Fields of a Uniformly Accelerated Charge

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1 Problem

Deduce the lab-frame potentials, electromagnetic fields and Poynting vector of an electric charge q with rest mass m that moves parallel to a uniform external electric field $\mathbf{E}_{\rm ext} = E_0\,\hat{\mathbf{x}}$. Comment on the physical character of these fields in the idealized case of an infinite time domain for the motion.

2 Solution

This problem has a long history, in which everyone admires the mathematical elegance of the formal solution, but opinions differ as to the physical significance of the idealization that the motion began infinitely far in the past and will continue indefinitely. And, the notion of a static electric field of infinite spatial extent is also somewhat problematic.

2.1 The Motion

The force on the charge in the (inertial) lab frame is constant, $\mathbf{F} = qE_0\,\hat{\mathbf{x}}$, which suggests that the motion is that for uniform acceleration. However, "uniform acceleration" cannot mean constant acceleration in the (inertial) lab frame, as this would eventually lead to faster-than-light motion. Rather, (following Born [1]) we note that for motion parallel to the electric field, the acceleration is uniform with respect to the instantaneous rest frame of the accelerated object, since the component of the electric field parallel to the motion is the same in this frame as in the lab frame.

Quantities in this frame will be designated with the superscript *.

From sec. 10 of Einstein's first paper on relativity [2] we have that for acceleration parallel to the velocity $\mathbf{v}$ of an object, the acceleration in the lab frame is related to that in the instantaneous rest frame according to,

$$\frac{dv}{dt} = (1 - v^2/c^2)^{3/2} \frac{dv^\star}{dt^\star}\,, \tag{1}$$

where c is the speed of light in vacuum. In this, two powers of $\sqrt{1 - v^2/c^2} \equiv 1/\gamma$ come from the transformation of relative velocity, and another comes from time dilation.

For uniform acceleration $a^\star = dv^\star/dt^\star = qE^\star/m = qE_0/m$ (in Gaussian units), eq. (1) can be integrated to find the velocity v. Thus, the acceleration in the lab frame is related to that in the instantaneous rest frame according to,

$$\frac{dv}{(1-v^2/c^2)^{3/2}} = a^\star\, dt, \qquad \frac{v}{\sqrt{1-v^2/c^2}} = \gamma v = a^\star t, \quad \text{and} \qquad \frac{dx}{dt} = v = \frac{a^\star t}{\sqrt{1+a^{\star\,2}t^2/c^2}}\,. \tag{2}$$

supposing that $v = 0$ when $t = 0$.[1] Integrating eq. (2), we obtain,

$$x = x_0 + \frac{c^2}{a^\star}\left(\sqrt{1+a^{\star\,2}t^2/c^2} - 1\right) = x_0 - \frac{c^2}{a^\star} + \sqrt{\left(\frac{c^2}{a^\star}\right)^2 + c^2t^2}, \tag{3}$$

where x_0 is the x-coordinate of the object at time $t = 0$.

We take $x_0 = c^2/a^\star \equiv b$, and write the motion as,

$$x(t) \equiv x_b = \sqrt{b^2 + c^2t^2}, \qquad y = 0 = z. \tag{4}$$

The (proper) time $t^\star$ on a clock carried by the accelerating object is related by,

$$dt^\star = dt\sqrt{1-v^2/c^2} = \frac{dt}{\sqrt{1+a^{\star\,2}t^2/c^2}}\,, \tag{5}$$

and hence,

$$t^\star = \frac{c}{a}\sinh^{-1}\frac{a^\star t}{c}\,, \qquad t = \frac{c}{a^\star}\sinh\frac{a^\star t^\star}{c}\,, \qquad ct = b\sinh\frac{a^\star t^\star}{c}\,. \tag{6}$$

Using this, eqs. (2) and (4) can be rewritten as,

$$v = c\,\tanh\frac{a^\star t^\star}{c} = \frac{c^2 t}{x_b}\,, \qquad \gamma = \cosh\frac{a^\star t^\star}{c}\,, \qquad \text{and} \qquad x(t) = x_b = b\cosh\frac{a^\star t^\star}{c}\,. \tag{7}$$

As such, uniformly accelerated motion is often called "hyperbolic motion".[2,3,4,5]

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[1]For completeness, we note that the acceleration in the inertial lab frame is $a = dv/dt = a^\star(1-v^2/c^2)/\sqrt{1+a^{\star\,2}t^2/c^2}$.

[2]Hyperbolic motion appears to have been first discussed briefly by Minkowski [3], and then more fully by Born [1] and Sommerfeld [4].

[3]An extended object that is subject to the same uniform acceleration at all of its points is observed to have the same length in the lab frame at all times; there is no Lorentz contraction observed in the lab frame in the case of uniform acceleration of an extended object. See, for example, the Appendices of [5].

[4]For times such that $|at| \ll c$, the position is well approximated by the Newtonian form,

$$x \approx b + \frac{at^2}{2} \qquad (|at| \ll c). \tag{8}$$

[5]To establish a uniformly accelerated frame of reference with coordinates $(x^\star, y^\star, z^\star, t^\star)$, it is customary to take $x^\star(t^\star = 0)$ to be at rest with respect to the inertial frame with coordinates (x, y, z, t). This requires shifting the above analysis for $x(t)$ by $b = c^2/a^\star$. The coordinate transformations are then,

$$x = \left(x^\star + \frac{c^2}{a^\star}\right)\cosh\frac{a^\star\, ct^\star}{c^2} - \frac{c^2}{a^\star}\,, \qquad y = y^\star, \qquad z = z^\star, \qquad ct = \left(z^\star + \frac{c^2}{a^\star}\right)\sinh\frac{a^\star\, ct^\star}{c^2}\,, \tag{9}$$

$$x^\star = \sqrt{\left(x + \frac{c^2}{a^\star}\right)^2 - c^2t^2} - \frac{c^2}{a^\star}\,, \qquad y^\star = y, \qquad z^\star = z, \qquad ct^\star = \frac{c^2}{a^\star}\tanh^{-1}\left(\frac{ct}{z + c^2/a^\star}\right). \tag{10}$$

2.2 The Potentials

The rest of this note largely follows the book of Schott (1912) [6].

We compute the electromagnetic potentials V and $\mathbf{A}$ of the uniformly accelerated charge via the prescription of Liénard [7] and Wiechert [8],

$$V(\mathbf{x},t) = \int \frac{q\delta(t'-t_r)}{R(t,t_r)}\,dt' = \frac{q}{R - \boldsymbol{\beta}_r \cdot \mathbf{R}}\,, \qquad \mathbf{A}(\mathbf{x},t) = \boldsymbol{\beta}_r V, \tag{11}$$

where[6]

$$\mathbf{R} = \mathbf{x}(t) - \mathbf{x}_b(t_r) = \left(x - \sqrt{b^2 + c^2 t_r^2}\right)\hat{\mathbf{x}} + \rho\,\hat{\boldsymbol{\rho}}, \tag{14}$$

is the position vector from the charge at the retarded time (called τ by Schott [6]),

$$t_r = t - \frac{R(t,t_r)}{c}\,, \tag{15}$$

to the observer at position $\mathbf{x}$ at the present time t, $\rho = \sqrt{y^2+z^2}$,

$$\boldsymbol{\beta}_r = \frac{\mathbf{v}(t_r)}{c} = \frac{ct_r}{x_b(t_r)}\,\hat{\mathbf{x}}. \tag{16}$$

Equations (14) and (15) combine to give a quadratic equation in t_r with solution,

$$ct_r = \frac{ct(x^2+b^2+\rho^2-c^2t^2) - xs}{2(x^2-c^2t^2)}\,, \tag{17}$$

where,

$$s = \sqrt{(x^2+\rho^2+b^2-c^2t^2)^2 - 4b^2(x^2-c^2t^2)} = \sqrt{(x^2+\rho^2-x_b^2)^2 + 4b^2\rho^2}, \tag{18}$$

and the minus sign has been chosen for the term xs so that $t_r < 0$ when $t = 0$. Then,

$$R = ct - ct_r = \frac{ct(x^2-b^2-\rho^2-c^2t^2) + xs}{2(x^2-c^2t^2)}\,, \tag{19}$$

$$x_b(t_r) = \sqrt{b^2+c^2t_r^2} = \frac{x(x^2+b^2+\rho^2-c^2t^2) - cts}{2(x^2-c^2t^2)}\,, \tag{20}$$

