# On the History of the Radiation Reaction[1]

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## 1 Introduction

Apparently, Kepler considered the pointing of comets' tails away from the Sun as evidence for radiation pressure of light [2].[2] Following Newton's third law (see p. 83 of [3]), one might suppose there to be a reaction of the comet back on the incident light. However, this theme lay largely dormant until Poincaré (1891) [38, 42] and Planck (1896) [47] discussed the effect of "radiation damping" on an oscillating electric charge that emits electromagnetic radiation.

Already in 1892, Lorentz [39] had considered the self force on an extended, accelerated charge $e$, finding that for low velocity $\mathbf{v}$ this force has the approximate form (in Gaussian units, where $c$ is the speed of light in vacuum), independent of the radius of the charge,

$$\mathbf{F}_{\rm self} = \frac{2e^2}{3c^3}\frac{d^2\mathbf{v}}{dt^2} = \frac{2e^2\ddot{\mathbf{v}}}{3c^3}\,. \qquad (v \ll c). \tag{1}$$

Lorentz made no connection at the time between this force and radiation, which connection rather was first made by Planck [47], who considered that there should be a damping force on an accelerated charge in reaction to its radiation, and by a clever transformation arrived at a "radiation-damping" force identical to eq. (1). Today, Lorentz is often credited with identifying eq. (1) as the "radiation-reaction force", and the contribution of Planck is seldom acknowledged.

This note attempts to review the history of thoughts on the "radiation reaction", which seems to be in conflict with the brief discussions in many papers and "textbooks".[3]

## 2 What is "Radiation"?

The "radiation reaction" would seem to be a reaction to "radiation", but the concept of "radiation" is remarkably poorly defined in the literature.

The author's view [181] is that electromagnetic radiation should be identified with the Poynting vector [28], $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$, where $\mathbf{E}$ and $\mathbf{B}$ are the electromagnetic fields. In the literature on the "radiation reaction", the notion of "radiation" is generally restricted to the Poynting vector at large distances from the source charge/current, as reinforced by

[1]An earlier set of comments by the author related to the radiation reaction is given in [1].

[2]Several inconclusive attempts to confirm Kepler's conjecture are reviewed in [117].

[3]In 1969, Ginzburg [129] noted that the question of the radiation reaction is something of a "perpetual problem", and like many others claimed that his discussion would settle the issue. The author is not so sanguine, and supposes that, despite this note, the radiation reaction will continue to be a "perpetual problem" for most readers.

the so-called "Sommerfeld radiation condition" [95, 166].[4] However, the flow of energy and momentum in electromagnetic fields far from their source does not necessarily reflect the flow at the source.[5]

## 3 Stewart and a Reaction to Radiation

In 1871, Balfour Stewart [17] discussed the interaction of a body in motion inside a cavity that contained blackbody radiation, stating: *The body in motion in the enclosure is not therefore giving the precise rays which it would have given it had it been at the same temperature and at rest.* And further: *It is not therefore, allowable to suppose that in such an enclosure the moving body retains all its energy of motion, and consequently such a body will have its energy of motion gradually stopped.*

These comments may be the first relatively clear statement of an effect of a "radiation reaction", although Stewart considered only thermal radiation/light by electrically neutral objects, and did not follow Maxwell in identifying light with electromagnetic radiation.[6]

In 1873, Stewart [20] briefly discussed "æthereal friction," which is a different way of characterizing Maxwell's radiation pressure (sec. 4 below).

## 4 Maxwell and Radiation Pressure

Although Maxwell famously identified light with electromagnetic waves (p. 22 of [11]), he did not live long enough to consider generation of such waves by electrical devices. In Arts. 792-793 of his *Treatise* (1873) [19] he discussed the "energy and stress of radiation", and noted that when a plane electromagnetic wave is absorbed by a plate, the effect on the plate is a kind of (radiation) pressure. He did not discuss the back reaction of the object on the electromagnetic wave, which occurs for a reflecting plate that can be considered to re-emit the absorbed wave, thereby doubling the radiation pressure.

[4]Although "radiation" in the broad sense of transfer of energy via an electromagnetic field [181] exists in all electrical devices that contain currents (including steady currents) in resistive materials, the concept of radiation played essentially no role in the early history of electricity and magnetism. While electromagnetic radiation in the form of waves exists in (almost) all electrical devices that have time-dependent charge and/or current distributions, and was present, for example, in Faraday's earliest studies [4] of electromagnetic induction, the wave character of the radiation plays little role unless the time dependence is rapid enough that the wavelength of the radiation is similar to (or smaller than) the size of the devices.

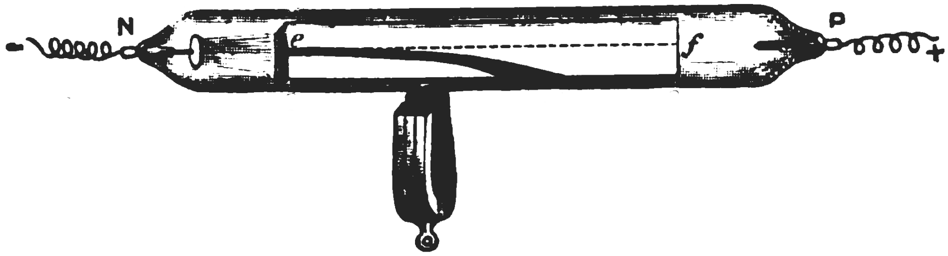
[5]An early, partial awareness of this issue underlies the debate, beginning with Born (1909), sec. 18 below, as to whether or not a uniformly accelerated charge "radiates", since the radiation-reaction force (1) vanishes in this case. For comments by the author on this theme, see [187]. Another example is the collapse of an electric dipole, initially at rest, in which "radiation" flows outward at large distances, but inward close to the dipole [191]. And in the realm of applied physics, there is the "radiation paradox" of Schelkunoff (sec. 6 of [113]) that since the electric field can only be normal to the surface of a good/perfect conductor, the Poynting vector can only be parallel to the surface, and no electromagnetic-field energy flows out of the surface of the conductors of an antenna (and flows into the surfaces of resistive conductors).

[6]Stewart did subscribe to a wave (undulatory) theory of heat, as opposed to a particle (emissive) theory, as he discussed in secs. 179 and 387 of [12].

## 5 Crookes and Cathode Rays

The radiation reaction is an effect related to elementary charged particles, but such entities were not part of Maxwell's vision of electrodynamics, in which electric charge was a continuous distribution $\rho$ associated with a "strain" in the æther, $\rho = \nabla \cdot \mathbf{D}$.[7] The notion that charge is a property of particles gained strength from dramatic experiments with evacuated tubes equipped with electrodes. This technology was introduced in the 1850's by Plücker and his assistant Geissler [9, 10],[8] and was popularized by Crookes in the 1870's [21, 24].

For example, in the apparatus sketched below (from [24]), a beam emanated from the cathode ($e$), and these "cathode rays" were deflected by a magnetic field perpendicular to the page, rather than striking the anode ($f$) as they did for zero field.




Such experiments suggested that negatively charged particles were being emitted into the vacuum (although it was not logically excluded that the beam was a continuous fluid).[9,10] They also provided evidence in favor of the $\mathbf{J} \times \mathbf{B}$ force (*i.e.*, the $q\mathbf{v} \times \mathbf{B}$ Lorentz force), discussed by Maxwell in Arts. 602-603 of his *Treatise* [19].

