Transverse Waves on an Inelastic Rotating String

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1 Problem

What are the frequencies of small oscillations of an inelastic string of length l and linear mass density λ that is constrained to move on a plane which rotates at angular velocity Ω about a fixed axis in that plane, with one end of the string connected to a point on that axis?

Gravity can be neglected in this problem. Also, $\Omega l \ll c$, where c is the speed of light.

2 Solution

The equilibrium state of the string is that is lies along the line in the rotating plane that passes through the point of connecting and is perpendicular to the axis. Let x measure the distance along this line, with $x = 0$ at the axis. Let $s(x, t)$ be the (small) transverse displacement (in the rotating plane) of the string from its equilibrium state.

Then, in the rotating frame, an segment dx of the string about point x experiences a "fictitious" outward force $\lambda\ dx\ \Omega^2 x$, which is balanced by the x-component of the tension $T(x)$ in the string. For small oscillations the x-component of T is well approximated as T, so,

$$T(x + dx) - T(x) = T'\,dx = \lambda\Omega^2 x\,dx, \qquad T' = \lambda\Omega^2 x, \tag{1}$$

and,

$$T(x) = \frac{\lambda\Omega^2}{2}(l^2 - x^2), \tag{2}$$

noting that the tension vanishes at the free end, $T(l) = 0$.

We ignore effects of the Coriolis force in the approximation that the motion is purely transverse.

The equation of transverse motion for a segment of the string is,

$$\lambda\,dx\,\ddot{s} = T(x + dx)s'(x + dx) - T(x)s'(x) = \frac{\partial(Ts')}{\partial x}\,dx = \frac{\lambda\Omega^2}{2}\frac{\partial[(l^2 - x^2)s']}{\partial x}\,dx. \tag{3}$$

For oscillations at angular frequency ω of the form $s(x, t) = s(x)e^{i\omega t}$, eq. (3) reduces to,

$$\frac{d[(l^2 - x^2)s']}{dx} + \frac{2\omega^2}{\Omega^2}s = 0. \tag{4}$$

Changing to the dimensionless variable $z = x/l$, this becomes,

$$\frac{d}{dz}\left[(1 - z^2)\frac{ds}{dz}\right] + \frac{2\omega^2}{\Omega^2}s = 0. \tag{5}$$

We recognize this as Legrendre's equation, whose solutions are the Legendre polynomials $P_m(z)$ where,

$$\frac{2\omega^2}{\Omega^2} = m(m+1), \tag{6}$$

for non-negative integers m.

The string obeys the boundary condition that $s(0) = 0$, which is satisfied only by Legendre polynomials of odd m. Hence the frequencies of small oscillation are,

$$\omega = \Omega\sqrt{\frac{m(m+1)}{2}} \qquad (m \text{ odd}), \tag{7}$$

$$\omega = \Omega, \sqrt{6}\Omega, \sqrt{15}\Omega, ... \tag{8}$$

The corresponding waveforms are,

$$s(z) = s_0 z, \frac{s_0}{2}(5z^3 - 3z), \frac{s_0}{8}(63z^5 - 70z^3 + 15z), ... \tag{9}$$

where s_0 is the amplitude of the oscillation at $z = 1$ ($x = l$).

We have obtained this solution to a second-order differential equation using only a single boundary condition. Note that by expanding eq. (5), and setting $z = 1$, we obtain,

$$\frac{ds}{dz} = \frac{\omega^2}{\Omega^2} s \qquad (z = 1), \tag{10}$$

which is a kind of "automatic" boundary condition that can be satisfied by a real string.

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