[6]The usual integration over the delta function is based on,

$$\int \frac{q\delta[f(t')]}{R}\,dt' = \int \frac{q\delta(f)}{R}\,\frac{df}{df/dt'} = \frac{q}{R\,df/dt'|_{f=0}}\,, \qquad \text{where} \qquad f = t' - t + \frac{R(t,t')}{c}\,, \tag{12}$$

so that $f = 0$ for $t' = t - R/c = t_r$, and using eqs. (14) and (16) we find,

$$R\frac{df}{dt'} = R + \frac{R}{c}\frac{dR}{dt'} = R + \frac{\mathbf{R}\cdot\mathbf{v}_r}{c} = R - \frac{\mathbf{x}(t) - \mathbf{x}_b(t')}{c}\cdot\frac{d\mathbf{x}_b(t')}{dt'} = R - \mathbf{R}\cdot\boldsymbol{\beta}_r. \tag{13}$$

$$\beta_r = \frac{ct_r}{x_b(t_r)} = \frac{ct(x^2+b^2+\rho^2-c^2t^2)-xs}{x(x^2+b^2+\rho^2-c^2t^2)-cts}\,, \tag{21}$$

$$\begin{aligned} R - \boldsymbol{\beta}_r\cdot\mathbf{R} &= R - \beta_r[x - x_b(t_r)] = ct - ct_r - \frac{ct_r}{x_b(t_r)}[x - x_b(t_r)] = ct - \beta_r x \\ &= \frac{s(x^2-c^2t^2)}{x(x^2+b^2+\rho^2-c^2t^2)-cts}\,, \end{aligned} \tag{22}$$

and the potentials (11) can be written as,[7]

$$V_{\rm Schott} = q\frac{x(x^2+b^2+\rho^2-c^2t^2)-cts}{s(x^2-c^2t^2)}\,, \qquad A_{{\rm Schott},x} = q\frac{ct(x^2+b^2+\rho^2-c^2t^2)-xs}{s(x^2-c^2t^2)}\,. \tag{23}$$

The potentials (and fields) are zero for $x < -ct$.[8]

These potentials appear to be singular at the planes $x = \pm ct$.[9] To see that the singularity occurs only for $x = -ct$ we follow Schott in multiplying and dividing eq. (23) by $x(x^2+\rho^2+b^2-c^2t^2)+cts$ (or by $ct(x^2+\rho^2+b^2-c^2t^2)+xs$) to find,

$$V_{\rm Schott} = q\frac{(x^2+\rho^2+b^2-c^2t^2)^2+4b^2c^2t^2}{s[x(x^2+\rho^2+b^2-c^2t^2)+cts]}\,, \qquad A_{{\rm Schott},x} = q\frac{4b^2c^2t^2-(x^2+\rho^2+b^2-c^2t^2)^2}{s[ct(x^2+\rho^2+b^2-c^2t^2)+xs]}\,, \tag{25}$$

Then, for $x = \pm ct$ where $s = b^2+\rho^2$ these become,

$$V_{\rm Schott}(x=\pm ct) = q\frac{(\rho^2+b^2)^2+4b^2c^2t^2}{(x+ct)(\rho^2+b^2)^2}\,, \qquad A_{{\rm Schott},x}(x=\pm ct) = q\frac{4b^2c^2t^2-(\rho^2+b^2)^2}{(x+ct)(\rho^2+b^2)^2}\,, \tag{26}$$

which are singular only for the plane $x = -ct$.

However, the potentials found above suffer from a defect apparently first noticed only in 1955 by Bondi and Gold [9],[10] that the corresponding electromagnetic fields do not satisfy Maxwell's equations in the plane $x + ct = 0$. This can be attributed to the creation of the charged particle at $t = -\infty$ with speed $v_x = -c$ and with singular fields and potentials, while the Liénard-Wiechert forms (11) tacitly assume there is no singular behavior at early

[7] These results are given on pp. 64-65 of [6].

[8] An "amusing" result holds for the special case that the retarded time is zero, $t_r = 0$, when the retarded position of the charge is $(b, 0, 0)$, and the retarded velocity is zero, $\boldsymbol{\beta}_r = 0$.

Then, $R = ct$ according to eq. (22), and according to eq. (11) the potentials at $t_r = 0$ are,

$$V(t_r = 0) = q/R\,, \qquad \mathbf{A}(t_r = 0) = 0. \tag{24}$$

That is, the scalar potential has the apparent form of the static Coulomb potential.

However, the form (24) holds, for the present time $t > 0$, only on a sphere of radius $R = ct$ about $(b, 0, 0)$ (= the location of the accelerated charge when it is instantaneously at rest), and not at other locations. That is, the potential $V_{\rm Schott}$ of eq. (23) is not the Coulomb potential in general, although it happens to be so at any time $t > 0$ on a sphere of radius ct about $(b, 0, 0)$.

[9] The potentials are also singular at the location of the charge, $x = x_b$, $\rho = 0$. Close to the charge, the potentials are approximately those of a uniformly moving charge, as discussed further in sec. 2.4.

[10] See also [10, 11, 12].

times. The defect can be remedied by expressing the singular behavior for $x = -ct$ in terms of delta functions,[11,12]

$$V(x = -ct) = -q \ln \frac{b^2 + \rho^2}{b^2} \delta(x + ct), \qquad A_x(x = -ct) = q \ln \frac{b^2 + \rho^2}{b^2} \delta(x + ct). \tag{32}$$

2.3 The Electromagnetic Fields

The electromagnetic fields follow from the potentials (23) according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}, \tag{33}$$

such that the nonzero field components for $x > -ct$ are,

$$E_x = q \frac{4b^2(x^2 - \rho^2 - x_b^2)}{s^3}, \qquad E_\rho = q \frac{8b^2 x \rho}{s^3}, \qquad B_\phi = q \frac{8b^2 ct \rho}{s^3} \qquad (x > -ct), \tag{34}$$

in cylindrical coordinates (ρ, ϕ, x) with $\hat{\boldsymbol{\phi}} = \hat{\mathbf{x}} \times \hat{\boldsymbol{\rho}}$.

The electric field lines $\mathbf{E}$ for times $t = \mp b/c$, when the charge is at $x = \sqrt{2}b$ are shown as solid lines in the figures on the next page (from paper I of [10]), with the dashed lines being the Poynting vector $\mathbf{S}$ (sec. 2.5). The plane $x = -ct$, on which the field lines appear to terminate, has moved to the left between figures (a) and (b).

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[11] As discussed in eqs. (15)-(16) of [12], the retarded time associated with the plane $x = -ct$ is $t_r = -\infty$, and the retarded distance (14) can be (delicately) approximated for $t_r = t' \to -\infty$ as,

$$R(t, t') = \left[\left(x - \sqrt{b^2 + c^2 t'^2} \right)^2 + \rho^2 \right]^{1/2} \approx [(x + ct')^2 + b^2 + \rho^2]^{1/2} \approx -(x + ct') - \frac{b^2 + \rho^2}{2ct'}, \tag{27}$$

so that the first form of eq. (11) gives the potential for $x = -ct$, due to the contribution at $t' = -\infty$, as,

$$V(x = -ct) = q \int_{t'=-\infty} \frac{\delta(ct - R - ct')}{R/c} dt' \approx -q \int \delta \left(x + ct + \frac{b^2 + \rho^2}{2ct'} \right) \frac{dt'}{t'} = q \int \delta(\xi - \eta) \frac{d\eta}{\eta}, \tag{28}$$

where $\xi = x + ct$, $\eta = -(b^2 + \rho^2)/2ct' > 0$ with $dt'/t' = -d\eta/\eta$ and $t' \to -\infty$ corresponding to the lower limit of the η integration. For fixed t we have $dx = d\xi$, such that,

