}' \cdot \mathbf{E}' &= 4\pi\left(\tilde{\rho}' - \frac{\mathbf{v} \cdot \tilde{\mathbf{J}}'}{c^2} + \frac{\boldsymbol{\omega} \cdot \mathbf{B}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{E}'}{\partial ct'}\right)\\
&= 4\pi\left(\tilde{\rho}' + \rho'_{\rm more}\right) \equiv 4\pi\rho'_{\rm total}\,, &(185)
\end{aligned}
$$

and,

$$
\begin{aligned}
\boldsymbol{\nabla}' \times \mathbf{B}' - \frac{\partial \mathbf{E}'}{\partial ct'} &= \frac{4\pi}{c}\left(\tilde{\mathbf{J}}' + \tilde{\rho}'\mathbf{v} + \left(\frac{\mathbf{E}'}{4\pi} \cdot \boldsymbol{\nabla}\right)\mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla})\frac{\mathbf{E}'}{4\pi} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{B}'}{\partial t'}\right)\\
&= \frac{4\pi}{c}\left(\tilde{\mathbf{J}}' + \mathbf{J}'_{\rm more}\right) \equiv \frac{4\pi}{c}\mathbf{J}'_{\rm total}\,, &(186)
\end{aligned}
$$

where,

$$
\begin{aligned}
\rho'_{\rm more} &= -\frac{\mathbf{v} \cdot \tilde{\mathbf{J}}'}{c^2} + \frac{\boldsymbol{\omega} \cdot \mathbf{B}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c} \cdot \frac{\partial \mathbf{E}'}{\partial ct'}\,, &(187)\\
\mathbf{J}'_{\rm more} &= \tilde{\rho}'\mathbf{v} + \left(\frac{\mathbf{E}'}{4\pi} \cdot \boldsymbol{\nabla}\right)\mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla})\frac{\mathbf{E}'}{4\pi} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{B}'}{\partial t'}\,. &(188)
\end{aligned}
$$

An awkwardness in eqs. (185)-(186) is that the contravariant total charge and current densities $\tilde{\rho}'$ and $\tilde{\mathbf{J}}'$ contain contributions from the contravariant polarization densities $\tilde{\mathbf{P}}'$ and $\tilde{\mathbf{M}}'$, whereas we have otherwise preferred to use covariant fields in the rotating frame. Hence, it is more consistent to use contravariant component only for the free charge and current densities $\tilde{\rho}'_{\rm free}$ and $\tilde{\mathbf{J}}'_{\rm free}$, and to rewrite the total charge and current densities as,

$$
\begin{aligned}
\tilde{\rho}' &= \tilde{\rho}'_{\rm free} + \tilde{\rho}'_{\rm bound} = \tilde{\rho}'_{\rm free} - \boldsymbol{\nabla}' \cdot \mathbf{P}' - \frac{2\boldsymbol{\omega} \cdot \mathbf{M}'}{c} + \frac{\mathbf{v}}{c} \cdot \boldsymbol{\nabla}' \times \mathbf{M}', &(189)\\
\tilde{\mathbf{J}}' &= \tilde{\mathbf{J}}'_{\rm free} + \tilde{\mathbf{J}}'_{\rm bound}\\
&= \tilde{\mathbf{J}}'_{\rm free} + \frac{\partial \mathbf{P}'}{\partial t'} + c\,\boldsymbol{\nabla}' \times \mathbf{M}' + \mathbf{v}(\boldsymbol{\nabla}' \cdot \mathbf{P}') + \frac{\mathbf{v}}{c} \times \frac{\partial \mathbf{M}'}{\partial t'} + (\mathbf{P}' \cdot \boldsymbol{\nabla})\mathbf{v} - (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{P}', &(190)
\end{aligned}
$$

recalling eqs. (145)-(146).

Using eqs. (189)-(190) we can write the total charge and current densities $\rho'_{\rm total}$ and $\mathbf{J}'_{\rm total}$ that were defined in eqs. (185)-(186) as,

