Self Trapping of Optical Beams

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(April 15, 2001)

1 Problem

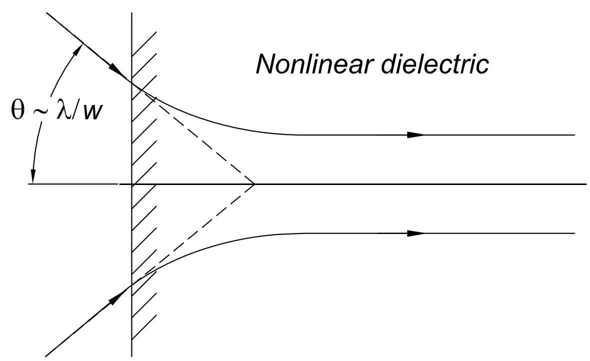
When an isotropic dielectric (for example, a gas) is placed in a strong electric field, the induced polarization is a nonlinear function,[1]

$$\mathbf{P} = \chi_1 \mathbf{E} + \chi_3 E^2 \mathbf{E} + ... \qquad \text{(Gaussian units)} \tag{1}$$

a) Give an order of magnitude estimate of the susceptibilities χ_1 and χ_2 for a nitrogen gas at STP, supposing the nonlinear polarization is comparable to the linear term in an electric field just strong enough to ionize the gas. Fact: liquid nitrogen has index of refraction 1.2 and density 0.8 g/cm^3.

If the applied electric field is due to a laser beam, the nonlinear polarizability results in an index of refraction that is larger where the electric field is stronger. The radial gradient of the index of refraction is similar to that in a fiberoptic cable, so the laser beam can become trapped in a kind of "light pipe" of its own making.

b) Suppose a laser beam of wavelength λ is focused at angle $\theta \approx \lambda/w$ into a slab of nonlinear dielectric over a radius of diameter w. What is the minimum power of the laser beam so that it becomes trapped in a channel of constant radius?



c) Deduce the pulse shape $E_x(y)$ of a 2-dimensional beam (which obeys $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$),

$$\mathbf{E} = E_x(y)\, \hat{\mathbf{x}}\, e^{i(k_z z - \omega t)}, \tag{2}$$

that propagates with an invariant transverse profile in the nonlinear medium, assuming that the pulsewidth in y is large compared to a wavelength so that E_x is determined by time-averaged quantities.

[1] In SI units one writes $\mathbf{P} = \epsilon_0 \chi_1 \mathbf{E} + ...$, with the unfortunate result that $\chi_{\rm SI} = 4\pi\chi_{\rm Gaussian}$.

2 Solution

This problem is based on a paper by R.Y. Chiao, E. Garmire and C.N. Townes, Phys. Rev. Lett. **13**, 479 (1964).[2]

We work in Gaussian units.

a) We recall that the electric displacement is related by,

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} = \epsilon\mathbf{E}, \tag{3}$$

so the dielectric "constant" ϵ follows from eq. (3) as,

$$\epsilon = 1 + 4\pi\chi_1 + 4\pi\chi_3 E^2 + ... \equiv \epsilon_1 + \epsilon_3 E^2 + ... \tag{4}$$

The index of refraction n is, of course, related by,

$$n = \sqrt{\epsilon} \approx 1 + 2\pi\chi_1 + 2\pi\chi_3 E^2 + ... \tag{5}$$

For a typical gas at STP, $n - 1 \approx$ few $\times 10^{-4}$, so we estimate the linear susceptibility as,

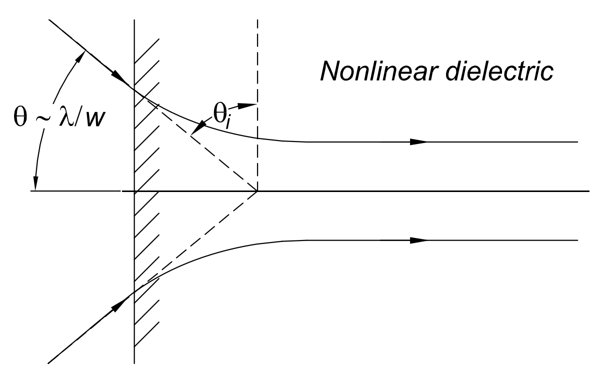
$$\chi_1 \approx \frac{n-1}{2\pi} \approx 10^{-4}. \tag{6}$$