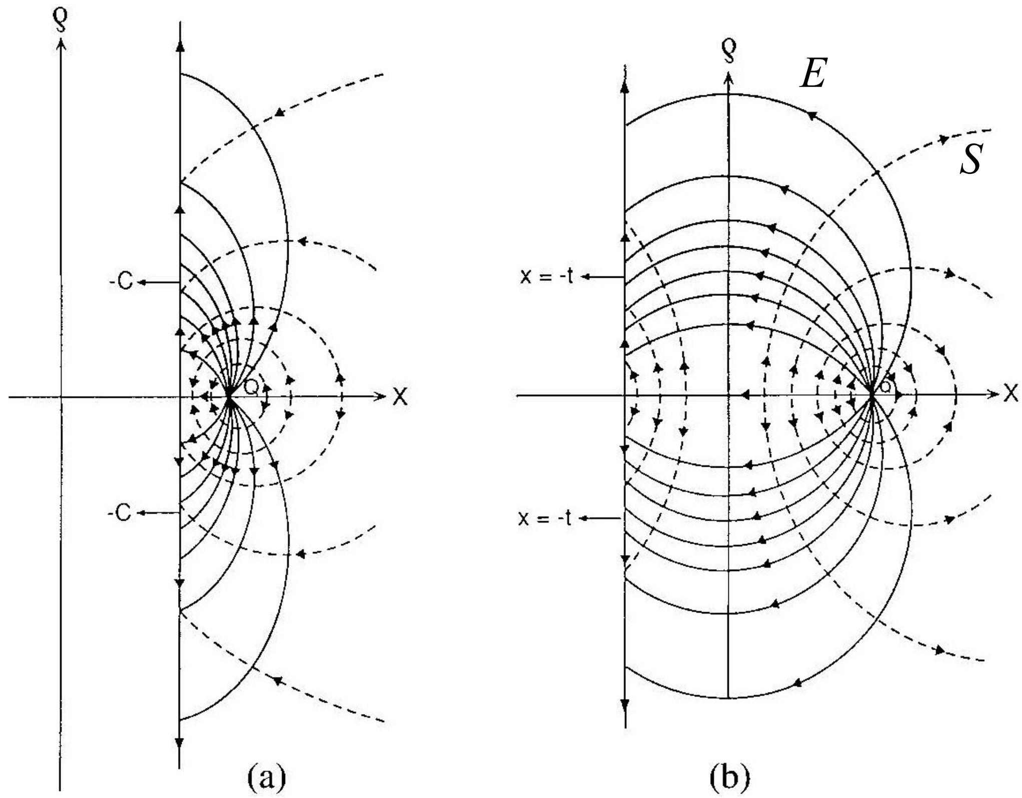
$$\int V(x = -ct)\ dx \approx q \int \int \delta(\xi - \eta) d\xi \frac{d\eta}{\eta} = q \int_{t' \to -\infty} \frac{d\eta}{\eta} = -q \ln \eta = -q \ln \frac{b^2 + \rho^2}{b^2} + q \ln \frac{-2ct'}{b^2}. \tag{29}$$

[12] Since the retarded velocity associated with $x = -ct$ is $-c\,\hat{\mathbf{x}}$, $A_x(x = -ct) = -V(x = -ct)$. Hence, the singular potentials at $x = -ct$ can be written as,

$$V(x = -ct) = -q \ln \frac{b^2 + \rho^2}{b^2} \delta(x + ct) + q \ln \frac{-2ct'}{b^2} \delta(x + ct), \tag{30}$$

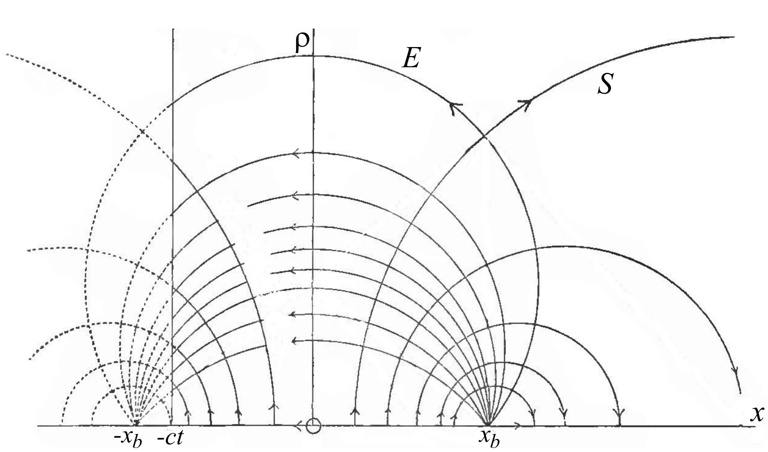
$$A_x(x = -ct) = q \ln \frac{b^2 + \rho^2}{b^2} \delta(x + ct) - q \ln \frac{-2ct'}{b^2} \delta(x + ct). \tag{31}$$

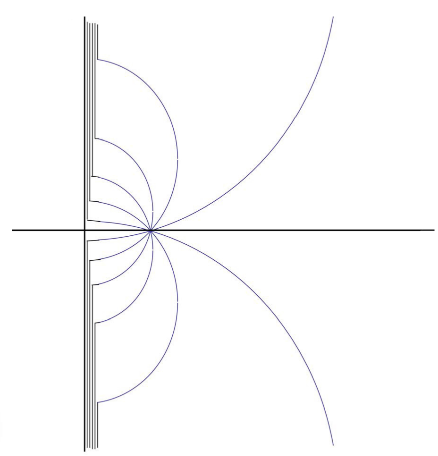
The awkward terms in $\ln(-2ct'/b^2)$ have no effect on the fields $\mathbf{E}$ and $\mathbf{B}$, and can be removed by the gauge transformation $V \to V - \partial\Lambda/\partial ct$, $A_x \to A_x - \partial\Lambda/\partial x$ with $\Lambda = q \ln(-2ct'/b^2)\delta(x + ct)$, leading at last to the forms (32). While the full potential (30) is positive (for positive q), the first, physical term is negative.



The figures suggest that the fields close to the charge resemble those of a uniformly moving charge, while away from the charge they curve towards the image charge at $x = -x_b$. This will be verified in sec. 2.4.

Note that the pattern of field lines is as if there were a (negative) image charge at $x = -x_b = -\sqrt{b^2 + c^2t^2}$ (beyond the "event horizon"), as also shown in the left figure below (from p. 68 of [6], with $ct = 4b/3$). Section 2.3.1 below will continue this theme.





In the plane $x = -ct$, where the potentials (32) are singular, the fields are those of a singular wavefront,

$$E_x = -q\frac{4b^2}{(b^2+\rho^2)^2}, \qquad E_\rho = 2q\frac{\rho}{b^2+\rho^2}\delta(x+ct) = -B_\phi \qquad (x = -ct), \tag{35}$$

where the form for E_x is the limit of that in eq. (34) as $x \to -ct$.

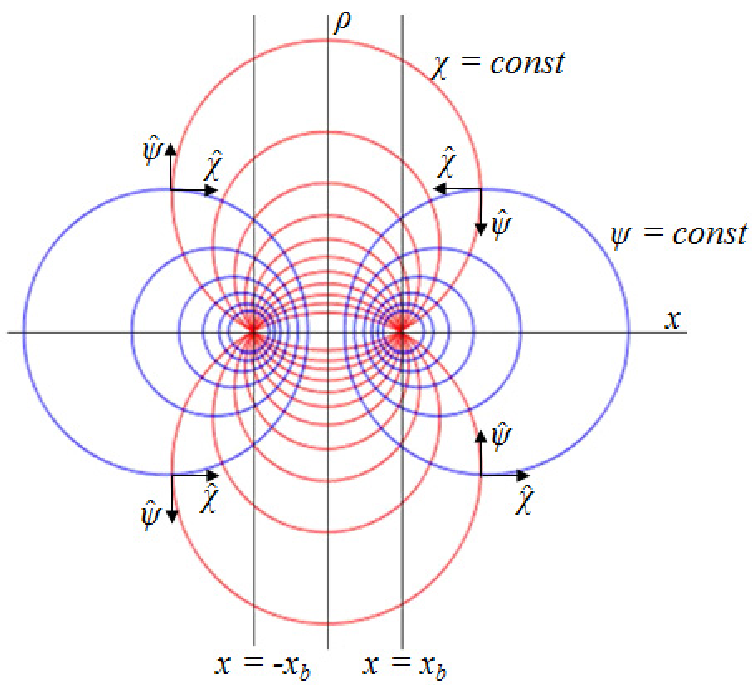
An electric field line of the accelerated charge for $x > -ct$ does not "end" where it intercepts the plane $x = -ct$, but has a kink there, and heads off within this plane to $\rho = \infty$. The fields in this plane are essentially transverse, in contrast to those for $x > -ct$, as emphasized in [11] from which the figure on the right above is taken.