$$\begin{aligned}
\rho'_{\rm total} &= \tilde\rho'_{\rm free} + \tilde\rho'_{\rm bound} + \rho'_{\rm more} \\
&= \tilde\rho'_{\rm free} - \boldsymbol\nabla'\cdot\mathbf{P}' - \frac{2\boldsymbol\omega\cdot\mathbf{M}'}{c} + \frac{\mathbf{v}}{c}\cdot\boldsymbol\nabla'\times\mathbf{M}' \\
&\qquad -\frac{\mathbf{v}}{c^2}\cdot\left(\tilde{\mathbf{J}}'_{\rm free} + \frac{\partial\mathbf{P}'}{\partial t'} + c\boldsymbol\nabla'\times\mathbf{M}'\right) + \frac{\boldsymbol\omega\cdot\mathbf{B}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial\mathbf{E}'}{\partial ct'} \\
&= \tilde\rho'_{\rm free} - \frac{\mathbf{v}}{c^2}\cdot\tilde{\mathbf{J}}'_{\rm free} - \boldsymbol\nabla'\cdot\mathbf{P}' + \frac{\boldsymbol\omega\cdot\mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial\mathbf{D}'}{\partial ct'} \\
&= \tilde\rho'_{\rm free,total} - \boldsymbol\nabla'\cdot\mathbf{P}', \qquad (191) \\
\mathbf{J}'_{\rm total} &= \tilde{\mathbf{J}}'_{\rm free} + \tilde{\mathbf{J}}'_{\rm bound} + \mathbf{J}'_{\rm more}, \\
&= \tilde{\mathbf{J}}'_{\rm free} + \frac{\partial\mathbf{P}'}{\partial t'} + c\boldsymbol\nabla'\times\mathbf{M}' + \mathbf{v}(\boldsymbol\nabla'\cdot\mathbf{P}') + \frac{\mathbf{v}}{c}\times\frac{\partial\mathbf{M}'}{\partial t'} + (\mathbf{P}'\cdot\boldsymbol\nabla)\mathbf{v} - (\mathbf{v}\cdot\boldsymbol\nabla)\mathbf{P}' \\
&\qquad +\mathbf{v}\left(\tilde\rho'_{\rm free} - \boldsymbol\nabla'\cdot\mathbf{P}' - \frac{2\boldsymbol\omega\cdot\mathbf{M}'}{c} + \frac{\mathbf{v}}{c}\cdot\boldsymbol\nabla'\times\mathbf{M}'\right) + \left(\frac{\mathbf{E}'}{4\pi}\cdot\boldsymbol\nabla\right)\mathbf{v} - (\mathbf{v}\cdot\boldsymbol\nabla)\frac{\mathbf{E}'}{4\pi} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial\mathbf{B}'}{\partial t'} \\
&= \tilde{\mathbf{J}}'_{\rm free} + \frac{\partial\mathbf{P}'}{\partial t'} + c\boldsymbol\nabla'\times\mathbf{M}' + \tilde\rho'_{\rm free}\mathbf{v} + \left(\frac{\mathbf{D}'}{4\pi}\cdot\boldsymbol\nabla\right)\mathbf{v} - (\mathbf{v}\cdot\boldsymbol\nabla)\frac{\mathbf{D}'}{4\pi} - \frac{\mathbf{v}}{4\pi c}\times\frac{\partial\mathbf{H}'}{\partial t'} \\
&= \tilde{\mathbf{J}}'_{\rm free,total} + \frac{\partial\mathbf{P}'}{\partial t'} + c\boldsymbol\nabla'\times\mathbf{M}', \qquad (192)
\end{aligned}$$

where the charge and current densities $\rho'_{\rm free,total}$ and $\mathbf{J}'_{\rm free,total}$ were defined in eqs. (175)-(178). Thus, we have multiple ways of accounting for the source terms in Maxwell's equations for the fields $\mathbf{E}'$ and $\mathbf{B}'$ in the rotating frame. Because the "other" source terms depend on the fields in the rotating frame, Maxwell's equations cannot in general be solved directly for the fields in this frame. Rather, an iterative approach is required, for which the somewhat inelegant forms (191)-(192) and may be of use.

B Appendix: Rotating Charged Cylinder (May 17, 2022)

For an example in which "fictitious" charge and current densities are significant, we consider an infinite cylindrical shell of radius a about the z-axis with uniform surface charge density[29] that rotates with constant angular velocity $\boldsymbol\omega = \omega\,\hat{\mathbf{z}}$, where $u = \omega a \ll c$ is the speed of a point on the rotating cylinder. The lab-frame polarization densities are zero, $\mathbf{P} = 0 = \mathbf{M}$.

