Greek Temple Seismograph

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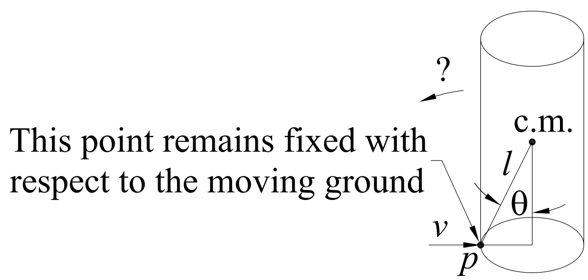
1 Problem

Following a major earthquake near Naples in 1857, R. Mallet suggested that a measure of the horizontal velocity of the ground during an earthquake could be deduced from the maximum height of cylindrical columns that remained standing [1].

Deduce the minimum horizontal velocity v needed to overturn a solid, vertical, cylindrical column whose diagonal has length $2l$ and makes angle θ to the vertical, assuming that a point p on the base remains fixed with respect to the moving ground, as sketched below.



If the cylinder is too squat (large θ), it can lose contact with the ground during its motion. Supposing that the velocity v is the minimum value found above, deduce a condition on the angle θ such that the cylinder always remains in contact with the ground as it falls over.

You may assume that the earthquake lasts long enough, with point p always at horizontal velocity v, that the column could fall over, although in reality the details can be much more complicated.

2 Solution

This problem is taken from secs. 174-175 of [2].

While we could suppose the ground is initially at rest, and is suddenly given horizontal velocity v by the earthquake, it is convenient to work in the (inertial) frame in which the ground (and the vertical cylinder) have initial horizontal velocity v, and the velocity of the ground is suddenly reduced to zero, such that point p on the cylinder is also brought to rest.

Using point p as the reference for a torque analysis, there is no torque on the cylinder about this point (due to the impulsive force of the earthquake that is applied at point p). Hence, angular momentum of the cylinder about point p is conserved during the impulse. Thus,

$$Ł_i = mvl\cos\theta = L_f = I_p\,\omega_0, \qquad \text{and so} \qquad \omega_0 = \frac{mvl\cos\theta}{I_p}\,, \tag{1}$$

where m is the mass of the cylinder, I_p is its moment of inertial about the fixed point, and ω_0 is the initial angular velocity just after the impulse. Recalling that the moment of inertia of a thin disc about an a diameter is $mr^2/4$, the moment of inertia I_p is, using the parallel axis theorem,

$$I_p = \int_0^h \frac{m\ dy}{h}\left(\frac{r^2}{4} + r^2 + y^2\right) = \frac{m}{4}\left(5r^2 + \frac{4h^2}{3}\right) = \frac{ml^2}{12}(15\sin^2\theta + 16\cos^2\theta) = \frac{ml^2}{12}(15 + \cos^2\theta), \tag{2}$$

noting that $r = l\sin\theta$ and $h = 2l\cos\theta$.

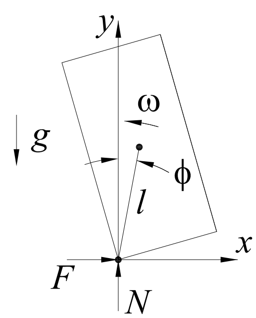
The column will fall over if the initial kinetic energy $I_p\omega_0^2/2$ just after the earthquake is sufficient that the center of mass of the column can rise from $h/2 = l\cos\theta$ to l. Hence, the minimum velocity of the ground needed to topple the column is related by,

$$mgl(1-\cos\theta) = \frac{I_p\omega_{0,\mathrm{min}}^2}{2} = \frac{m^2 v_{\mathrm{min}}^2 l^2\cos^2\theta}{2I_p}\,, \tag{3}$$

and so,

$$v_{\mathrm{min}}^2 = \frac{2gI_p(1-\cos\theta)}{ml\cos^2\theta} = \frac{gl}{6}\frac{(1-\cos\theta)(15+\cos^2\theta)}{\cos^2\theta}\,. \tag{4}$$

As the column rotates about the fixed point with angular velocity $\omega(\phi)$, where ϕ is the angle of the diagonal to the vertical, it will lose contact with the ground if the normal force N goes to zero.



Referring to the figure above, the y-equation of motion of the center of mass of the cylinder is,

$$F_y = m\ddot{y} = m\frac{d^2}{dt^2}(l\cos\phi) = ml\frac{d}{dt}(\omega\sin\phi) = ml(\dot{\omega}\sin\phi - \omega^2\cos\phi) = N - mg, \tag{5}$$

noting that $d\phi/dt \equiv \dot{\phi} = -\omega$. The normal force goes to zero if there is an angle ϕ such that,

$$g = l(\omega^2\cos\phi - \dot{\omega}\sin\phi). \tag{6}$$

As the column rotates (about the z-axis), conservation of energy relates ω and ϕ according to,

$$\omega^2 = \omega_0^2 - 2\frac{mgl}{I_p}(\cos\phi - \cos\theta). \tag{7}$$

If we restrict our attention to the case that the velocity of the ground is the minimum value (4), then using eq. (3) in (7) yields,