The fields are zero for $x < -ct$, and the (moving) plane $x = -ct$ is a kind of "event horizon" in the limited sense that observers at $x < -ct$ cannot be aware of the accelerated charge prior to time $t = -x/c$. This contrasts with the case of a charge in uniform motion, whose field lines fill all space at all times.

The premise of uniformly accelerated motion for all times is that somehow the charge is brought into existence at $(x, y, z, t) = (\infty, 0, 0, -\infty)$ with initial velocity $\mathbf{v} = -c\,\hat{\mathbf{x}}$. The "initial" field lines are largely a "pancake" of transverse lines as in eq. (35), with $E = B$ and $\mathbf{E} \cdot \mathbf{B} = 0$, as for "radiation" fields.[13]

As time increases (from $t = -\infty$), these transverse "radiation" fields, in the plane $x = -ct$, slowly pull away from the charge (which is at $x_b = \sqrt{b^2 + c^2t^2}$), and the fields are nonzero in the region $x \ge -ct$. For $x > -ct$ the electric field lines are approximately those associated with a uniformly moving charge at x_b plus an image charge at $-x_b$, while these lines bend into the plane $x = -ct$ and the field is zero for $x < -ct$.

2.3.1 Electric Field in Bipolar Coordinates



The electric field lines (but not the scalar potential) for $x > -ct$ lie on surfaces of constant coordinate in a bipolar coordinate system with foci at $x = \pm x_b(t)$, illustrated in the figure

[13]Following an earlier discussion by Fermi [15], Weizsäcker [16] and Williams [17] noted that the electromagnetic fields of an electron in uniform relativistic motion are predominantly transverse, with $\mathbf{E} \approx \mathbf{B}$ (in Gaussian units). This is very much like the fields of a plane wave, so one is led to regard a fast electron as carrying with it a cloud of virtual photons that it can shed (radiate) if perturbed. See also [18].

above.[14] Following pp. 66-67 of [6], we define,

$$x = \frac{x_b \sinh\psi}{\cosh\psi - \cos\chi}\,, \qquad \rho = \frac{x_b \sin\chi}{\cosh\psi - \cos\chi}\,. \tag{36}$$

For use below we note that,

$$\beta^2 = \frac{v^2}{c^2} = \frac{c^2 t^2}{x_b^2} = \frac{c^2 t^2}{b^2 + c^2 t^2}\,, \qquad \text{so that} \qquad \frac{1}{\gamma^2} = 1 - \beta^2 = \frac{b^2}{x_b^2}\,, \tag{37}$$

and,

$$\begin{aligned} s &= \sqrt{(x^2 + \rho^2 - x_b^2)^2 + 4b^2\rho^2} \\ &= \frac{x_b^2}{(\cosh\psi - \cos\chi)^2}\sqrt{[\sinh^2\psi + \sin^2\chi - (\cosh\psi - \cos\chi)^2]^2 + 4(1-\beta^2)\sin^2\chi(\cosh\psi - \cos\chi)^2} \\ &= \frac{x_b^2}{(\cosh\psi - \cos\chi)^2}\sqrt{4\cos^2\chi(\cosh\psi - \cos\chi)^2 + 4(1-\beta^2)\sin^2\chi(\cosh\psi - \cos\chi)^2} \\ &= \frac{2x_b^2}{\cosh\psi - \cos\chi}\sqrt{1 - \beta^2\sin^2\chi}. \end{aligned} \tag{38}$$

The so-called scale factors for bipolar coordinates are,

$$h_\psi = h_\chi = \frac{x_b}{\cosh\psi - \cos\chi}\,, \tag{39}$$

and the unit vectors (with directions shown in the figure on the previous page) are,

$$\hat{\boldsymbol{\psi}} = \frac{1}{h_\psi}\frac{\partial \mathbf{r}}{\partial\psi} = \frac{1 - \cosh\psi\cos\chi}{\cosh\psi - \cos\chi}\,\hat{\mathbf{x}} - \frac{\sinh\psi\sin\chi}{\cosh\psi - \cos\chi}\,\hat{\boldsymbol{\rho}}, \tag{40}$$

$$\hat{\boldsymbol{\chi}} = \frac{1}{h_\chi}\frac{\partial \mathbf{r}}{\partial\chi} = -\frac{\sinh\psi\sin\chi}{\cosh\psi - \cos\chi}\,\hat{\mathbf{x}} - \frac{1 - \cosh\psi\cos\chi}{\cosh\psi - \cos\chi}\,\hat{\boldsymbol{\rho}}, \tag{41}$$

where $\mathbf{r} = x\,\hat{\mathbf{x}} + \rho\,\hat{\boldsymbol{\rho}}$.

Then from,

$$\mathbf{E} = E_x\,\hat{\mathbf{x}} + E_\rho\,\hat{\boldsymbol{\rho}} = E_\psi\,\hat{\boldsymbol{\psi}} + E_\chi\,\hat{\boldsymbol{\chi}}, \tag{42}$$

we have that,

$$\begin{aligned} E_\psi &= E_x\,\hat{\mathbf{x}}\cdot\hat{\boldsymbol{\psi}} + E_\rho\,\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\psi}} = q\frac{4b^2(x^2 - \rho^2 - x_b^2)}{s^3}\frac{1 - \cosh\psi\cos\chi}{\cosh\psi - \cos\chi} - q\frac{8b^2 x\rho}{s^3}\frac{\sinh\psi\sin\chi}{\cosh\psi - \cos\chi} \\ &= \frac{4qb^2x_b^2}{s^3}\frac{(\sinh^2\psi - \sin^2\chi - (\cosh\psi - \cos\chi)^2)(1 - \cosh\psi\cos\chi)}{(\cosh\psi - \cos\chi)^3} - \frac{8qb^2x_b^2}{s^3}\frac{\sinh^2\psi\sin^2\chi}{(\cosh\psi - \cos\chi)^3} \\ &= -\frac{8qb^2x_b^2}{s^3}\frac{(1 - \cosh\psi\cos\chi)^2 + \sinh^2\psi\sin^2\chi}{(\cosh\psi - \cos\chi)^3} = -\frac{8qb^2x_b^2}{s^3(\cosh\psi - \cos\chi)} \end{aligned}$$

[14] The 3-dimensional coordinate system obtained by rotating the 2-dimensional bipolar coordinate system about the x-axis is called a bispherical coordinate system.

$$= -\frac{qb^2(\cosh\psi - \cos\chi)^2}{x_b^4(1-\beta^2\sin^2\chi)^{3/2}} = -q\frac{(\cosh\psi - \cos\chi)^2}{\gamma^2 x_b^2(1-\beta^2\sin^2\chi)^{3/2}}\,, \tag{43}$$

$$E_\chi = E_x\,\hat{\mathbf{x}}\cdot\hat{\boldsymbol{\chi}} + E_\rho\,\hat{\boldsymbol{\rho}}\cdot\hat{\boldsymbol{\chi}} = -q\frac{4b^2(x^2-\rho^2-x_b^2)}{s^3}\frac{\sinh\psi\sin\chi}{\cosh\psi-\cos\chi} - q\frac{8b^2 x\rho}{s^3}\frac{1-\cosh\psi\cos\chi}{\cosh\psi-\cos\chi}$$

$$= -\frac{4qb^2x_b^2}{s^3}\frac{[\sinh^2\psi - \sin^2\chi - (\cosh\psi-\cos\chi)^2]\sinh\psi\sin\chi}{(\cosh\psi-\cos\chi)^3}$$

$$-\frac{8qb^2x_b^2}{s^3}\frac{\sinh\psi\sin\chi(1-\cosh\psi\cos\chi)}{(\cosh\psi-\cos\chi)^3} = 0. \tag{44}$$

This confirms that the electric field lines follow lines of constant χ, which are circles that pass through $x = \pm x_b$, as if an image charge $-q$ were at $x = -x_b$ in addition to the actual charge q at $x = x_b$. Of course, the physical electric field exists only for $x \ge -ct > -x_b$, and only the field for $x > -ct$ is described by eqs. (43)-(44).