## 6 J.J. Thomson and Electromagnetic Mass

In 1881, J.J. Thomson (age 25) [25] was inspired by Crookes' results to consider the electromagnetic fields of a slowly moving charged particle (which surprisingly had not been previously discussed in the literature). In particular, the magnetic field is proportional to the velocity of the charge, so the magnetic field energy, $U_B = \int B^2 \, d\mathrm{Vol}/8\pi$, goes as the square of the velocity, as does the kinetic energy of the particle. Thomson took the bold step of supposing that this magnetic energy is effectively part of the particle's mass, which is perhaps the first example of what has come to be called mass "renormalization",[11] as well

[7] In contrast, elementary charged particles were central to Weber's electrodynamics (1846) [6].

[8] Plucker noted that the pattern of the glow in discharge tubes was affected by a magnetic field, and characterized this phenomenon as "magnetic light".

[9] In 1894 these "atoms of electricity" were named "electrons" by Stoney [43].

[10] The above experiment of Crookes was not qualitatively distinct from that of Thomson (1897) [48], which is credited with the "discovery" of the electron.

[11] Already in 1843, Stokes (age 24) had argued, p. 106 of [5], that *when a cylinder oscillates in an infinitely extended fluid, the effect of the inertia of the fluid is to increase the mass of the cylinder by that of the fluid displaced.* Stokes did not propose that the mass of the displaced fluid be considered as part of the mass of the cylinder.

Following present usage, we could say that the cylinder was "dressed" by the fluid close it, which increased its effective mass. The cylinder + interacting fluid could then be regarded as a "quasiparticle".

The theme of an effective mass of "sunlight", was pursued by W. Thomson in 1855 [7], without associating light with electromagnetism.

as of the concept of “relativistic mass”, $m = m_0/\sqrt{1 - v^2/c^2}$, where $m_0$ is the rest mass of particle with speed $v$.[12]

In eq. (3) of [25], Thomson identified the quantity $ke^2/ac^2$, where $a$ is the radius of a charged sphere, as having dimensions of mass, but he did not go so far as to suppose that the entire mass of a charge is thereby explained (which would imply that $a/k = e^2/mc^2 =$ the so-called classical charge radius).

# 7 FitzGerald and Radiation by an Oscillating Current Loop

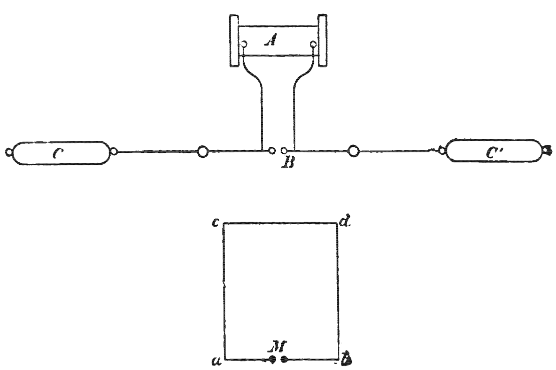
The first calculation of the radiation electromagnetic energy by an electrical current was given by FitzGerald in 1883 [27, 180], one year before Poynting [28] introduced his vectorial measure of the flow of electromagnetic field energy. FitzGerald’s result for the time-averaged power $\langle P\rangle$ radiated by a current with time dependence $I_0 \cos\omega t$ can be written (in Gaussian units) as,

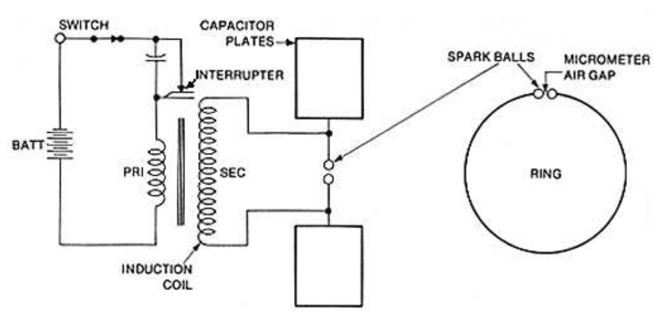
$$\langle P\rangle = \frac{I_0^2 R_{\rm rad}}{2}, \qquad \text{where} \qquad R_{\rm rad} = \frac{2\pi^2}{3c}\left(\frac{L}{\lambda}\right)^4 = 197\left(\frac{L}{\lambda}\right)^4 \Omega, \tag{2}$$

where $R_{\rm rad}$ is the so-called radiation resistance, where $L \ll \lambda$ is the circumference of the loop, and $\lambda = 2\pi c/\omega$.[13]

# 8 Hertz and Electric-Dipole Radiation

In 1887, Hertz began experiments [29, 30, 31] (sketched below) with what is now called an electric-dipole antenna, driven by an induction (Ruhmkorff) coil/spark gap, together with a loop antenna at whose gap sparks could be induced by waves from the first antenna.




[12] There exists a campaign [163] to deny the existence of “relativistic mass”, or at least that Einstein had anything to do with this concept, and hence that it should not be discussed.

[13] The “natural” unit of electric resistance is $1/c = 30\,\Omega$. The concept of “natural” units was popularized by Planck (1899) in [51], where the Planck length $= \sqrt{\hbar G/c^3} \approx 10^{-33}$ cm was first introduced.

In 1889, he gave an analysis [34, 35], using the Poynting vector, that the instantaneous radiated power can be written as,

$$P = \frac{2p_0^2\,\omega^4 \cos^2 \omega t}{3c^3}\,, \tag{3}$$

where $L \ll \lambda$ is the total length of the antenna, $p = p_0 \cos \omega t$ is its electric-dipole moment, which is related to the current $I_0 \sin \omega t$ in its spark gap by $p_0 = I_0 L/2\omega$.

*Hertz did not describe the time-averaged radiated power as,*

$$\langle P \rangle = \frac{I_0^2 R_{\rm rad}}{2}\,, \qquad where \qquad R_{\rm rad} = \frac{2\pi^2}{3c}\left(\frac{L}{\lambda}\right)^2, \tag{4}$$

*and $R_{\rm rad}$ is the radiation resistance of the antenna.*[14]

# 9 Lodge and Radiation Resistance

In 1889, Lodge translated Hertz' paper [34] into English [35], and repeated many of Hertz' experiments [36]. On p. 153 of [36], Lodge remarked that the power radiated in the apparatus of Hertz "may be compared with the form $V^2/R$", where $V$ is the "difference in potential between its ends" and $R$ is a resistance. This appears to be the first conception of the radiation resistance associated with an electrical circuit that emits radiation.[15]

# 10 Poincaré and *Amortissement*

Hertz' achievements had considerable immediate impact, including commentaries by Poincaré, (1891) [38] and secs. 47-48, pp. 92 ff, of [42], that mentioned the *amortissement* (damping) of a Hertzian electrical oscillator due to its emission of energy in the form of radiation.[16]

# 11 Radiation Resistance *vs.* Radiation Reaction

The radiation resistance, which is to be included in circuit analyses of devices that emit radiation, is the most practical application of the "radiation reaction", as discussed in sec. 2.2 of [176].

The radiation resistance (2) for a loop of radius $r = L/2\pi$ can be deduced from the radiation-reaction force $\mathbf{F}_{\rm react} = -e^2 r^2 \dddot{\mathbf{v}}/6c^5$ on electrons of charge $e$ moving in that loop.

[14]In early experiments on antennas with spark gaps it was hard to distinguish the radiation resistance from the resistance of the spark (plasma). See, for example, [60].