$$\omega^2 = \frac{2mgl}{I_p}(1-\cos\phi). \tag{8}$$

Taking the time derivative of eq. (8) we find that,

$$\dot{\omega} = -\frac{mgl}{I_p} \sin\phi, \tag{9}$$

and the condition (6) becomes,

$$\frac{I_p}{mgl^2} = \frac{15 + \cos^2\theta}{12} = 2\cos\phi(1 - \cos\phi) + \sin^2\phi = -3\cos^2\phi + 2\cos\phi + 1, \tag{10}$$

or,

$$3\cos^2\phi - 2\cos\phi + \frac{3 + \cos^2\theta}{12} = 0. \tag{11}$$

Thus, the column will loose contact with the ground if,

$$\cos\phi = \frac{2 \pm \sqrt{4 - 12(3 + \cos^2\theta)/12}}{6} = \frac{2 \pm \sin\theta}{6}. \tag{12}$$

As the column rotates, angle ϕ is less than θ (and $\phi_0 = \theta$) so that $\cos\phi \geq \cos\theta$. If the column is to remain in contact with the ground at all times, we must have that neither solution (12) is greater than $\cos\theta$, *i.e.*,

$$\cos\theta > \frac{2 + \sin\theta}{6}. \tag{13}$$

The critical angle is roughly 61.3°, so if the height of the column is more than 1.83 times its diameter it will remain in contact with the ground at all times while it falls over after an earthquake that is minimally capable of causing this.

A Appendix: Use of the Center of Mass as the Reference Point

The Appendix added July 14, 2023 at the suggestion of Ralph Wang.

As reviewed in [3], it is generally a good strategy in torque analyses of rigid-body motion to use the center of mass of the body as the reference point for computation of the torque.

The initial angular momentum $L_{i,\mathrm{cm}}$ of the cylinder about its center of mass is zero, and its angular momentum just after the impulse of the earthquake is $L_{f,\mathrm{cm}} = I_{\mathrm{cm}}\,\omega_0$, where the moment of inertia of the cylinder about a horizontal line through its center of mass is,

$$I_{\mathrm{cm}} = \int_{-h/2}^{h/2} \frac{m\ dy}{h} \left(\frac{r^2}{4} + y^2\right) = m\left(\frac{r^2}{4} + \frac{h^2}{12}\right) = ml^2\left(\frac{\sin^2\theta}{4} + \frac{\cos^2\theta}{3}\right) = \frac{ml^2}{12}(3 + \cos^2\theta), \tag{14}$$

recalling that $r = l\sin\theta$ and $h = 2l\cos\theta$.

The change in angular momentum about the center of mass during the impulse equals the torque impulse about the center of mass, $\mathbf{\Delta P} \times \mathbf{l}$, where $\mathbf{\Delta P} = \mathbf{P}_f - \mathbf{P}_i$ is the change in momentum of the cylinder during the earthquake, and $\mathbf{l}$ is the distance vector from the center of mass to point p.

Before the earthquake the cylinder has horizontal momentum $P_i = mv$, and just after the earthquake is has momentum $\mathbf{P}_f = m\mathbf{v}_f = m\boldsymbol{\omega}_0 \times \mathbf{l}$, which makes angle θ to the horizontal.

The torque equation about the center of mass is,

$$\Delta\mathbf{L}_{\rm cm} = \mathbf{L}_{f,\rm cm} - \mathbf{L}_{i,\rm cm} = I_{\rm cm}\,\boldsymbol{\omega}_0 = \mathbf{P}_f \times \mathbf{l} - \mathbf{P}_i \times \mathbf{l}. \tag{15}$$

The magnitude of $\mathbf{P}_f \times \mathbf{l}$ is $ml^2\,\omega_0$, and the magnitude of $\mathbf{P}_i \times \mathbf{l}$ is $mvl\cos\theta$. Noting the signs of the various terms in eq. (15), it can be rewritten as,

$$I_{\rm cm}\,\omega_0 + ml^2\omega_0 = \frac{ml^2\,\omega_0}{12}(3+\cos^2\theta) + ml^2\,\omega_0 = \frac{ml^2\,\omega_0}{12}(15+\cos^2\theta) = I_p\,\omega_0 = mvl\cos\theta, \tag{16}$$

as previously found (more quickly) in eq. (1).

References

[1] R. Mallet, *The First Principles of Observational Seismology* (Chapman and Hall, 1862), Vol. 1, Chap. 16, http://kirkmcd.princeton.edu/examples/mechanics/mallet_chap16.pdf

[2] E.J. Routh, *The Elementary Part of a Treatise on the Dynamics of a System of Rigid Bodies*, 7$^{\rm th}$ ed. (Macmillan, 1905), http://kirkmcd.princeton.edu/examples/mechanics/routh_elementary_rigid_dynamics.pdf

[3] K.T. McDonald, *Comments on Torque Analyses* (April 28, 2019), http://kirkmcd.princeton.edu/examples/torque.pdf