The magnetic field (34) for $x > -ct$, written in bipolar coordinates, is,

$$B_\phi = q\frac{8b^2 ct\rho}{s^3} = q\frac{\beta\sin\chi(\cosh\psi-\cos\chi)^2}{\gamma^2 x_b^2(1-\beta^2\sin^2\chi)^{3/2}} = \beta\sin\chi E_\psi\,. \tag{45}$$

2.4 Potentials and Field Close to the Charge

To discuss the potentials (and fields) close to the charge we introduce the distance $\mathbf{r}$ from the present position of the charge to the observation point, and the angle θ between $\mathbf{r}$ and the positive x-axis,

$$r = \sqrt{(x-x_b)^2+\rho^2}, \qquad x = x_b + r\cos\theta, \qquad \rho = r\sin\theta. \tag{46}$$

Then, from eq. (18), for small r we have,

$$s = \sqrt{(-2x_b r\cos\theta + r^2\cos^2\theta)^2 + 4b^2r^2\sin^2\theta} \approx 2r\sqrt{x_b^2\cos^2\theta + x_b^2(1-\beta^2)\sin^2\theta}$$

$$= 2rx_b\sqrt{1-\beta^2\sin^2\theta}, \tag{47}$$

noting that $\beta^2 = v^2/c^2 = c^2t^2/x_b^2 = c^2t^2/(b^2+c^2t^2)$, so that $1/\gamma^2 = 1-\beta^2 = b^2/x_b^2$. Then, eq. (23) becomes (for small r where $x^2 - c^2t^2 \approx x_b^2 - c^2t^2 = b^2$),

$$V \approx \frac{2x_b b^2}{sb^2} = \frac{1}{r\sqrt{1-\beta^2\sin^2\theta}}\,, \qquad A_x \approx \frac{2ctb^2}{sb^2} = \beta V, \tag{48}$$

which are the potentials of a uniformly moving charge with velocity $\boldsymbol{\beta} = \beta\,\hat{\mathbf{x}}$. Similarly, from eq. (34),

$$E_x \approx q\frac{4b^2(x^2-x_b^2)}{s^3} \approx q\frac{8b^2x_b(x-x_b)}{8r^3x_b^2(1-\beta^2\sin^2\theta)^{3/2}} = q\frac{x-x_b}{\gamma^2r^3(1-\beta^2\sin^2\theta)^{3/2}}\,, \tag{49}$$

$$E_\rho \approx q\frac{8b^2x_b\rho}{s^3} = q\frac{\rho}{\gamma^2r^3(1-\beta^2\sin^2\theta)^{3/2}}\,, \qquad \mathbf{E} \approx q\frac{\mathbf{r}}{\gamma^2r^3(1-\beta^2\sin^2\theta)^{3/2}}\,, \tag{50}$$

$$B_\phi \approx q\frac{8b^2ct\rho}{s^3} = q\frac{\beta\rho}{\gamma^2r^3(1-\beta^2\sin^2\theta)^{3/2}} = \beta E_\rho, \qquad \mathbf{B} \approx \boldsymbol{\beta}\times\mathbf{E}, \tag{51}$$

which are the electromagnetic fields of a uniformly moving charge.[15]

2.5 Poynting Vector

The Poynting vector for $x > -ct$ is, using the fields (43) and (45),

$$\mathbf{S}_{x>-ct} = \frac{c\mathbf{E}\times\mathbf{B}}{4\pi} = \frac{cE_\psi\,\hat{\boldsymbol{\psi}}\times B_\phi\,\hat{\boldsymbol{\phi}}}{4\pi} = \frac{cE_\psi B_\phi}{4\pi}\,\hat{\boldsymbol{\chi}} = -\frac{cq^2\beta}{4\pi}\frac{(\cosh\psi-\cos\chi)^4\sin\chi}{\gamma^4 x_b^4(1-\beta^2\sin^2\chi)^3}\,\hat{\boldsymbol{\chi}}. \qquad (54)$$

Lines of the Poynting vector for $x > -ct$ point along $\hat{\boldsymbol{\chi}}$, following circular paths (which are orthogonal to the circular lines of the electric field), as shown in the figures on p. 5. The Poynting vector does not emanate from either the present or the retarded position of the charge! Furthermore, the Poynting vector is proportional to $\beta = v/c = ct/x_b$, and so vanishes for all $x > 0$ at the time $t = 0$ when the charge is instantaneously at rest.[16]

[15]We can also consider the case when the observer is on the x-axis, along which the charge moves. Then, coordinate ρ of the observer is zero, and the (positive) quantity s of eq. (18) simplifies to $s(\rho = 0) = \left|x^2 - b^2 - c^2t^2\right|$, where $x_b = \sqrt{b^2 + c^2t^2}$ is the position of the charge q at time t. The potentials (23) for $\rho = 0$ become,

$$V = q\frac{x(x^2+b^2-c^2t^2)-cts}{s(x^2-c^2t^2)}\,, \qquad A_x = \frac{v_r}{c}V = q\frac{ct(x^2+b^2-c^2t^2)-xs}{s(x^2-c^2t^2)}\,, \qquad (\rho=0). \qquad (52)$$

We now take the additional limit that $x,\ c\,|t| \gg b$, for which $v \approx \pm c$, and $s = \pm(x^2 - c^2t^2)$ for $x \gtrless c\,|t|$,

$$V \approx \pm q\left(\frac{x\mp ct}{x^2-c^2t^2}\right) = \frac{\pm q}{x\pm ct}\,, \qquad A_x = -q\left(\frac{x\mp ct}{x^2-c^2t^2}\right) = \frac{-q}{x\pm ct} \qquad (\rho=0,\ x,\ c\,|t|\gg b). \qquad (53)$$

The interpretation of these results requires care.

For $t \approx -\infty$ the charge is at $x \approx -ct \gg 0$ and has velocity $v \approx -c$ (in the $-x$ direction). As discussed in sec. 2.2, the potentials are zero for $x < -ct$, *i.e.*, for $x \sim x_b$ when $t \approx -\infty$. At such early times, the potentials are nonzero only for $x > -ct \approx x_b$, so only the upper sign in the last of eq. (53) is physically meaningful. Then, the potentials along the axis (for $x > x_b \gg 0$ are $V \approx q/(x + ct) \approx q/(x - x_b) \approx -A_x$, as expected for a charge in uniform motion (where in this case $v \approx -c$ along the x-axis).

For $t \approx +\infty$, the charge has velocity $v \approx +c$ along the x-axis, and its position is $x_b \approx ct(1+b^2/3c^2t^2) > ct$. Since the charge is moving at close to lightspeed, the potential for $x > x_b > ct$ is little affected by the recent location of the charge, and rather is a "memory" of the charge at times much earlier than t, since the retarded time of eq. (16) goes to zero for $x > x_b \approx ct$. Indeed, the potentials for $x \gtrsim x_b \approx ct$ according to eq. (53) are $V \approx q/(ct + x) \approx q/2x \approx -A_x$, which are the (tiny) retarded potentials for a charge with uniform velocity from times $t \lesssim 0$, when the charge was near $x = b$ but its velocity was still large ($v \approx -c$). On the other hand, for $x < ct \approx x_b$, the potentials is $V \approx q/(ct - x) \approx q/(x_b - x) \approx A_x$, which has the form of the potentials of charge position $x_b \approx ct$ with uniform velocity $v \approx c$, according to an observer at position x. There remains the case where $ct < x < x_b$, which is a very small region for which the observer is close to the charge at time t. In this region, $V \approx q/(x - ct)$, which is very large, similar to the case of an observer close to and "behind" a charge moving with uniform velocity $v \approx c$, for which the potential is very large in the plane $x = x_b$.