In the inertial lab frame, and in cylindrical coordinates (r, ϕ, z), the cylinder has surface charge $\lambda_{\rm free}$ per unit length in z, and hence azimuthal current $I_{\rm free} = \lambda_{\rm free}\,\omega/2\pi$ per unit length. The corresponding (static) electromagnetic fields are,

$$\mathbf{E} = \begin{cases} 0 & (r < a), \\ \frac{2\lambda_{\rm free}}{r}\,\hat{\mathbf{r}} & (r > a), \end{cases} \qquad \mathbf{B} = \begin{cases} \frac{2\lambda_{\rm free}\,\omega}{c}\,\hat{\mathbf{z}} & (r < a), \\ 0 & (r > a). \end{cases} \qquad (193)$$

[29] We call this charge density "free" in the sense that it is not associated with charge bound in atoms.

In the frame that rotates with angular velocity $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ about the z-axis, the free charge and current (per unit length along z, at radius $r' = r = a$) are, according to eq. (42),

$$\lambda'_{\rm free} = \lambda_{\rm free}, \qquad I'_{\rm free} = 0, \tag{194}$$

and the electromagnetic fields are, according to eq. (43),

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} = \begin{cases} \frac{2\lambda_{\rm free}\omega^2 r'}{c^2}\,\hat{\mathbf{r}}' & (r' < a), \\ \frac{2\lambda_{\rm free}}{r'}\,\hat{\mathbf{r}}' & (r' > a), \end{cases} \qquad \mathbf{B}' = \mathbf{B}, \tag{195}$$

noting that $r' = r$ and $v = \omega r'$. These fields do not obey the static Maxwell equations,

$$\boldsymbol{\nabla}' \cdot \mathbf{E}' = 4\pi\rho'_{\rm free}, \qquad \boldsymbol{\nabla}' \times \mathbf{B}' = \frac{4\pi}{c}\mathbf{J}'_{\rm free}, \tag{196}$$

even though $\boldsymbol{\nabla}' = \boldsymbol{\nabla}$. To have (static) Maxwell equations in the rotating frame with the same general form as in the inertial lab frame, we write,

$$\boldsymbol{\nabla}' \cdot \mathbf{E}' = 4\pi\rho'_{\rm free,total}, \qquad \boldsymbol{\nabla}' \times \mathbf{B}' = \frac{4\pi}{c}\mathbf{J}'_{\rm free,total}, \tag{197}$$

with,

$$\rho'_{\rm free,total} = \rho'_{\rm free} + \rho'_{\rm other} \qquad \mathbf{J}'_{\rm free,total} = \mathbf{J}'_{\rm free} + \mathbf{J}'_{\rm other} \tag{198}$$

where $\rho'_{\rm other}$ and $\mathbf{J}'_{\rm other}$ are charge and current densities apparent to observers in the rotating frame, but not to observers in the inertial lab frame.

From the symmetry of this example, the "other" charge and current densities are only functions of r'. Then, the fields (195) and the Maxwell equations (197) require that,

$$\rho'_{\rm other} = \begin{cases} \frac{1}{4\pi}\boldsymbol{\nabla}' \cdot \mathbf{E}' = \frac{1}{4\pi r'}\frac{d}{dr'}(r' E_{r'}) = \frac{\lambda_{\rm free}\omega^2}{\pi c^2} = \frac{\omega \cdot B'}{2\pi c} & (r' < a), \\ 0 & (r' > a), \end{cases} \tag{199}$$

and, since the magnetic field is the same in the lab frame and the rotating frame we must have a surface current on the cylinder in the rotating frame with,

$$I'_{\rm other} = I_{\rm free} = \frac{\lambda_{\rm free}\,\omega}{2\pi}\,. \tag{200}$$

Recalling the general forms (53) and (54) for $\rho'_{\rm other}$ and $\mathbf{J}'_{\rm other}$, we see that eq. (199) corresponds to the second term in eq. (53) (and the other two terms are zero in the present example), while eq. (200) corresponds to the first term in eq. (54) (and the other three terms are zero in the present example).

The physical character of the "other" charges and currents of eqs. (199)-(200) is unclear. For example, a device (ammeter) to detect conduction currents would detect nothing if used at rest in the rotating frame. And, the region $r' < a$ in the rotating frame could be vacuum, such that the charge density $\rho'_{\rm other}$ is not associated with charged matter.[30]