For a somewhat more precise estimate we use the given facts about nitrogen. The index of liquid nitrogen is related to its susceptibility χ_L via $n_L = 1.2 \approx \sqrt{1 + 4\pi\chi_L}$, so $\chi_L \approx 0.44/4\pi = 0.035$. The susceptibility is proportional to the density, so the susceptibility of nitrogen gas, whose density is $\rho_{\rm gas} = 28$ g/22.4 l $= 1.25 \times 10^{-3}$ g/cm^3, is given by $\chi_1 = \chi_L \rho_{\rm gas}/\rho_L = 0.035 \cdot 1.25 \times 10^{-3}/0.8 = 5.5 \times 10^{-5}$. The index of the gas is $n \approx \sqrt{1 + 4\pi\chi_1} \approx 1 + 2\pi\chi_1 \approx 1.00035$. The actual value for the index of nitrogen gas at STP is $n = 1.00030$.

We estimate that the nonlinear term in the polarizability is comparable to the linear term when the external field is barely sufficient to ionize a nitrogen atom. For this the field strength would need to be about 6 V/angstrom (since air is transparent for light up to about 6 eV, and the size of a nitrogen atom is about 1 angstrom). Recalling that 1 V = 300 statvolt, we need field $E_{\rm ionize} \approx (6/300) \times 10^8 = 2 \times 10^6$ statvolt/cm. Thus, we estimate,

$$\chi_3 \approx \frac{\chi_1}{E_{\rm ionize}^2} \approx \frac{5 \times 10^{-5}}{(2 \times 10^6)^2} \approx 1.25 \times 10^{-17}. \tag{7}$$

b) We consider the fate of a ray that is initially at angle $\theta \approx \lambda/w$ to the axis of the laser beam, shown as a dashed line in the figure.



[2] http://kirkmcd.princeton.edu/examples/optics/chiao_prl_13_479_64.pdf

The index of refraction is varying with position along this ray, reaching a value $n_{\max}$ on the axis of the laser beam. In a simplified view, this ray would like to continue across the axis of the beam back into the region of low index of refraction, where $n \approx 1$. However, a ray is subject to total internal relection when going from high index to low index if,

$$n_{\max} \sin \theta_i > 1, \tag{8}$$

according to Snell's law. From the figure, we see that,

$$\sin \theta_i \approx \cos \theta. \tag{9}$$

Taking $n_{\max}$ from eq. (4), the ray will be totally internally reflected, and hence trapped in a tube, if,

$$(1 + 2\pi\chi_1 + 2\pi\chi_3 E^2)\left(1 - \frac{\theta^2}{2}\right) \approx 1. \tag{10}$$

For $2\pi\chi_1 \approx 2\pi\chi_3 E^2$ and $\theta \approx \lambda/w$, this requires a laser electric field strength of,

$$E^2 \approx \frac{\theta^2}{8\pi\chi_3} \approx \frac{(\lambda/w)^2}{8\pi\chi_3}. \tag{11}$$

The corresponding laser power P is given in terms of the Poynting vector S as,

$$\begin{aligned} P &= S \cdot \text{Area} \approx \frac{c}{4\pi} E^2 w^2 \approx \frac{c(\lambda/w)^2 w^2}{2(4\pi)^2 \chi_3} = \frac{c}{2\chi_2}\left(\frac{\lambda}{4\pi}\right)^2 \approx \frac{3 \times 10^{10} \cdot (5 \times 10^{-6})^2}{2 \times 10^{-17}} \\ &\approx 3 \times 10^{16} \text{ erg/s} = 3 \times 10^9 \text{ W} = 3 \text{ GW}. \end{aligned} \tag{12}$$

Such powers can be achieved in table-top lasers, where a pulse of is compressed to 100 ps and amplified to 1 J.