[15]In 1898, Abraham (age 23) [49] considered the *Dämpfung durch Ausstrahlung* of an electric-dipole antenna (consisting of two halves of a prolate spheroid). In 1902, eq. (9) of [59], Abraham gave the power radiated by a particular antenna in a form equivalent to $I_0^2 R/2$ with $R = 36.6\ \Omega$, but did not describe this result as involving a resistance. This value was called the *Strahlungswiderstand* (radiation resistance) on p. 459 of [85] (1908).

[16]The damping of an electrical oscillator due to Joule heating had been noted in [37].

See secs. 2.3-4 of [176]. This example involves magnetic-dipole radiation (rather than electric-dipole radiation), in which case the leading term, proportional to $\ddot{\mathbf{v}}$, vanishes in Lorentz' series expansion for the self force, and the next-to-leading term, proportional to $\dddot{\mathbf{v}}$, dominates. However, the physics literature on the "radiation reaction" is almost exclusively concerned with single electrical charges, which latter cannot be well described in classical electrodynamics. In contrast to the widespread application of the concept of the radiation resistance in antenna engineering, there is essentially no experimental evidence for effects of the classical radiation reaction on single electric charges, which topic remains almost purely a "theoretical" concern, with the character of a "perpetual problem" [129], about which contentious literature (perhaps including this note) continues to be generated.

# 12 Lorentz' Self Force on an Accelerated Charge

In 1892, Lorentz enunciated his "classical electron theory" [39], that electric charge resides on "particles" (rather than in the æther/fields surrounding such particles, as according to Maxwell [18]).[17] In sec. 120 of [39], Lorentz approximated the retarded potentials of Lorenz [14] and Riemann [13] to order $1/c^3$ to deduce the self force on an accelerated, extended charge $e$ with low velocity ($v \ll c$), finding in his eq. (111) the famous result (in Gaussian units),

$$\mathbf{F}_{\mathrm{self}} = \frac{3e^2}{3c^3}\frac{d^2\mathbf{v}}{dt^2} = \frac{2e^2\ddot{\mathbf{v}}}{3c^3} \qquad (v \ll c). \tag{5}$$

That is, Lorentz considered the equation of motion for the charge, of mass $m$, when subject to an external force $\mathbf{F}_{\mathrm{ext}}$, to be,

$$m\dot{\mathbf{v}} = \mathbf{F}_{\mathrm{ext}} + \mathbf{F}_{\mathrm{self}}. \tag{6}$$

There was no mention of radiation in Lorentz' derivation.

Lorentz continued his development of "electron theory" in [44], and gave a version of his argument for eq. (5) in sec. 20 of [68], and in Note 18 (in English) of [81]. In sec. 21 of [68], and between eqs. (35) and (36) of Note 18 [81], it was clarified that part of the self force could be written as $-ke^2\,\dot{\mathbf{v}}/r_0c^2$, where $r_0$ is the radius of the charge and $k$ is a constant of order unity. This was interpreted as the negative of the acceleration $\dot{\mathbf{v}}$ times the "electromagnetic mass" $ke^2/r_0c^2$ (first noted by J.J. Thomson [25]), and was therefore absorbed ("renormalized") into the left side of eq. (6), where the effective mass became $m + ke^2/r_0c^2$.[18,19]

[17] For a review of Lorentz' thinking about charges and field prior to [39], see [128].

[18] The suggestion that the mass of an electron is entirely due to its electromagnetic field energy may be due to Wien (1900) [58]. In 1902, Kaufmann [65] argued that he had demonstrated this to be so experimentally. Poincaré pointed out in 1905 [79, 80] that the mass of a charge cannot all be electromagnetic, in that for the charge to be stable against the Coulomb repulsion of its parts, there must exist internal stresses that are also associated with mass/energy. These stresses provide a resolution of the so-called 4/3 problem, as reviewed in sec. 91 of [126]. Lorentz later argued [97] that the Poincaré stresses would not render a charge stable against deformations.

[19] Lorentz' low-velocity derivation of the self force (5) was embellished by Abraham in 1902 [63], with the

Lorentz may have first noted a relation between his self force (5) and radiation in note 10, pp. 60-62 of [76] (1905). Then, on p. 49 of [81] (1906) he inverted Planck's argument [47] to show that the time integral $\int \mathbf{F}_{\rm self} \cdot \mathbf{v}\, dt$ can be integrated by parts to give a term $-\int (2e^2\dot{\mathbf{v}}^2/3c^2)\, dt$, without mentioning that this result corresponds to the negative of the radiated power. A connection of this term to radiation was, however, made on p. 259, Note 22 of [81].

In sec. 37 of [81], Lorentz added the comment that his result (5) holds only if the motion of the charge does not change "sensibly" during the time light takes to cross the (extended) charge. *He did not mention that this restriction would exclude so-called "runaway" solutions that have since been considered for the equation of motion of a classical charge. Nor did he mention that this restriction implies that the strengths of the external electric and magnetic fields cannot be arbitrarily large.*[20]

### 12.1 Should Lorentz' Self Force Be "Renormalized" Away?

In view of eq. (5), Lorentz' equation of motion (6) of an electron with "mechanical" momentum $\mathbf{p}_{\rm mech} = m\mathbf{v}$ could be written as,

$$m\dot{\mathbf{v}} = \frac{d\,\mathbf{p}_{\rm mech}}{dt} = \mathbf{F}_{\rm ext} + \mathbf{F}_{\rm self} = \mathbf{F}_{\rm ext} + \frac{d}{dt}\frac{2e^2\dot{\mathbf{v}}}{3c^2}\,, \qquad \frac{d}{dt}\left(\mathbf{p}_{\rm mech} - \frac{2e^2\dot{\mathbf{v}}}{3c^2}\right) = \mathbf{F}_{\rm ext}. \qquad (7)$$

This suggests that the effective momentum of an electron might be regarded as $\mathbf{p}_{\rm eff} = \mathbf{p}_{\rm mech} - 2e^2\dot{\mathbf{v}}/3c^2$, in which case the "renormalized" equation of motion might be simply $d\,\mathbf{p}_{\rm eff}/dt = \mathbf{F}_{\rm ext}$, and the peculiar form of the Lorentz' self force (5) would have no impact on the observable behavior of an electron.

While this procedure has some appeal in giving a "trivial" resolution of the issue of the self force on an electron, it seems inappropriate once one is aware that the self force (5) is related to the reaction on an accelerated charge due to its radiation of electromagnetic energy, as noted by Planck (sec. 13 below). Hence, people have hesitated to "renormalize away" the self force (5), leaving the topic of "classical electron theory" to wrestle with its consequences, which to this day seems (to this author) to be somewhat imperfectly resolved.[21]

An "engineering" argument for continued use of Lorentz' self force (5) is that it leads to the radiation resistance of antennas, such as eqs. (2) and (4), which is the most "practical"

explicit assumption that the mass of a charge is entirely electromagnetic. Versions of Lorentz' argument are given in sec. 21-7 of [125], and in sec. 16.3 of [169]. The spirit of Lorentz' argument is well illustrated in the pedagogic paper [147] (although this paper is too naïve in its claim that the energy of the radiation associated with an accelerated charge must come from its kinetic energy, whereas it can also come from the electromagnetic field energy of that charge, as noted by Schott [96] and discussed in sec. 19 below.