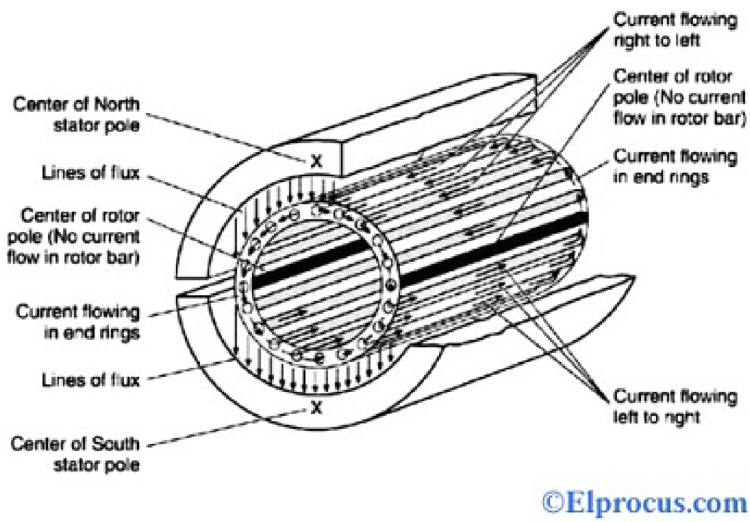
[30]Perhaps $\rho'_{\rm other}$ is more like Maxwell's vision of electric charge as a strain in the ether.

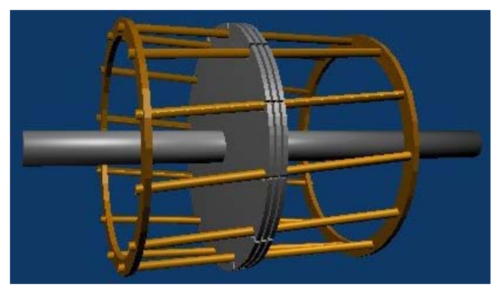
C Appendix: Measurements in the Rotor Frame of an Induction Motor (May 19, 2022)

This Appendix was inspired by e-discussions with Luca Zilberti.

For an example of measurements in a rotating frame where "fictitious" charge and current densities are negligible, we consider an induction motor.[31]

The concept of a 1-phase induction motor is shown the left sketch below.[32] The outer portion of the motor, the stator (an electromagnet), is fixed in the lab frame and generates a transverse, time-dependent magnetic field in the interior of the motor. There, the rotor includes a conducting "squirrel cage" of copper bars/rods whose ends are connected to copper rings, a version of which is sketched on the right below. The "squirrel cage" is filled with steel laminations (to enhance the **B** field compared to **H**, which latter is driven by the electrical windings in the stator), 3 of which are shown in the center of the right figure below.[33]





The time-dependent magnetic field of the stator induces electrical currents in the conducting loops of the "squirrel cage," such that the $\mathbf{I} \times \mathbf{B}$ force on these currents torques the rotor, leading to rotation of the latter at almost, but not quite, that of the angular velocity of the (rotating) magnetic field of the stator.[34] Hence, in the rotating frame of the rotor, the magnetic field has a rather low frequency, around 1 Hz for a 50-60 Hz induction motor.

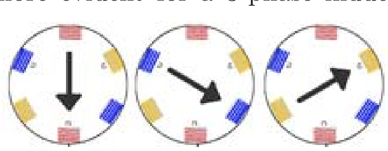
Electrical Measurements [53]-[57] have been made in the rotating frame of the rotor of a

[31]https://en.wikipedia.org/wiki/Induction_motor. For a historical review, see chap. IV of [52].

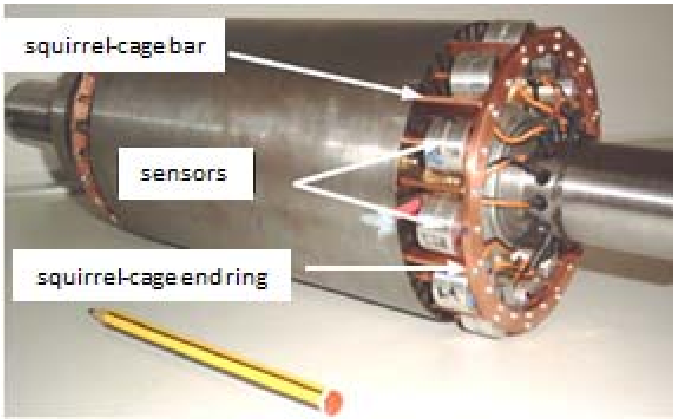
[32]A 1-phase induction motor is not self-starting without additional components, while induction motors with 2 or more phase are self starting, and are more common.