Hence, the potentials are close to those of a uniformly moving charge with $|v| \approx c$ in the limit that $\rho = 0$ and $|t| \approx \infty$, but only for observers "behind" the charge ($x < x_b$ for $t > 0$, and $x > x_b$ for $t < 0$). The main argument of this section is perhaps to be preferred in this regard, with its results (48)-(51) that the potentials and fields close to the charge are those of a charge in uniform motion with the instantaneous velocity of the accelerated charge at any time t (and for nonzero ρ).

[16]The magnetic field also vanishes at time $t = 0$ according to eq. (34), as noted in eq. (250) of [25]. And, the electric field $\mathbf{E}(x, \rho; t = 0)$ is along the line from the retarded position $x_b(t_r) = (x^2 + \rho^2 + b^2)/2x$ (recalling

The Poynting vector for $x = -ct$ is, using the fields eq. (34) extrapolated onto this plane (as suggested by Schott [6]),

$$\begin{aligned} \mathbf{S}_{x=-ct,\text{Schott}} &= \frac{c}{4\pi}(E_x\,\hat{\mathbf{x}} + E_\rho\,\hat{\boldsymbol{\rho}}) \times B_\phi\,\hat{\boldsymbol{\phi}} = \frac{c}{4\pi}(E_\rho B_\phi\,\hat{\mathbf{x}} - E_x B_\phi\,\hat{\boldsymbol{\rho}}) \\ &= -\frac{c}{4\pi}\frac{32q^2b^4x\rho}{(b^2+\rho^2)^5}\left(\frac{2x\rho}{b^2+\rho^2}\,\hat{\mathbf{x}} + \hat{\boldsymbol{\rho}}\right). \end{aligned} \tag{56}$$

The Poynting vector (56) vanishes at $x = 0$ at all times, such that no energy is transported across this plane (which is not crossed by the accelerated charge), so it would be inconsistent to have nonzero fields for $x < 0$ at $t > 0$. This defect is remedied by the forms (32) to the potentials for $x = -ct$, for which the Poynting vector (57) has a nonzero component in the $-x$ direction at all times. Using eqs. (35) one finds [9, 12],

$$\begin{aligned} \mathbf{S}_{x=-ct} &= \frac{c}{4\pi}(E_x\,\hat{\mathbf{x}} + E_\rho\,\hat{\boldsymbol{\rho}}) \times B_\phi\,\hat{\boldsymbol{\phi}} = \frac{c}{4\pi}(E_\rho B_\phi\,\hat{\mathbf{x}} - E_x B_\phi\,\hat{\boldsymbol{\rho}}) \\ &= -\frac{cE_\rho^2}{4\pi}\,\hat{\mathbf{x}} - \frac{c}{4\pi}\frac{8q^2b^2\rho^2}{(b^2+\rho^3)^3}\delta(x+ct)\,\hat{\boldsymbol{\rho}}, \end{aligned} \tag{57}$$

where $S_x = -E_\rho^2 c/4\pi = -uc$ and u is the field energy density. That is, the flux of energy in the $-x$ direction in the plane $x = -ct$ is just the product of the energy density in that plane and its velocity. In eq. (57) the field energy density u in the plane is infinite, but as shown in paper III of [10], the total field energy in the plane $x = -ct$ can be written as,

$$U_{x=-ct} = \int u_{x=-ct}\,d\text{Vol} = U_{-\infty} - \frac{2q^2a^2t}{3c^3}\,, \tag{58}$$

where $U_{-\infty}$ is the infinite field energy created along with the accelerated charge at $t = -\infty$. The field energy in the plane $x = -ct$ decreases with time and approaches zero as $t \to \infty$. The energy lost by the plane $x = -ct$ appears as an increase of the field energy in the region $x > -ct > 0$ for $t < 0$, and in the region $-ct < x < 0$ for $t > 0$.

Energy flows radially inward on the plane $x = -ct$, according to eq. (57). This flow can be said to exit the plane through its "bounding surface" $x = -ct^+$, leading to the increase of field energy in the region $x > -ct$ noted above.

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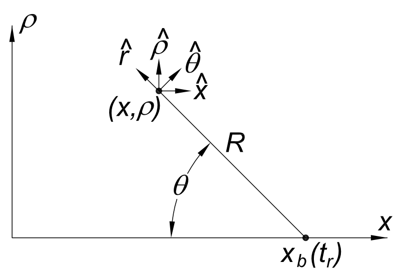
eq. (20)) of the accelerated charge. To see this, refer to the figure below and note that $\hat{\mathbf{x}} = \hat{\boldsymbol{\theta}}\sin\theta - \hat{\mathbf{r}}\cos\theta$ and $\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\theta}}\cos\theta + \hat{\mathbf{r}}\sin\theta$, such that $\mathbf{E} = E_x\,\hat{\mathbf{x}} + E_\rho\,\hat{\boldsymbol{\rho}} = (E_x\sin\theta + E_\rho\cos\theta)\,\hat{\boldsymbol{\theta}} - (E_x\cos\theta - E_\rho\sin\theta)\,\hat{\mathbf{r}}$ and, since $x_b = b$ at $t = 0$,

$$E_\theta(x,\rho;t=0) = q\frac{4b^2(x^2-\rho^2-x_b^2)}{s^3}\frac{b}{R} + q\frac{8b^2x\rho}{s^3}\frac{x_b(t_r)-x}{R} = 0. \tag{55}$$



2.6 Does a Uniformly Accelerated Charge "Radiate"?

That the Poynting vector (54) does not flow out from the charge is consistent with the fact that the "radiation-reaction" force,[17]

$$\mathbf{F}_{\text{rad react}} = \frac{2q^2}{3c^3}\ddot{\mathbf{v}}, \tag{59}$$

vanishes for uniform acceleration.[18] This has led many people to conclude that a uniformly accelerated charge does not "radiate" [1, 25, 26, 27, 28].[19] On the other hand, the flux of energy associated with the Liénard-Wiechert fields (34) across a sphere of large radius R at time $t' = t + R/c$ whose center is at the location of the charge at time t is $2q^2a^2/3c^3$ [29], which indicates that it is reasonable to say that the accelerated charge does "radiate", according to the so-called Sommerfeld criterion [30].[20,21]

The author considers that the term "radiation" should be used wherever the Poynting vector is nonzero [14], and that a uniformly accelerated charge involves "radiation" even though the Poynting vector does not emanate from the charge.[22,23,24]

Sept. 29, 2023. In sec. 32 of [25], Pauli appeared to argue that since $\mathbf{B} = 0$ at time $t = 0$ for the motion of the accelerated charge as discussed here, the Poynting vector was zero at

[17]For commentary by the author on the "radiation reaction", see [19, 20].

[18]Equation (59) was first deduced by Lorentz [21, 22] by arguments that did not mention radiation, but strictly holds only for $v \ll c$. While the nonrelativistic radiation-reaction force, eq. (59), vanishes for uniform acceleration, it is not immediately obvious that this also holds for large velocities. The relativistic version of the radiation-reaction force was first deduced by Abraham [23, 20],

$$\mathbf{F}_{\text{rad react}} = \frac{2q^2\gamma^2}{3c^3}\left(\ddot{\mathbf{v}} + \frac{\gamma^2\mathbf{v}(\mathbf{v}\cdot\ddot{\mathbf{v}})}{c^2} + \frac{3\gamma^2\dot{\mathbf{v}}(\mathbf{v}\cdot\dot{\mathbf{v}})}{c^2} + \frac{3\gamma^4\mathbf{v}(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^4}\right), \tag{60}$$

and verified by von Laue [24] to follow from the nonrelativistic result (59) via a Lorentz transformation.

From eq. (2) we have that $\dot{v} = a^\star/\gamma^3$, $\ddot{v} = -3a^\star v/\gamma^4 c^2$, which imply that $\mathbf{F}_{\text{rad react}} = 0$ for any speed v in case of uniform, linear acceleration.

Surprisingly, Schott did not realize this in [6], as inferred from his comments on pp. 63 and 245-246.

[19]Many of the opinions on this issue are reviewed in [10], particularly paper II.

[20]The fields of the electric dipole of charge q at $x_b(t)$ and $-q$ at $-x_b(t)$ do not satisfy the Sommerfeld radiation condition [30], as noted in [1, 13].