[20]Another indication of the approximate character of Lorentz' self force (5) comes from the extension of Planck's argument (sec. 13) for a radiation-reaction force to higher-order multipole radiation. A single charge not at the origin has nonzero electric and magnetic moments of all orders. If the charge is accelerated, it emits electric- and magnetic-multipole radiation of all orders, of which electric-dipole radiation is typically the strongest. As indicated in [164], and in sec. 2.3 of [176], the radiation-reaction force associated with higher-order multipole radiation depends on higher-order derivatives of the charge's velocity. In extreme cases, the higher-order derivatives could be larger than $\ddot{\mathbf{v}}$, such that a higher-order radiation-reaction force is the largest, and corrections to eq. (5) would be important.

[21]This theme will be considered further in secs. 19, 26 and 30 below.

application of this concept [176].[22]

## 13 Planck and the Radiation Damping Force

In the mid 1890's, Planck became interested in the relation between electromagnetism and thermodynamic aspects of emission and absorption of radiation, as pioneered by Kirchhoff [8].[23] Planck was inspired by Hertz' analysis [31, 34] of his experiments (sec. 8 above) that generated and detected electromagnetic waves at a single frequency, in which the source of these waves was a "resonator" that Hertz interpreted as an oscillating electric dipole. Planck noted (1895) [45] that such a "resonator" could be modeled as a single electric charge $e$ tied to a force center by a "spring", and that in the steady state (thermal equilibrium) the work done by the driving (external) force $F_{\rm ext}$, presumably an electromagnetic field, must equal the energy radiated.

In a subsequent paper (1896) [47], Planck gave a more analytic discussion.[24] He argued that, for the model where the oscillator is an electric charge $e$ of mass $m$ tied to a spring of constant $k$ in, say, an external electromagnetic wave with electric field $\mathbf{E}$ of the resonant angular frequency $\omega = \sqrt{k/m}$, if the equation of motion were simply,

$$m\ddot{x} + kx = F_{\rm ext} = eE\cos(\omega t), \tag{8}$$

then the amplitude of the oscillation would grow indefinitely with time. To avoid this physical impossibility, the equation of motion must include a damping term, that is plausibly related to the radiation of energy by the "resonator", which causes a loss of energy similar to the case of ordinary friction. That is, the equation of motion must actually have the form,

$$m\ddot{x} + kx = F_{\rm ext} + F_{\rm damping}, \tag{9}$$

and that in the steady state the time-average work done by the external force equals the radiated energy, such that the time-average work done by the damping force is the negative of this.

The resulting motion of the charge, if steady, has the form $x = x_0\cos(\omega t + \phi)$, for which the corresponding electric-dipole moment about the origin is $p = ex$. According to Hertz (1889) [34], the "resonator" continuously radiates energy at the rate,[25]

$$P = \frac{dU_{\rm rad}}{dt} = \frac{2\ddot{p}^2}{3c^3} = \frac{2e^2\ddot{x}^2}{3c^3}\,. \tag{10}$$

[22] If an antenna emits net momentum at rate $d\mathbf{P}_{\rm rad}/dt$, then there is a back reaction force on the antenna equal and opposite to this [149], which contrasts with the effect of the self force (5) in generating the radiation resistance experienced by the conduction electrons.

[23] For a review of Planck's thinking in this era, see [188].

[24] Planck mentioned being inspired by Poincaré's discussion, [38] and secs. 47-48, pp. 92 ff, of [42], of the *amortissement* (translated by Planck as *Dampfung durch Strahlung* = radiation damping) of a Hertzian oscillator.

[25] While Hertz' result (9) agrees with the Larmor formula [46], the latter was deduced only in 1897, after Planck's analysis [47].

Integrating over the period $T = 2\pi/\omega$ of the steady oscillation, the work done $W$ by the damping force is,

$$\begin{aligned} W &= \int_0^T F_{\rm damping}\, \dot x\, dt = -\int_0^T \frac{dU_{\rm rad}}{dt}\, dt = -\frac{2e^2}{3c^3}\int_0^T \ddot x^2\, dt = -\frac{2e^2}{3c^3}\dot x\,\ddot x\Big|_0^T + \frac{2e^2}{3c^3}\int_0^T \dot x\,\dddot x\, dt \\ &= \frac{2e^2}{3c^3}\int_0^T \dot x\,\dddot x\, dt. \end{aligned} \tag{11}$$

Hence, it is consistent to identify the damping force as,[26]

$$F_{\rm damping} = \frac{2e^2}{3c^3}\dddot x = \frac{2e^2}{3c^3}\ddot v, \tag{12}$$

which is the same as Lorentz' self force (5) that was deduced by a very different argument, although Planck did not mention this in [47].

Planck's derivation can be regarded as transforming a "radiation reaction" via an integration by parts into "radiation damping" (*Dampfung durch Strahlung*).

Planck did remark that the equation of motion (9) in the case of zero external force and zero spring constant becomes $\ddot x = \tau \dddot x$, where $\tau = 2e^3/3mc^3$, which has the "runaway" solution $x = x_0\, e^{t/\tau}$,[27] but which he dismissed (sec. 4 of [47]) as having no physical meaning (*keine Bedeutung*).

Planck's efforts to understand thermal radiation soon led him to infer that the energy of his "resonators" was quantized [55], after which he seldom discussed radiation damping (and his broader interest in "classical electron theory" also diminished with time).[28]

## 14 Related Efforts around 1900

Following the "discovery" of the electron by Thomson (1897) [48] and others, interest increased in charges with high velocity.[29] The early effort by Thomson (1881) [25] on the fields of a charge with low velocity had been generalized to arbitrary velocity by Heaviside (1889) [32], who found that the equipotential surfaces are ellipsoids contracted by the factor $1/\sqrt{1 - v^2/c^2}$ in the direction of motion. FitzGerald (1889) [33] and Lorentz (1892) [40] then argued that this implies the length of material objects in motion to be so contracted (although at the time they considered the contraction only to order $v^2/c^2$). This was considered to be an electromechanical effect in what we now call the (inertial) lab frame. Only in 1905 did Einstein develop the theory of special relativity [77] in which the contraction is

[26] The logic of eq. (11) is crisp only for periodic motion.

[27] Planck's result came shortly before J.J. Thomson [48] determined the ratio $e/m$ for an electron, with the implication that $\tau \approx 10^{-23}$ s.

[28] Some mention by Planck of radiation damping appears in sec. III of the 1906 edition of [83], but this was omitted in the second (1914) edition that most readers are more familiar with.

[29] The emerging "electron theory" was the topic of the 1900 *Festschrift* for Lorentz [54]. Efforts around this time related to the question of electromagnetic mass (but not the radiation reaction) are reviewed in chap. 1 of [154].

regarded as an effect of the contraction of the spatial coordinates (along with an expansion of the time scale) of a moving (inertial) frame of reference relative to those in the lab frame.

Meanwhile, an important step was the deduction of the potentials and fields of an accelerated charge with arbitrary velocity (less than $c$) by Liénard (1898) [50], and by Wiechert (1900) [57]. Liénard also deduced the power radiated by an accelerated charge with high velocity, eq. (21) of [50], which was independently deduced by Heaviside (1902) [64].[30] These results had the implication that atoms could not be tiny "solar systems" with electrons orbiting a central positive charge, as such orbits would quickly decay via emission of radiation, which led Thomson (1903) to consider "plum pudding" models in which a sphere of positive charge at rest contained rings of moving negative charge that did not emit radiation [69, 75].[31]

The first convincing experimental evidence for the radiation pressure of light was given by Lebedev in 1901 [61], and confirmed by Nichols and Hull [62, 67].[32]

# 15 Abraham and the Radiation Reaction at High Velocity

In the early 1900's, Abraham followed Lorentz in considering classical models of a spherical electron, first [63] discussing the self force (for low velocity) in the manner of Lorentz, without mention of radiation.[33] Then, in 1904 [70] he began to speak of the *Strahlungsdruckes* (radiation pressure) exerted on an accelerated charge (of arbitrary velocity) as a reaction to its emission of radiation.[34] It seems (to this author) that his results were more clearly presented in his 1905 "textbook" [72], in which sec. 15 is titled *Die Rückwirkung der Strahlung auf ein bewegtes Elektron* (the back reaction of radiation on a moving electron). If we also translate *Rückwirkung der Strahlung* as "radiation reaction", this would be the earliest use of that term.

