Dielectric Cylinder
That Rotates in a Uniform Magnetic Field

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1 Problem

A cylinder of relative dielectric constant ε_r (and relative permeability $\mu_r = 1$) rotates with constant angular velocity ω about its axis. A uniform magnetic field $\mathbf{B}$ is parallel to the axis, in the same sense as $\boldsymbol{\omega}$. Find the resulting dielectric polarization $\mathbf{P}$ in the cylinder and the surface and volume charge densities σ and ρ, neglecting terms of order $(\omega a/c)^2$, where a is the radius of the cylinder.

This problem can be conveniently analyzed by starting in the rotating frame, in which $\mathbf{P}' = \mathbf{P}$ and $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$, when $(v/c)^2$ corrections are neglected. Consider also the electric displacement $\mathbf{D}$.

(Nov. 11, 2020.) This note has been largely superseded by http://kirkmcd.princeton.edu/examples/wilson.pdf, *which discusses how the result of the present note is not valid when $\mu_r \neq 1$.*

2 Solution

The $\mathbf{v} \times \mathbf{B}$ force on an atom in the rotating cylinder is radially outwards, and increasing linearly with radius, so we expect a positive radial polarization $\mathbf{P} = P\,\hat{\mathbf{r}}$ in cylindrical coordinates.

There will be an electric field $\mathbf{E}$ inside the dielectric associated with this polarization. We now have a "chicken-and-egg" problem: the magnetic field induces some polarization in the rotating cylinder, which induces some electric field, which induces some more polarization, *etc.*

One way to proceed is to follow this line of thought to develop an iterative solution for the polarization. This is done somewhat later in the solution. Or, we can avoid the iterative approach by going to the rotating frame, where there is no interaction between the medium and the magnetic field, but where there is an effective electric field $\mathbf{E}'$.

2.1 Solution via the Rotating Frame

However, we must be cautious when using the rotating frame as to what part of the lore of nonrotating frames still applies.

In the rotating frame, any polarization charge density is at rest, and so does not interact with the magnetic field. Individual molecules are polarized by the effective field $\mathbf{E}'$ according to $\mathbf{p}' = \alpha\mathbf{E}'$, where α is the (scalar) molecular polarizability, whose value is that same in any frame in which the molecules are at rest. Summing up the microscopic polarization, we obtain the macroscopic polarization density (in the rotating frame),

$$\mathbf{P}' = \chi \mathbf{E}', \tag{1}$$

where $\mathbf{E}'$ and $\mathbf{P}'$ are the electric field and dielectric polarization in the rotating frame, and χ is the (scalar) dielectric susceptibility. If $v = \omega r \ll c$, where c is the speed of light, then the electric field in the rotating frame is related to lab frame quantities by,

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}, \tag{2}$$

where $\mathbf{E}$ is the electric field due to the polarization that we have yet to find. Since polarization is charge times distance, in the nonrelativistic limit the polarization is the same in the lab frame and the rotating frame: $\mathbf{P}' = \mathbf{P}$.[1]

We do NOT expect that $\mathbf{E}' = 0$ (as would hold in the rest frame of a conductor with no external $\mathcal{EMF}$'s,[2]), since the polarization would vanish in this case.

The velocity has magnitude $v = \omega r$, and is in the azimuthal direction. Thus, $\mathbf{v} \times \mathbf{B} = \omega B\,\mathbf{r}$, so that,

$$\mathbf{P} = \chi \left(\mathbf{E} + \frac{\omega B}{c} \mathbf{r} \right) . \tag{3}$$

We need an additional relation to proceed. The suggestion is to consider the electric displacement $\mathbf{D}$. But, in which frame? This is the trickiest point in the problem. In the rotating (rest) frame of the dielectric, we expect that $\mathbf{D}' = \varepsilon_r \mathbf{E}'$ and (naively) that $\nabla' \cdot \mathbf{D}' = 4\pi \rho'_{\rm free}$, where $\rho'_{\rm free} = \rho_{\rm free}$ in the nonrelativistic limit. Since $\rho_{\rm free} = 0$ in the lab frame for this problem, the preceding arument would imply that $\mathbf{D}' = 0$, and hence that $\mathbf{E}' = 0$, which in turn implies that $\mathbf{P}' = \mathbf{P} = 0$, which is not the case!

It's safer to consider the displacement in the lab frame, where we know that $\rho_{\rm free} = 0$, and hence that $\mathbf{D} = 0$ since it has no sources. But we do not necessarily expect that $\mathbf{D} = \varepsilon_r \mathbf{E}$ in the lab frame, because in this frame we consider that the magnetic field is causing some of the polarization. So, we invoke the basic relation between $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{P}$ to write

$$\mathbf{D} = 0 = \mathbf{E} + 4\pi \mathbf{P}. \tag{4}$$

Thus,

$$\mathbf{E} = -4\pi \mathbf{P} \tag{5}$$

is the additional relation that we need. Recalling that $\chi = (\varepsilon_r - 1)/4\pi$, eq. (3) leads to,

$$\mathbf{P} = \frac{\chi}{1 + 4\pi\chi} \frac{\omega B}{c} \mathbf{r} = \frac{\varepsilon_r - 1}{4\pi\varepsilon_r} \frac{\omega B}{c} \mathbf{r}. \tag{6}$$