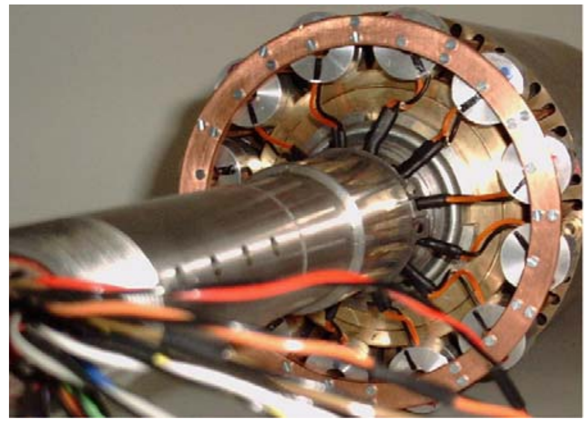
[33]The bars/rods of the squirrel cage are typically skewed, as shown in the right figure above, to enhance the motor's performance in a manner not relevant to this Appendix.

[34]The rotation of the stator field is more evident for a 3-phase induction motor, as illustrated in an animation at the Wiki page cited above.

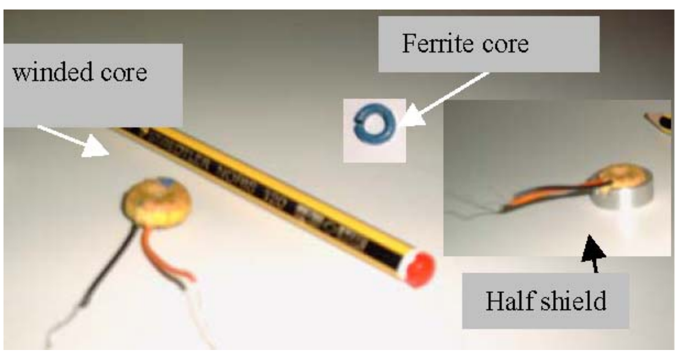


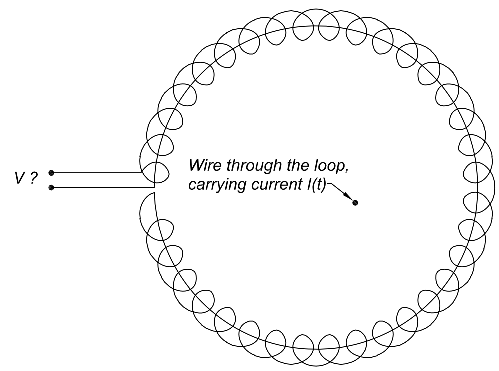
3-phase induction motor,[35] with the goal of determining the (time-dependent) current in the copper "bars" (or rods) of the "squirrel cage" of the rotor. As shown in the photos below, the 24 copper bars of a special "squirrel cage" extended some distance beyond the laminations of the rotor, permitting "amp-clamp"-type sensors to be placed on 12 of the bars.



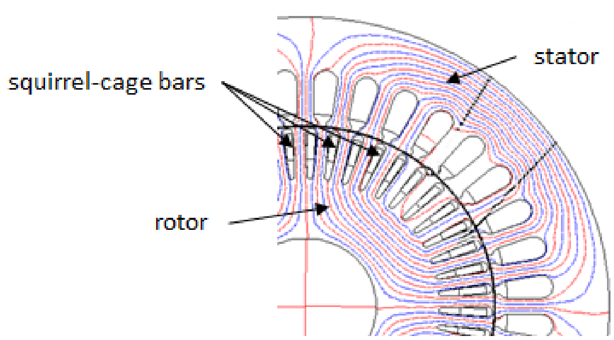


The sensors were ferrite cores with a toroidal winding, encased in an aluminum Faraday cage for electrical (but not magnetic) shielding, as shown in the photograph below. The sensors observe the azimuthal magnetic field around the axis of the ferrite cores, which is interpreted as a conduction current in the "squirrel-cage" bar that passes through the core. A sketch of the concept of an "amp-clamp" is shown on the right below, from [58].





The magnetic field lines in one quadrant of the induction motor are sketched below, at one time (and which lines rotate about the axis of the motor at 50-60 Hz in the lab frame). The 24 copper bars are in every second of the 48 slots in the rotor. The three phases of windings are in the 36 slots of the stator.