[30] Liénard discussed the self force, sec. III, p. 53 of [50], but only displayed the term that contributes to the "electromagnetic mass".

[31] A classical electron would gain infinite kinetic energy as it spiraled in to a point nucleus (during a finite, short time), which is a type of "runaway" solution. The inclusion of Lorentz' self force (5) in the equation of motion does not prevent this "runaway" behavior [173].

The gain in kinetic energy of the inspiraling electron is compensated by a reduction in the cross term in electric field energy of the electron and the nucleus. However, this early clue as to the importance of such cross terms seems not to have been noticed around 1900.

*The gain in kinetic energy associated with the "friction" due to radiation is analogous to the "satellite paradox", that a satellite in low-Earth orbit speeds up, and falls inwards, due to atmospheric drag [119].*

[32] The famous Crookes radiometer [22] does not demonstrate electromagnetic radiation pressure, as argued by Schuster [23]. See also [127].

[33] For a commentary (1908) on the work of Lorentz and Abraham, see [84]. Other early commentaries include [93, 100, 101], and more recently [130, 134].

A parallel effort by Sommerfeld led to an integral form for the self force, sec. 11 of [73]. See also [74]. This work had little impact, but survives in discussions of the "memory equation" in, for example, chap. 7 of [174].

[34] The title of a brief report from 1904 (in English) by Abraham [71] is *The Reaction of the Radiation on a Moving Electron.*

On pp. 71-73 of [72] (pp. 69-70 of the 1908 edition [86]), Abraham reviewed Planck's derivation of radiation damping [45], but without attribution.[35]

In eqs. (77d-e), p. 111, of [72], he gave the Poynting vector [28], $\mathbf{S} = (c/4\pi)\mathbf{E} \times \mathbf{B}$, at large distances from an accelerated charge $e$. He then obtained the total power $P_{\rm rad}$ that crosses a sphere of large radius $r$ in eq. (82), p. 118, with $\gamma = 1/\sqrt{1 - v^2/c^2}$,[36,37]

$$P_{\rm rad} = \frac{dU_{\rm rad}}{dt} = \frac{2e^2\gamma^4}{3c^3}\left[\dot{\mathbf{v}}^2 + \gamma^2 \frac{(\mathbf{v} \cdot \dot{\mathbf{v}})^2}{c^2}\right]. \tag{13}$$

For $\mathbf{v}$, $\dot{\mathbf{v}}$ and $\gamma$ at time $t$ at the charge, the power $P$ would be that observed at time $t_{\rm obs} = t + r/c$, upon integration of the Poynting vector over the whole sphere.[38] Abraham considered that the power $P$ comes from the energy $U_e$ of the charge, so his eq. (83) actually had the form,

$$\frac{dU_e}{dt} = -\frac{dU_{\rm rad}}{dt} = -P_{\rm rad} = -\frac{2e^2\gamma^4}{3c^3}\left[\dot{\mathbf{v}}^2 + \gamma^2 \frac{(\mathbf{v} \cdot \dot{\mathbf{v}})^2}{c^2}\right]. \tag{14}$$

On pp. 107-115 of [72], Abraham established for charges with low velocity that the power $P_{\rm rad}$ transmitted across the large sphere is accompanied by a transmission of momentum, $d\mathbf{P}_{\rm rad}/dt$, related by,

$$\frac{d\mathbf{P}_{\rm rad}}{dt} = P_{\rm rad}\frac{\mathbf{v}}{c^2} = \frac{dU_{\rm rad}}{dt}\frac{\mathbf{v}}{c^2}. \tag{15}$$

In this he used Thomson's [41, 185] and Poincaré's [56] identification that $\mathbf{p} = \mathbf{S}/c^2$ is the density of momentum in the electromagnetic field.[39,40] Then, for motion with arbitrary

[35] Abraham cited Planck's paper [47] in [49] (1898). Because [72] was a "textbook" and not a research paper, Abraham may have felt it unnecessary to reference a then-well-known paper. Whatever the reason for Abraham not citing Planck, an unfortunate consequence is that most present "textbooks" also omit acknowledgment of Planck's (and Poincaré's) introduction of the notion of a damping force in reaction to radiation. Abraham associated Planck's radiation-damping force (12) with Lorentz' self force (5), perhaps giving the impression, as implied in some later textbooks, that Lorentz derived this result by the method of Planck.

As will be seen below, Abraham's derivation of the radiation damping force for charges with arbitrary velocity followed Planck, rather than Lorentz, but it is sometimes implied (as in sec. 16.3 of [169]) that Abraham used Lorentz' considerations of the self force on an extended charge in reaching his results for arbitrary velocity. However, Abraham did not reproduce Lorentz' "direct derivation" (*direktere Ableitung*) of eq. (5) in [72], but referred the reader to sec. 20 of Lorentz' 1903 article [68].

[36] Abraham used the symbol $\mathcal{X}$ for the quantity $\sqrt{1 - v^2/c^2} = 1/\gamma$, and wrote $\eta$ as the angle between the velocity vector $\mathbf{v}$ and the acceleration vector $\dot{\mathbf{v}}$.

[37] The result (13) had been deduced by Liénard (1898) in eq. (21) of [50], and by Heaviside (1902) in eq. (10) of [64].

[38] According to the "Sommerfeld condition" [95, 166], eq. (13) is the power radiated by the charge at time $t$. However, this interpretation is misleading, in that the power $P$ need not have come from the charge itself, but could have come from the electromagnetic field energy associated with the charge.

[39] Poincaré's paper [56] is cited by Abraham on p. 31 of [72]. The relation $\mathbf{p} = \mathbf{S}/c^2$ had been deduced by Abraham on pp. 124-125 of [66], then appeared just after eq. (33b), p. 273 of [70], and was given in eq. (18), p. 27 of [72].