The surface charge density is,

$$\sigma_{\rm pol} = \mathbf{P}(a) \cdot \hat{\mathbf{r}} = \frac{\varepsilon_r - 1}{4\pi\varepsilon_r} \frac{\omega B a}{c} , \tag{7}$$

[1]For a more extensive discussion, see K.T. McDonald, *Electrodynamics of Rotating Systems* (Aug, 8, 2008), http://kirkmcd.princeton.edu/examples/rotatingEM.pdf

[2]K.T. McDonald *Conducting Sphere That Rotates in a Uniform Magnetic Field* (Mar. 13, 2002), http://kirkmcd.princeton.edu/examples/rotatingsphere.pdf

where a is the radius of the cylinder. As well as this surface charge density, there is a volume charge density,

$$\rho_{\rm pol} = -\boldsymbol{\nabla}\cdot\mathbf{P} = -\frac{1}{r}\frac{\partial r P_r}{\partial r} = -\frac{\varepsilon_r - 1}{2\pi\varepsilon_r}\frac{\omega B}{c}\,, \tag{8}$$

so that the cylinder remains neutral over all.

Both the surface and volume charge densities are proportional to $v(r)/c$, and are moving at velocity $v(r)$. Hence, the magnetic field created by these charges is of order v^2/c^2, and we neglect it in this analysis.

This example is perhaps noteworthy in that a nonvanishing, static volume charge density arises in a linear dielectric material (with no external charges). In pure electrostatics this cannot happen, since $\mathbf{P} = \chi\mathbf{E}$ together with $\boldsymbol{\nabla}\cdot\mathbf{D} = 0 = \nabla\cdot\mathbf{E} + 4\pi\boldsymbol{\nabla}\cdot\mathbf{P}$ imply that $\rho_{\rm pol} = -\boldsymbol{\nabla}\cdot\mathbf{P} = 0$.

We can now go back and examine the fields $\mathbf{E}'$ and $\mathbf{D}' = \varepsilon_r\mathbf{E}'$ in the rotating frame. Combining eqs. (2), (5) and (6) we find,

$$\mathbf{E} = -(\varepsilon_r - 1)\frac{\omega B}{c\varepsilon_r}\mathbf{r}, \qquad \mathbf{E}' = \frac{\omega B}{c\varepsilon_r}\mathbf{r}, \qquad \text{and hence} \qquad \mathbf{D}' = \frac{\omega B}{c}\mathbf{r}. \tag{9}$$

If the relative dielectric constant ε_r were unity (as if the cylinder were a vacuum), then eq. (9) tells us that the lab electric field would vanish, as expected. The result that $\mathbf{D}' = \omega B\,\mathbf{r}/c$ is independent of the dielectric constant, and holds even if the cylinder were empty. The fact that $\boldsymbol{\nabla}'\cdot\mathbf{D}' = 2\omega B/c \neq 0$ would imply that $\rho'_{\rm free} \neq 0$ IF $\boldsymbol{\nabla}'\cdot\mathbf{D}' = 4\pi\rho'_{\rm free}$. Since this cannot be, we must re-examine our assumptions.

A useful excerise is to transform the lab-frame Maxwell equation $\nabla\cdot\mathbf{D} = 4\pi\rho_{\rm free}$ into the rotating frame. For this, we note that $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$ transforms (in the nonrelativistic limit) to $\mathbf{E}' - \mathbf{v}/c\times\mathbf{B}' + 4\pi\mathbf{P}'$, and that $\rho_{\rm free}$ transforms to $\rho'_{\rm free}$. Hence, our transformed Maxwell equation is $\boldsymbol{\nabla}'\cdot(\mathbf{E}' - \mathbf{v}/c\times\mathbf{B}' + 4\pi\mathbf{P}') = 4\pi\rho'_{\rm free}$. If we suppose that the electric displacement in the rotating frame obeys the basic definition $\mathbf{D}' = \mathbf{E}' + 4\pi\mathbf{P}'$, then,

$$\boldsymbol{\nabla}'\cdot\mathbf{D}' = 4\pi\rho'_{\rm free} + \boldsymbol{\nabla}'\cdot\frac{\mathbf{v}}{c}\times\mathbf{B}'. \tag{10}$$

In the present problem, $\mathbf{B}' = \mathbf{B}$ to first order, and $\boldsymbol{\nabla}' = \boldsymbol{\nabla}$, so $\mathbf{v}/c\times\mathbf{B}' = \omega B\,\mathbf{r}/c$, whose divergence is $2\omega B/c$, which is the value for $\nabla\cdot\mathbf{D}'$ found above. Hence, we retain consistency with $\rho'_{\rm free} = 0$ while having a nonzero displacement $\mathbf{D}'$ in the rotating frame.