[21]As argued by Schott [31], p. 51, in the case of uniform acceleration "the energy radiated by the electron is derived entirely from its acceleration energy; there is as it were internal compensation amongst the different parts of its radiation pressure, which causes its resultant effect to vanish." This view is somewhat easier to follow if "acceleration energy" (now often called the Schott energy) means energy stored in the near and induction zones of the electromagnetic field [10, 29, 32], as Schott was unaware of the transfer of energy from the plane $x = -ct$ into the region $x > -ct$. See also sec. 2.8.

[22]In the view that any nonzero Poynting vector is "radiation", DC circuits with a battery and resistor involve "radiation" which flows from the battery to the resistor. Also, a charge with uniform velocity involves "radiation", which is consistent of the virtual-photon concept advocated by Fermi [15] and developed further by Weizsäcker [16] and Williams [17].

[23]Teitelboim [33] has developed a Lorentz-invariant partition of the field energy-momentum tensor (of a single electric charge) into pieces he calls "bound" and "radiated". In this view, a uniformly accelerated charge is a sink of bound energy-momentum and a source of radiated energy-momentum, with the fluxes of these two being equal and opposite close to the charge. See also paper II of [10].

[24]See sec. 4.2 of [19] for commentary as to how Hawking-Unruh radiation (a quantum effect) by an accelerated charge supports the existence of "ordinary" radiation by that charge.

this time, and hence there could be no radiation at any time. However, the Poynting vector is nonzero (for $x > -ct$ at any other time), so the vanishing of the Poynting vector is for a "set of measure zero", and has no physical implication.

Pauli also stated in sec. 32 of [25] that a uniformly accelerated charge does radiate if the uniform acceleration exists for only a finite time interval. This implies that the existence or not of radiation at the present time, near the charge, depends on the behavior of the charge at, say, one billion years earlier when the charge was nearly a billion light years from the charge and the observer. This seems very implausible to the present authors, who take the view that a uniformly accelerated charge is associated with radiation, although the energy of this radiation does not come from the charge itself.[25]

Pauli further stated in sec. 32 of [25] that the fields of a uniformly accelerated charge have no "wave zone", presumably meaning there exists no region in which the electromagnetic fields predominantly fall off as $1/r$ from the charge. From our eq. (34) we see that the magnetic field $\mathbf{B}$ varies are $1/\rho^2$ for large ρ, so indeed there is no "wave zone" for the fields of a uniformly accelerated charge.[26]

2.7 Field Momentum and Electromagnetic Mass

If we suppose the charge q is a spherical shell of radius r_0 when at rest, then when at velocity v the shell is Lorentz contracted in the x-direction, and is an oblate spheroid of semiminor axis r_0/γ. On p. 69 of [6] Schott computes the electromagnetic field momentum outside that oblate spheroid to be $2q^2\gamma v/3r_0$ to lowest order, and identifies the electromagnetic mass as $2q^2/3r_0$, which value he attributes to Lorentz without reference.[27]

A more complete calculation of the field energy and momentum of the fields of the accelerated charge is given in sec. IV, paper III of [10],

$$U_{x>-ct} = \frac{q^2\gamma}{2r_0}\left(1 + \frac{v^2}{3c^2}\right) + \frac{2q^2\gamma a^\star v}{3c^3}, \qquad \mathbf{P}_{x>-ct} = \frac{2q^2\gamma}{3r_0}\mathbf{v} - \frac{2q^2\gamma a^\star}{3c^3}\mathbf{v}. \tag{61}$$

[25]Another example of this behavior is discussed in sec. 2.5 of [34], concerning an initially static electric dipole that decays exponentially starting at, say, $t = 0$. Then, a distant observer detects radiation as expected from a time-dependent dipole, which the Poynting vector near the dipole converges on it. That is, the electromagnetic energy initially stored in the static electric field of the dipole is partially radiated away to large distances, and partially converted to the kinetic energy of the collapsing dipole.

[26]The term "wave zone", has the unfortunate implication that "radiation" exists only in the "wave zone" and not in the "near zone". If so, an ordinary antenna has "radiation" in the "wave zone" but not close to the antenna in the "near zone". Then, the "radiation" of the antenna is somehow mysteriously created at some unspecifiable distance from the charges and currents in the antenna (or the"radiation" exists only at "infinity", in the spirit of an argument of Sommerfeld [30]).

[27]Probably the missing reference is to eq. (28) of [35], which considers the field momentum of a uniformly moving shell of charge. This had been considered earlier by J.J. Thomson in [36] for $v \ll c$, and in sec. 16 of [37] for arbitrary but constant v.

On p. 61 of [6] Schott refers to the "relativistic mass" γm as the "Lorentz mass". This is likely a reference to the statement at the end of sec. 12 of [35], "the masses of all particles are influenced by a translation to the same degree as the electromagnetic masses of the electrons." However, Lorentz distinguished between "longitudinal" and "transverse" masses, and the possible role of γm as "the" relativistic mass was not emphasized until 1912 (the publication year of [6]) by Tolman [38]. That Schott does not mention Einstein in this context is perhaps a precursor of the present trend [39] to deny the existence of "relativistic mass", or at least that Einstein had anything to do with this concept.

The field energy and momentum in the plane $x = -ct$ are,[28]

$$U_{x=-ct} = U_{-\infty} - \frac{2q^2\gamma a^\star v}{3c^3}\,, \qquad \mathbf{P}_{x=-ct} = -U_{-\infty} c\,\hat{\mathbf{x}} + \frac{2q^2\gamma a^\star}{3c^3}\mathbf{v}, \tag{62}$$

and the total field energy and momentum are,

$$U_{\rm total} = U_{-\infty} + \frac{q^2\gamma}{2r_0}\left(1 + \frac{v^2}{3c^2}\right), \qquad \mathbf{P}_{\rm total} = -U_{-\infty} c\,\hat{\mathbf{x}} + \frac{2q^2\gamma}{3r_0}\mathbf{v}. \tag{63}$$

The terms in $1/r_0$ can be interpreted ("renormalized") as aspects of the energy and momentum of the particle. In this view, the field energy and momentum not associated with the energy/momentum of the particle are constant, ending up all in the region $x > -ct$ "behind" the wavefront for large positive times.

2.8 The Schott/Interaction Field Energy-Momentum

Examination of the total field energy and momentum of the accelerated charge in sec. III, paper III of [10] led to the identifications of terms called the Schott energy and momentum, where the Schott energy was first deduced by a different argument in [31].

The Liénard-Wiechert fields **E** and **B** [7, 8] of an accelerated charge each have two terms, which we call the "Coulomb" and "radiation" fields. Only the latter depend on the acceleration **a**.

When computing the field energy and momentum, which are quadratic in the fields, one gets three terms, which can be called "Coulomb only", "radiation only", and "Coulomb-radiation interference", as in $\mathbf{E} = \mathbf{E}_{\rm Coulomb} + \mathbf{E}_{\rm rad}$,

$$U_E = \int \frac{E^2}{8\pi}\, d{\rm Vol} = \int \frac{E^2_{\rm Coulomb}}{8\pi}\, d{\rm Vol} + \int \frac{\mathbf{E}_{\rm Coulomb}\cdot\mathbf{E}_{\rm rad}}{4\pi}\, d{\rm Vol} + \int \frac{E^2_{\rm rad}}{8\pi}\, d{\rm Vol}, \qquad etc. \tag{64}$$

For a model of an electric charge being a spherical shell of radius r_0 in its rest frame (assuming no distortion of its shape even if accelerated), the field energy and momentum (at some time t) are,

$$U = \frac{q^2\gamma}{2r_0}\left(1 + \frac{v^2}{3c^2}\right) - \frac{2q^2\gamma^4\mathbf{a}\cdot\mathbf{v}}{3c^3} + U_{\rm rad}\,, \quad \mathbf{P} = \frac{2q^2\gamma}{3r_0}\mathbf{v} - \frac{2q^2\gamma^2}{3c^3}\left\{\mathbf{a} + \frac{\gamma^2(\mathbf{a}\cdot\mathbf{v})\mathbf{v}}{c^2}\right\} + \mathbf{P}_{\rm rad}. \tag{65}$$

As discussed in sec. 2.7, the first terms of U and **P** are to be absorbed/"renormalized" (along with the energy-momentum of the Poincaré stresses [40] that hold the shell of charge together) into the "mechanical" energy-momentum of the charge.