[40] A justification via special relativity of the relation (15) is given in sec. 2.2.6 of [148].

velocity, eqs. (13) and (15) lead to,

$$\frac{d\mathbf{P}_{\rm rad}}{dt} = \frac{2e^2\gamma^4\mathbf{v}}{3c^5}\left[\dot{\mathbf{v}}^2 + \gamma^2\frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\right]. \tag{16}$$

The quantity $d\mathbf{P}_{\rm rad}/dt$ has the dimensions of force, and Abraham argued that there should be a back reaction force on the charge given by,[41]

$$\mathbf{F}_{\rm rad} = -\frac{d\mathbf{P}_{\rm rad}}{dt} = -\frac{2e^2\gamma^4\mathbf{v}}{3c^5}\left[\dot{\mathbf{v}}^2 + \gamma^2\frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\right]. \tag{17}$$

However, the supposed reaction force (17) vanishes as velocity $v \to 0$, and does not equal Lorentz' self force (5).[42]

In sec. 15 of [72], Abraham found a way to reconcile Lorentz' result (5) with eqs. (14) and (17), following the spirit of Planck's derivation of the radiation damping force (sec. 13 above).[43]

Abraham supposed that the charge is accelerated only during the interval $t_1 < t < t_2$, and argued that the changes in energy and momentum of the charge in reaction to its radiation should be,

$$\begin{aligned} \Delta U &= \int_{t_1}^{t_2}\frac{dU_e}{dt}\,dt = -\int_{t_1}^{t_2} P_{\rm rad}\,dt = -\frac{2e^2}{3c^3}\int_{t_1}^{t_2}\left[\gamma^4\dot{\mathbf{v}}^2 + \gamma^6\frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\right]dt & (18)\\ \Delta\mathbf{P} &= \int_{t_1}^{t_2}\mathbf{F}_{\rm rad}\,dt = -\int_{t_1}^{t_2}\frac{d\mathbf{P}_{\rm rad}}{dt}\,dt = -\frac{2e^2}{3c^5}\int_{t_1}^{t_2}\left[\gamma^4\mathbf{v}\,\dot{\mathbf{v}}^2 + \gamma^6\frac{\mathbf{v}(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\right]dt. & (19) \end{aligned}$$

These quantities can be related to another form, which we call the radiation damping force, $\mathbf{F}_{\rm damping}$, by,

$$\Delta U = \int_{t_1}^{t_2}\mathbf{F}_{\rm damping}\cdot\mathbf{v}\,dt, \qquad \Delta\mathbf{P} = \int_{t_1}^{t_2}\mathbf{F}_{\rm damping}\,dt, \tag{20}$$

which are to be obtained from eqs. (18)-(19) via integration by parts. Abraham found (sec. 15 of [72]),[44]

$$\mathbf{F}_{\rm damping} = \frac{2e^2}{3c^3}\left[\gamma^2\ddot{\mathbf{v}} + \frac{\gamma^4\mathbf{v}(\mathbf{v}\cdot\ddot{\mathbf{v}})}{c^2} + \frac{3\gamma^4\dot{\mathbf{v}}(\mathbf{v}\cdot\dot{\mathbf{v}})}{c^2} + \frac{3\gamma^6\mathbf{v}(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^4}\right], \tag{21}$$

which equals Lorentz' form (5) when $\mathbf{v} = 0$.[45] This suggests, but does not demonstrate, that the form of eq. (21) is the desired generalization Lorentz' self force $\mathbf{F}_{\rm self}$ of eq. (5) to

[41]Equations (14) and (17) were the final results of Abraham's 1904 paper [70].

[42]Since the quantity $d\mathbf{P}_{\rm rad}/dt$ is related to the fields far from the charge, it is not necessarily the case that the momentum at large distances is directly related to the momentum of the accelerated charge. Indeed, in the example of a collapsing electric dipole [191], the charges gain, not lose, momentum from the field (while far from the collapsing dipole there is an outward flow of field energy. Hence, we should not expect eq. (17) to represent the radiation-reaction force on the accelerated charge.

[43]Abraham chose to transform eq. (17) into a form like eq. (5), but he could have instead transformed eq. (5) in to the form (17), as illustrated in Appendix A.

[44]The details of the integrations by parts were omitted from the 2nd (1908) edition [86] of [72].

[45]The result (21) was given by Abraham in eq. (12) of a brief report (in English) in 1904 [71], which he described as "the reaction of radiation on a point charge".

arbitrary velocity. Such a demonstration was provided by Abraham (1908, p. 387 of [86]) and by von Laue (1908) [89] via a Lorentz transformation of eq. (5), as reviewed in sec. 17 below, and by Schott (1912) [94] using Lorentz' argument based on retarded potentials, as reviewed in sec. 19.1 below.

Abraham noted (sec. 15 of [72]) that it is actually easier to go from eq. (20), using $\mathbf{F}_{\rm damping}$ from eq. (21), to eqs. (18)-(19) than *vice versa*. That is,

$$\int_{t_1}^{t_2} f\dot g\,dt = fg\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} g\dot f\,dt, \tag{22}$$

$$\int_{t_1}^{t_2} \gamma^2\ddot{\mathbf{v}}\,dt = \int_{t_1}^{t_2} \gamma^2\frac{d\dot{\mathbf{v}}}{dt}\,dt = \gamma^2\dot{\mathbf{v}}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{2\gamma^4\dot{\mathbf{v}}(\mathbf{v}\cdot\dot{\mathbf{v}})}{c^2}\,dt, \tag{23}$$

$$\begin{aligned}\int_{t_1}^{t_2} \frac{\gamma^4}{c^2}(\mathbf{v}\cdot\ddot{\mathbf{v}})\,\mathbf{v}\,dt &= \int_{t_1}^{t_2} \frac{\gamma^4}{c^2}\left(\mathbf{v}\cdot\frac{d\dot{\mathbf{v}}}{dt}\right)\mathbf{v}\,dt = \frac{\gamma^4}{c^2}(\mathbf{v}\cdot\dot{\mathbf{v}})\,\mathbf{v}\Big|_{t_1}^{t_2} \\ &\quad - \int_{t_1}^{t_2}\left[\frac{\gamma^4\mathbf{v}\,\dot{\mathbf{v}}^2}{c^2} + \frac{\gamma^4\dot{\mathbf{v}}(\mathbf{v}\cdot\dot{\mathbf{v}})}{c^2} + \frac{4\gamma^6\mathbf{v}(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^4}\right]dt, \end{aligned}\tag{24}$$

$$\int_{t_1}^{t_2}\left[\gamma^2\ddot{\mathbf{v}} + \frac{\gamma^4}{c^2}(\mathbf{v}\cdot\ddot{\mathbf{v}})\,\mathbf{v}\right]dt = -\int_{t_1}^{t_2}\left[\frac{\gamma^4\mathbf{v}\,\dot{\mathbf{v}}^2}{c^2} + \frac{3\gamma^4\dot{\mathbf{v}}(\mathbf{v}\cdot\dot{\mathbf{v}})}{c^2} + \frac{4\gamma^6\mathbf{v}(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^4}\right]dt, \tag{25}$$

Then, using eq. (25) in the time integral of eq. (21), we arrive at the rightmost form of eq. (19).

For completeness, we note that,

$$\begin{aligned}\mathbf{F}_{\rm damping}\cdot\mathbf{v} &= \frac{2e^2}{3c^3}\left[\gamma^2\mathbf{v}\cdot\ddot{\mathbf{v}}\left(1+\gamma^2\frac{v^2}{c^2}\right) + \frac{3\gamma^4(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\left(1+\gamma^2\frac{v^2}{c^2}\right)\right] \\ &= \frac{2e^2}{3c^3}\left[\gamma^4\mathbf{v}\cdot\ddot{\mathbf{v}} + \frac{3\gamma^6(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\right].\end{aligned}\tag{26}$$

Via a partial integration, we have that,

$$\begin{aligned}\int_{t_1}^{t_2}\gamma^2\mathbf{v}\cdot\ddot{\mathbf{v}}\,dt &= \int_{t_1}^{t_2}\gamma^2\mathbf{v}\cdot\frac{d\dot{\mathbf{v}}}{dt}\,dt \\ &= \gamma^2\mathbf{v}\cdot\dot{\mathbf{v}}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\left[\gamma^2\dot{\mathbf{v}}^2 + \frac{4\gamma^6(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\right]dt,\end{aligned}\tag{27}$$

such that the time integral of $\mathbf{F}_{\rm damping}\cdot\mathbf{v}$ equals the rightmost form of eq. (18).