We note that for the focused laser beam to be just below the ionization threshold, the focus angle θ obeys,

$$\theta \approx \sqrt{8\pi\chi_3 E^2} \approx \sqrt{8\pi\chi_1} \approx \sqrt{14 \times 10^{-4}} \approx \frac{1}{25}. \tag{13}$$

Since the focus angle is roughly the ratio D/f of the diameter D of the focusing lens to its focal length f, we see that a "long" lens of $f/D \approx 25$ must be used. The spot size w, and hence the approximate diameter of the self-trapped filament of light, is given by $w \approx \lambda/\theta \approx 25\lambda \approx 12\ \mu$m.

c) The wave equation in the nonlinear medium is,

$$\nabla^2 \mathbf{E} = \frac{n^2}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{\epsilon_1 + \epsilon_3 E^2}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}, \tag{14}$$

where $\epsilon_1 = 1 + 4\pi\chi_1$ and $\epsilon_3 = 4\pi\chi_3$. Inserting the pulse (2) into the wave equation, we find,

$$E_x'' - k_z^2 E_x = -\frac{\omega^2}{c^2}(\epsilon_1 + \epsilon_3 E^2) E_x, \tag{15}$$

where $E_x' = dE_x/dy$.

We suppose that the slowly varying waveform E_x responds only to the time average of E^2 that appears in eq. (15); of course $\langle E^2 \rangle = E_x^2/2$.

We define,

$$k \equiv \frac{\omega}{c}\,, \qquad k_1 \equiv \frac{\omega}{c/n_1} = \frac{\sqrt{\epsilon_1}\omega}{c}\,, \qquad \text{and} \qquad \Gamma^2 \equiv k_z^2 - k_1^2. \tag{16}$$

Then, eq. (15) becomes,

$$E_x'' = \Gamma^2 E_x - \frac{\epsilon_3 k^2}{2} E_x^3 \equiv \Gamma^2 \left(E_x - \frac{2}{a^2} E_x^3 \right), \tag{17}$$

where the constant a is defined by,

$$a = \frac{2\Gamma}{\sqrt{\epsilon_3} k}. \tag{18}$$

Perhaps surprisingly, this nonlinear differential equation is easily integrated after multiplying by E_x',

$$E_x' E_x'' = \frac{1}{2}\frac{dE_x'^2}{dx} = \Gamma^2 \left(E_x E_x' - \frac{2}{a^2} E_x^3 E_x' \right) = \Gamma^2 \left(\frac{1}{2}\frac{dE_x^2}{dx} - \frac{2}{a^2}\frac{1}{4}\frac{dE_x^4}{dx} \right). \tag{19}$$

Hence,

$$E_x'^2 = \Gamma^2 \left(E_x^2 - \frac{1}{a^2} E_x^4 \right) + C. \tag{20}$$

For a **pulse**, we desire that $E_x(\pm\infty) = 0 = E_x'(\pm\infty)$, so the integration constant C is zero. Then,

$$E_x' = \frac{dE_x}{dy} = \frac{\Gamma}{a} E_x \sqrt{a^2 - E_x^2}, \tag{21}$$

which integrates to,

$$\frac{\Gamma}{a} y + \frac{D}{a} = \int \frac{dE_x}{E_x \sqrt{a^2 - E_x^2}} = -\frac{1}{a} \text{sech}^{-1} \frac{E_x}{a}\,, \tag{22}$$

or,

$$E_x = a \ \text{sech}(\Gamma y + D). \tag{23}$$

For a pulseshape that is symmetric about $y = 0$, we set the integration constant D to zero, and have,

$$E_x = \frac{2\Gamma}{\sqrt{\epsilon_3} k} \text{sech}\ \Gamma y. \tag{24}$$

This pulse exhibits the hyperbolic secant shape found in many distortionless solutions (solitons) to nonlinear wave equations. Note also that the pulse amplitude is larger for narrower pulses.