Experts will note that the result $\mathbf{D}' = \omega B\,\mathbf{r}/c$ is consistent with the (nonrelativistic) field transformation,[3]

$$\mathbf{D}' = \mathbf{D} + \frac{\mathbf{v}}{c}\times\mathbf{H}, \tag{11}$$

since in the present problem $\mathbf{D} = 0$ and $\mathbf{H} = \mathbf{B}$. Further, experts know that the lab frame relation of the displacement $\mathbf{D}$ to the field $\mathbf{E}$ involves the magnetic field $\mathbf{H}$ as well, according to (in the nonrelativistic limit),

$$\mathbf{D} = \varepsilon_r\mathbf{E} + (\varepsilon_r\mu_r - 1)\frac{\mathbf{v}}{c}\times\mathbf{H}. \tag{12}$$

Then, using $\mathbf{E}$ from eq. (9), plus $\mu_r = 1$ and $\mathbf{B} = \mathbf{H}$ again leads to the result that $\mathbf{D} = 0$ in the lab frame.

[3]See chap. E III of R. Becker, *Electromagnetic Fields and Interactions* (Dover, 1964).

For the record, we pursue the consequence of supposing that since there is no free charge in this problem, the displacement obeys $\mathbf{D}' = 0$ in the rotating frame. Then, since $\mathbf{D}' = \varepsilon \mathbf{E}'$ we have that $\mathbf{E}' = 0$, and eq. (1) implies that $\mathbf{P}' = \mathbf{P} = 0$ also. But, eq. (2) now tells us that $\mathbf{E} = -\omega B\,\mathbf{r}/c \neq 0$, so that $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \mathbf{E}$, and $\nabla \cdot \mathbf{D} = -2\omega B/c \neq 0$, independent of the dielectric constant. The rotating cylinder could be imaginary, and the above analysis still should hold. This is implausible.

2.2 Iterative Solution in the Lab Frame

The preceding analysis via the rotating from was somewhat tricky, so it is desirable to confirm the results by another method. Hence, we consider an iterative solution.

The axial magnetic field acts on the rotating molecules to cause a $\mathbf{v} \times \mathbf{B}$ force radially outwards. This can be described by an effective electric field,

$$\mathbf{E}_0 = \frac{\omega B}{c}\,\mathbf{r}. \tag{13}$$

This field causes polarization,

$$\mathbf{P}_0 = \chi \mathbf{E}_0 = \chi \frac{\omega B}{c}\,\mathbf{r}. \tag{14}$$

Associated with this is the uniform (bound) volume charge density,

$$\rho_0 = -\nabla \cdot \mathbf{P}_0 = -2\chi\omega B. \tag{15}$$

According to Gauss' Law, this charge density sets up a radial electric field,

$$\mathbf{E}_1 = 2\pi\rho_0\,\mathbf{r} = -4\pi\chi\omega B\,\mathbf{r}. \tag{16}$$

At the next iteration, the total polarization is,

$$\mathbf{P}_1 = \chi(\mathbf{E}_0 + \mathbf{E}_1) = \chi(1 - 4\pi\chi)\frac{\omega B}{c}\,\mathbf{r}. \tag{17}$$

This polarization implies a bound charge, density ρ_1, which leads to a correction to field $\mathbf{E}_0$ that we call $\mathbf{E}_2$, ...

At the nth iteration, the polarization will have the form,

$$\mathbf{P}_n = k_n \frac{\omega B}{c}\,\mathbf{r}. \tag{18}$$

Then, the bound charge density is,

$$\rho_n = -\nabla \cdot \mathbf{P}_n = -2k_n\omega B, \tag{19}$$

which implies that the correction to the electric field becomes,

$$\mathbf{E}_{n+1} = 2\pi\rho_n\,\mathbf{r} = -4\pi k_n\omega B\,\mathbf{r}. \tag{20}$$

The effective electric field at iteration $n+1$ is the sum of $\mathbf{E}_0$ due to the $\mathbf{v} \times \mathbf{B}$ force and $\mathbf{E}_{n+1}$ due to the polarization charge. Thus,

$$\mathbf{P}_{n+1} = \chi(\mathbf{E}_0 + \mathbf{E}_{n+1}) = \chi(1 - \pi k_n)\frac{\omega B}{c}\,\mathbf{r}. \tag{21}$$

But, by the definition (18),

$$\mathbf{P}_{n+1} = k_{n+1} \frac{\omega B}{c} \, \mathbf{r}. \tag{22}$$

Hence,

$$k_{n+1} = \chi(1 - 4\pi k_n). \tag{23}$$

If this sequence converges to the value k, then we must have,

$$k = \chi(1 - 4\pi k), \tag{24}$$

so that,

$$k = \frac{\chi}{1 + 4\pi\chi} = \frac{\varepsilon_r - 1}{4\pi\varepsilon_r}\,, \qquad \mathbf{P} = k \frac{\omega B}{c} \, \mathbf{r} = \frac{\varepsilon_r - 1}{4\pi\varepsilon_r} \frac{\omega B}{c} \, \mathbf{r}, \tag{25}$$

which agrees with eq. (6) for the polarization.