Abraham achieved a significant success in arriving at eq. (21) for the radiation damping $\mathbf{F}_{\rm damping}$ of eq. (21), but it remained somewhat unsettling that this does not equal $\mathbf{F}_{\rm rad}$ of eq. (17), which seemed like the proper candidate for the "radiation-reaction force". An insight as to how this might be physically consistent was obtained later by Schott, as discussed in sec. 19.2 below.

In 1908, p. 387 of [86], Abraham used a Lorentz transformation of Planck's damping force, eq. (5) above (eq. (58), p. 70 of [86]), in an inertial frame where the charge is instantaneously at rest, to deduce the damping force in the lab frame where the charge has velocity $\mathbf{v}$, finding the same form as eq. (21) for $\mathbf{F}_{\rm damping}$.

While Plank's argument for the radiation-damping force requires that the acceleration vanish at $t \to \pm\infty$, this is not required in Lorentz' derivation of the self force (5). So, we infer that Abraham's expression (21) holds even for the (mathematical) case that the charge is accelerated in both limits $t \to \pm\infty$.

## 16 Hadamard and Ehrenfest

We digress slightly to note that in 1908, Hadamard [87, 106] considered radiation in two spatial dimensions, finding that in the Lorentz gauge the potential $V(\mathbf{x}, t)$ is not simply a function of the charge density at $\mathbf{x}'$ at the retarded time $t' = t - |\mathbf{x} - \mathbf{x}'|/c$, but at all earlier times as well.[46] As a consequence the form of the radiation reaction is somewhat different in two spatial dimensions than in three.

This led Ehrenfest (1917) [98, 102] to argue that this is why we live in a 3-dimensional space.

The theme of radiation, and the radiation reaction in other than three spatial dimensions is the subject of ongoing discussion, as recently reviewed in [198].

## 17 von Laue and the Lorentz Transformation

In 1908, von Laue [89] (still writing as M. Laue) independently used a Lorentz transformation of Lorentz' self force $\mathbf{F}_{\text{self}}$ (= Planck's damping force), eq. (5), to find Abraham's result eq. (21) for $\mathbf{F}_{\text{damping}}$.

In 1911, von Laue published a "textbook" on the theory of relativity [92] that included a section on uniformly accelerated (hyperbolic) motion (see also sec. 18 below), which ended with a footnote (p. 116): *For hyperbolic motion the self force (Zusatzkraft) $2e^2\ddot{v}/3c^3$ is zero. In this case there is no radiation in unbounded hyperbolic motion.*

## 18 Born and Uniformly Accelerated Motion

An example that has led to extensive discussion regarding the character of the radiation reaction is uniformly accelerated motion of an electric charge, due perhaps to a uniform external electric field. Minkowski (1909) [88] is sometimes credited for the first discussion of this example in a relativistic context, but perhaps Born (1909) [90] deserves the credit

[46] See also [184, 194, 196].

here.[47,48] The hyperbolic motion of a uniformly accelerated charge was also treated by Sommerfeld (1910) in sec. 8 of [91], who also referenced Minkowski for discussions of relativistic hyperbolae.[49]

Of these works, only Born [90] mentioned the self force/radiation reaction, writing on p. 5: *Bemerkenswert ist, daß ein Elektron bei einer Hyperbelbewegung, so groß auch ihre Beschleunigung sein mag, Reine eigentliche Strahlung veranlaßt, sondern sein Feld mit sich führt, was bis jetzt nur für gleichförmig bewegte Elektronen bekannt war. Die Strahlung und der Widerstand der Strahlung treten erst bei Abweichungen von der Hyperbelbewegung auf.* This can be translated as: *It is remarkable, that an electron causes no actual radiation, as great as its acceleration may be, but it drags its field along with it, which was up to now only known for uniformly moving electrons. The radiation and the resistance of the radiation only arise at deviations from hyperbolic motion.*

Born's is perhaps the earliest claim that a uniformly accelerated charge doesn't "radiate" (which claim was quickly endorsed by von Laue on p. 116 of [92]).

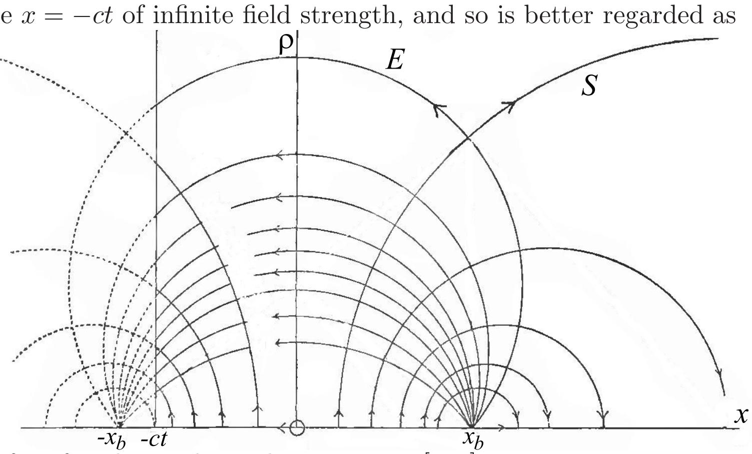
# 19 Schott

The radiation reaction was the subject of the 1908 Adams Prize (Cambridge), and the winning essay, by Schott, was enlarged and published as a book in 1912 [94].

## 19.1 Self Force

In Appendix D of [94], Schott followed Lorentz' method [39, 81] of computing the self force on an extended, accelerated charge, but considered the velocity $\mathbf{v}$ to be arbitrary, and arrived (sec. 225) at the same form for $\mathbf{F}_{\text{self}}$ as eq. (21) for Abraham's $\mathbf{F}_{\text{damping}}$.[50] Surprisingly, Schott did not realize in [94] that the damping/self force (21) is zero for uniformly accelerated

[47]Born's solution did not contain a plane $x = -ct$ of infinite field strength, and so is better regarded as holding for a pair of equal and opposite charges at $\pm x_b$, with equal and opposite, uniform acceleration, as pointed out in [104] (see sec. 21 below). This is illustrated by the figure on p. 68 of [94], which shows the electric field lines for a positive charge at $x_b$, which terminate on the plane $x = -ct$, but could be mathematically extended to terminate on a negative charged at $-x_b$. Lines of the Poynting vector $\mathbf{S}$ are also shown in the figure.




For additional discussion by the author of uniformly accelerated motion, see [187].

[48]The motion of a uniformly accelerated charge is a hyperbola on a space-time diagram, which diagrams were introduced by Minkowski [88], who did consider hyperbolae on such diagrams.

[49]The efforts of Minkowski and Sommerfeld related to uniformly accelerated motion are reviewed in [190].

[50]The author has the impression that many people suppose Abraham [72] to have followed Lorentz' method [39, 68, 81] (rather than that of Planck [47]), and omit giving Schott credit for being to the first to arrive at eq. (21) by Lorentz' approach.

motion,[51] as inferred from his comments on pp. 63 and 245-246. Only in sec. 8 of [96] (1915) did he remark that $F_{\rm self} = 0$ for uniformly accelerated motion.