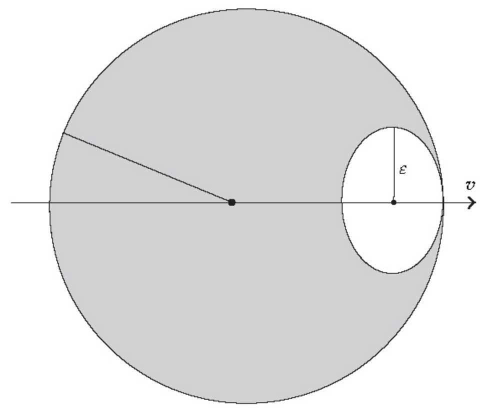
The middle terms of eq. (65) form a 4-vector $U_{\rm Schott}$ (whose time component was first identified in eq. (6) of [31] and is now often called the Schott energy),

$$U_{{\rm Schott},\mu} = -\frac{2q^2}{3}\,a_\mu = (U_{\rm Schott}, c\mathbf{P}_{\rm Schott}) = -\frac{2q^2}{3c^2}\left\{\gamma^4\mathbf{a}\cdot\frac{\mathbf{v}}{c}\,,\ \gamma^2\mathbf{a} + \gamma^4\left(\mathbf{a}\cdot\frac{\mathbf{v}}{c}\right)\frac{\mathbf{v}}{c}\right\}, \tag{66}$$

[28] Recall eq. (2) that $\gamma v = a^\star t$.

where a_μ is the acceleration 4-vector. That is, the Schott energy-momentum is associated with the interference terms, between the "Coulomb" and the "radiation" fields, in the field energy-momentum.[29,30]

Furthermore, Eriksen and Grøn were able to show, sec. 6, paper IV of [10], that for a model of the charge as a small sphere in its rest frame, which appears as a Lorentz-contracted ellipsoid in the lab frame, the Schott energy resides inside the smallest sphere centered on the various retarded positions of the center of the charge such that the sphere completely encloses the charge at its present position.



That is, an observer outside this sphere could/should consider that the "radiation" which he detects flows from inside this sphere, but not directly off the charge itself.

Since the Schott energy-momentum is localized so close to the charge (inside its Compton wavelength), it would seem reasonable to "renormalize" the Schott energy-momentum into the "mechanical" energy-momentum of the charge. Then, the so-call Lorentz-Abraham-Dirac equation of motion (see, for example, secs. 22 and 24 of [20]),

$$m\frac{du^\mu}{d\tau} = F^\mu_{\rm ext} + F^\mu_{\rm self} = eF^{\mu\nu}_{\rm ext}u_\nu - \frac{dU^\mu_{\rm Schott}}{d\tau} - \frac{dU^\mu_{\rm rad}}{d\tau} = eF^{\mu\nu}_{\rm ext}u_\nu + \frac{2e^2}{3c^2}\frac{d^2u^\mu}{d\tau^2} - \frac{P_{\rm rad}u^\mu}{c^2}\,, \qquad (67)$$

where $F^{\mu\nu}_{\rm ext}$ is the electromagnetic field 3-tensor, supposing the external force to be electromagnetic, would simplify to,

$$\frac{d}{d\tau}(m_{\rm eff}\,u^\mu) = F^\mu_{\rm ext} + F^\mu_{\rm rad} = eF^{\mu\nu}_{\rm ext}u_\nu - \frac{P_{\rm rad}u^\mu}{c^2}\,, \qquad (68)$$

and a century of debate over eq. (67) would become largely irrelevant.

These results were summarized in a pedagogic paper by Grøn [44], but he also could not bring himself to recommend "renormalizing" the Schott energy-momentum into the "mechanical" energy-momentum, as in eq. (68).

2.8.1 A Charge Moving in a Uniform Magnetic Field

Another example in which the Schott energy-momentum plays a role is the motion of a charge q in a uniform magnetic field $\mathbf{B}$. The Lorentz force $q\mathbf{v}/c \times \mathbf{B}$ is always perpendicular to the

[29]We find the above discussion in [10] more pertinent than subsequent attempts [41, 42] to "simplify" the argument for the "student".

[30]For other examples in which interference terms in the field energy play an important role, see [43].

velocity $\mathbf{v}$, so this force does not change the magnitude of $\mathbf{v}$. In the absence of radiation, the charge would move in a helical trajectory with constant radius about an axis parallel to $\mathbf{B}$.

The momentum radiated by the charge at time t in this helical motion is always parallel to $\mathbf{v}(t)$, according to observers at distance R from the charge (at time t, such the this is the retarded time with respect to the observation of the radiation at time $t_{\rm obs} = t + R/c$). However, the radiation-reaction force, eq. (59) in the low-velocity limit, is along $\ddot{\mathbf{v}}$, which is not along $\mathbf{v}$, but along $\mathbf{v}_\perp$, the component of the velocity perpendicular to $\mathbf{B}$. Hence, the change in momentum of the radiating charge is not the negative of the momentum radiated (according to the distant observers); the "radiation reaction" is not necessarily a simple application of Newton's third law, but can include more subtle effects of rearrangement of the field energy-momentum of an accelerated charge.[31]

This led to various discussion in the 1960's and early 1970's [48, 49, 50, 51, 52, 53, 54, 56, 55, 57], the most dramatic of which was the claim in [49] that the charge could radiate more energy than its initial kinetic energy. This incorrect result was traced to improper treatment of the Schott field energy [50, 51, 57]. A more minor issue was that a photon, emitted in the forward direction in the frame in which the trajectory is planar, is not in the forward direction in the lab frame [52, 53, 54, 55], which illustrates the complexity of Lorentz transformations of radiation patterns.[32]

2.9 Additional Remarks

(Mar. 2019)

The electromagnetic fields of a uniformly accelerated electromagnetic multipole of any order are discussed in [59].

It is demonstrated in [60, 61] that the exterior fields of a uniformly accelerated shell of charge, which is spherical in its rest frame, are the same as those of a uniformly accelerated charge. Series expansions of the fields of such a charged shell under any acceleration are given in sec. 57 of [62], and in Appendix C of [63].

(Apr. 28, 2022)

An "amusing" example of a classical "runaway" solution has been given in [64], that an electric dipole consisting of point charges $\pm e$ with mass m, somehow held apart by distance $r < e^2/2mc^2$ by a massless rod can have uniform acceleration transverse to the rod with no external force on the system.

According to the equivalence principle, such a dipole would levitate in a gravitation field if the dipole axis is horizontal (because the gravitational pull on the electrostatic field bends the field lines such that the line from one charge to the other is curved, resulting in a upward force on each charge, as shown in Fig. 1 of [65]).

[31]A noteworthy example of this for radiation involving two charges is the case of an electric dipole, initially at rest, but which collapses exponentially after time, say, $t = 0$. While the radiation in the far field is correctly described by the Larmor formula, the Poynting vector close to the dipole points inwards! The radiated energy does not flow from the dipole, but from the energy stored in the initial, electrostatic field thereof, some of which flows back onto the charges as the dipole collapses [45, 46]. For additional comments on the flow of energy in examples with pulsed dipoles, see [34, 47].

[32]For general comments on this theme, see [58].

However, these examples are unphysical in that the separation r must be less than the classical electron radius of the charges.

(Dec. 28, 2022)

Poynting's theorem states that,

$$\frac{\partial u_{\rm EM}}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{S} = \mathbf{J} \cdot \mathbf{E}, \tag{69}$$

where $u_{\rm EM} = (E^2 + B^2)/8\pi$ is the density of energy in the electromagnetic field, and $\mathbf{J}$ is the electric current density.

For the uniformly accelerated charge considered here, $\mathbf{E} = E_0\,\hat{\mathbf{x}}$, so $\mathbf{J} \cdot \mathbf{E}$ is negative for $t < 0$ and positive for $t > 0$. Close to the charge, both $\partial u_{\rm EM}/\partial t$ and $\boldsymbol{\nabla} \cdot \mathbf{S}$ are nonzero.

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