[35] https://en.wikipedia.org/wiki/Induction_motor

As noted above, a sensor (which is at rest in the rotating frame of the rotor) measures the sum of the azimuthal magnetic flux in the turns of the coil of the sensor. The sensors are filled with a ferrite that we take to have unit relative dielectric constant, $\epsilon = 1$, and large, constant relative permeability, $\mu \gg 1$. Then, fields inside the sensor (in the rotating frame) obey $\mathbf{D}' = \mathbf{E}'$ and $\mathbf{B}' = \mu\mathbf{H}' + (\mu - 1)\mathbf{v}/c \times \mathbf{H}' \approx \mu\mathbf{H}'$, according to eq. (61), where the approximation holds since $v \ll c$ for points on the rotating rotor. The Maxwell equation for $\mathbf{H}'$ is, from eqs. (52) and (54),

$$\boldsymbol{\nabla}' \times \mathbf{H}' - \frac{\partial \mathbf{E}'}{\partial ct'} = \frac{4\pi}{c}\left(\mathbf{J}'_{\rm free} + \rho'_{\rm free}\mathbf{v} + \left(\frac{\mathbf{E}'}{4\pi}\cdot\boldsymbol{\nabla}\right)\mathbf{v} - (\mathbf{v}\cdot\boldsymbol{\nabla})\frac{\mathbf{E}'}{4\pi} - \frac{\mathbf{v}}{4\pi c} \times \frac{\partial \mathbf{H}'}{\partial t'}.\right), \qquad (201)$$

Since $\rho'_{\rm free} = \rho_{\rm free} \approx 0$ in the ferrite core, we can neglect this term. The electric field $\mathbf{E}'$ in the rotor is induced via Faraday's law (in the rotating frame),

$$\boldsymbol{\nabla}' \times \mathbf{E}' = -\frac{1}{c}\frac{\partial \mathbf{B}'}{\partial t'} = -\frac{\mu}{c}\frac{\partial \mathbf{H}'}{\partial t'} \approx \omega'\frac{\mu}{c}\mathbf{H}' \approx \frac{1}{50}\frac{v}{r}\frac{\mu}{c}\mathbf{H}', \qquad \mathbf{E}' \approx \frac{\mu}{50}\frac{v}{c}\mathbf{H}', \qquad (202)$$

recalling that in the rotating frame the angular frequency ω' of the magnetic field is about 1/50 of the angular frequency $\omega = v/r$ of a point on the rotor in the lab frame. Hence, the third, fourth and fifth terms on the right side of eq. (201) all are of order v^2/c^2 and can be neglected in the present example. Similarly, the term $\partial\mathbf{E}'/\partial ct'$ is small (of order v/c^2) and we neglect it as well. In this (good) approximation, the Maxwell equation for $\mathbf{H}'$ is,

$$\boldsymbol{\nabla}' \times \mathbf{H}' = \frac{4\pi}{c}\mathbf{J}'_{\rm free} \left(= \boldsymbol{\nabla} \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_{\rm free}\right), \qquad (203)$$

recalling eqs. (42)-(43) and (135). Hence, the measurement of the magnetic flux in the ferrite-core sensors in the rotating frame is well interpreted as a measurement of the (free) conduction current in the bars of the "squirrel cage" of the rotor, just as it is for measurements in the inertial lab frame [58].

That is, in the measurement of this example, effects of "fictitious" currents in the rotating frame are unimportant.[36]

Acknowledgment

The author thanks J. Castro, V. Hnizdo and C. Ridgely for many e-discussions of this topic.

[36]If measurements had been made of the electric field $\mathbf{E}' = \mathbf{D}'$ in the rotating frame, these would include the effect of the "fictitious" charge density $\rho'_{\rm other}$, eq. (53),

$$\boldsymbol{\nabla}' \cdot \mathbf{E}' = 4\pi(\rho'_{\rm free} + \rho'_{\rm other}) \approx 4\pi\rho'_{\rm other}, \qquad \rho'_{\rm other} = -\frac{\mathbf{v}\cdot\mathbf{J}'_{\rm free}}{c^2} + \frac{\boldsymbol{\omega}\cdot\mathbf{H}'}{2\pi c} - \frac{\mathbf{v}}{4\pi c}\cdot\frac{\partial \mathbf{D}'}{\partial ct'} \approx -\frac{\mathbf{v}\cdot\mathbf{J}_{\rm free}}{c^2}, \quad (204)$$

where the approximation holds for the present example, where $\mathbf{H}'$ is largely perpendicular to the angular velocity $\boldsymbol{\omega}$, and $\mathbf{v}/4\pi c \cdot \partial\mathbf{D}'/\partial ct'$ is of order v^2/c^3 recalling eq. (202). The term $-\mathbf{v}\cdot\mathbf{J}_{\rm free}/c^2$ is very small, of order v/c^2, and so is largely negligible. That is, $\boldsymbol{\nabla}' \cdot \mathbf{E}' \approx 0$ in this example.

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