## 19.2 Uniformly Accelerated Charge and the Schott/Acceleration Energy

Schott treated the example of a charge uniformly accelerated by a constant external electric field in secs. 43-49, pp. 63-69 of [94], obtaining noteworthy results.[52]

In secs. 145-146 and 235 of [94] he considered the power delivered $\mathbf{v} \cdot \mathbf{F}$ delivered to the charge, and identified a term proportional to the time derivative of what is now called the Schott energy,

$$U_{\rm Schott} = -\frac{2e^2\gamma^4\mathbf{v}\cdot\dot{\mathbf{v}}}{3c^2}\,. \tag{28}$$

His comments in sec. 235 suggests that at the time (1912), he did not regard eq. (28) as part of the "electromagnetic energy" of the charge (which is to be "renormalized" into the its effective mass), but as somehow part of its "radiation".

Schott continued his arguments in 1915 [96], where he noted that eq. (26) can be rewritten as,

$$\mathbf{F}_{\rm self}\cdot\mathbf{v} = -P_{\rm rad} - \frac{dU_{\rm Schott}}{dt} = \frac{dU_e}{dt} - \frac{dU_{\rm Schott}}{dt} \tag{29}$$

where $P_{\rm rad} = -dU_e/dt$ of eqs. (15)-(16) is the power "radiated" into the far zone by the accelerated charge, and the Schott energy is given by eq. (28).

In [96], Schott defined $Q = 2e^3\gamma^4\mathbf{v}\cdot\dot{\mathbf{v}}/3c^2$, and said: *hence $-Q$ must be regarded as work stored in the electron in virtue of its acceleration, so that we may speak of it as acceleration energy*". And, at the beginning of sec. 146 of [94], Schott had said; *If we adopt the usual convention that the kinetic energy depends on the velocity only, and not on the acceleration, we must exclude the second small term* (*i.e.*, the Schott energy) *from the kinetic energy, and regard it as reversible radiant energy.*

Thus, it appears that Schott did not support the idea that his "acceleration energy" could/should be "renormalized" into the energy-momentum of the charge. And, it also seems that he did not consider how, according to Liénard [50], the fields of an accelerated charge have terms proportional to the (retarded) acceleration $\dot{\mathbf{v}}$, and terms that do not involve $\dot{\mathbf{v}}$, such that the field energy and momentum, which are quadratic in the fields, contain "cross terms" proportional to the acceleration. This last theme was only explored much later, by Teitelboim [131, 132, 151], and by Eriksen and Grøn [172, 179] who identify these "cross terms" with the Schott energy/momentum, as reviewed in sec. 30 below.

In any case, Schott's discussion of the "acceleration energy" introduced the notion that the electromagnetic energy "radiated" by a charge can come from its electromagnetic field,

[51] To verify this, note that if $a^\star$ is the constant acceleration in the instantaneous (inertial) rest frame of the charge, then the lab-frame acceleration is $\dot{v} = a^\star/\gamma^2$, and $\ddot{v} = -3a^\star v/\gamma^4 c^2$.

[52] For a review by the author of these results, see [187].

as well as flowing off the charge itself.[53]

From Schott's description of the quantity $P_{\rm rad}$ (Schott's $R$ of eq. (7) [96]), one infers that Schott considered a uniformly accelerated charge to radiate (and was unaware of the claims of Born [90] and von Laue [92] to the contrary).

Schott's result, eq. (29), gives an additional perspective on the argument of Abraham (sec. 15 above), that,

$$\int_{t_1}^{t_2} \mathbf{F}_{\rm self} \cdot \mathbf{v}\, dt = \int_{t_1}^{t_2} \frac{dU_e}{dt}\, dt - \int_{t_1}^{t_2} \frac{dU_{\rm Schott}}{dt}\, dt = \int_{t_1}^{t_2} \frac{dU_e}{dt}\, dt - U_{\rm Schott}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \frac{dU_e}{dt}\, dt, \tag{30}$$

when the acceleration $\dot{\mathbf{v}}$ is nonzero only for $t_1 < t < t_2$.

### 19.2.1 Schott Momentum

*This subsection contains results that follow from Schott's arguments, but were not deduced by him.*

Schott did not remark that the self force (21) could be rewritten to include a total time derivative as,

$$\mathbf{F}_{\rm self} = -\frac{d\mathbf{P}_{\rm rad}}{dt} - \frac{d\mathbf{P}_{\rm Schott}}{dt} = -\frac{P_{\rm rad}\mathbf{v}}{c^2} - \frac{d\mathbf{P}_{\rm Schott}}{dt}\,, \tag{31}$$

where the Schott momentum is defined as,

$$\mathbf{P}_{\rm Schott} = -\frac{2e^2}{3c^3}\left(\gamma^2\dot{\mathbf{v}} + \frac{\gamma^4\mathbf{v}\cdot\dot{\mathbf{v}}}{c^2}\mathbf{v}\right). \tag{32}$$

The Schott energy and momentum form a 4-vector,[54]

$$U^\mu_{\rm Schott} = -\frac{2e^2}{3c^2}\, a^\mu = (U_{\rm Schott}, c\mathbf{P}_{\rm Schott}) = -\frac{2e^2}{3c^2}\left\{\gamma^4\mathbf{a}\cdot\frac{\mathbf{v}}{c},\ \gamma^2\mathbf{a} + \gamma^4\left(\mathbf{a}\cdot\frac{\mathbf{v}}{c}\right)\frac{\mathbf{v}}{c}\right\}, \tag{33}$$

where we use the notation that the position 4-vector is $x^\mu = (ct, \mathbf{x})$, the velocity 4-vector is $u^\mu = dx^\mu/d\tau = \gamma\, dx^\mu/dt = \gamma(c, \mathbf{v})$, the acceleration 4-vector is,

$$a^\mu = \frac{du^\mu}{d\tau} = \gamma\,\frac{du^\mu}{dt} = \gamma\dot\gamma(c, \mathbf{v}) + \gamma^2(0, \dot{\mathbf{v}}) = \left(\frac{\gamma^4\mathbf{v}\cdot\dot{\mathbf{v}}}{c}, \gamma^2\dot{\mathbf{v}} + \frac{\gamma^4(\mathbf{v}\cdot\dot{\mathbf{v}})\mathbf{v}}{c^2}\right), \tag{34}$$

noting that,

$$\dot\gamma = \frac{\gamma^3\mathbf{v}\cdot\dot{\mathbf{v}}}{c}\,, \tag{35}$$

and the metric is $(1, -1, -1, -1)$. For completeness, we record that,

$$u^\mu u_\mu = c^2, \qquad a^\mu a_\mu = \frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau} = -\gamma^4\dot{\mathbf{v}}^2 - \gamma^6\,\frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2} = -\mathbf{a}^{\star 2}, \tag{36}$$

[53] This phenomenon is nicely illustrated in the example of an electric dipole, initially at rest, that decays starting at, say, $t = 0$ [191]. For the case of exponential decay, see [133, 167], as well as sec. 2.5 of [177] and sec. 2.7 of [182].

[54] Schott seemed skeptical of the theory of relativity, and avoided use of 4-vectors, prefering separate discussion of their space and time components.

where $\mathbf{a}^\star$ is the acceleration in the instantaneous (inertial) rest frame of the charge. Then, recalling eqs. (13) and (15), we see that the rate $P_{\rm rad}$ of radiated energy and momentum is a Lorentz invariant,

$$P_{\rm rad} = \frac{dU_{\rm rad}}{dt} = \frac{2e^2}{3c^3}\left[\gamma^4\dot{\mathbf{v}}^2 + \frac{\gamma^6(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^2}\right] = -\frac{2e^2}{3c^3}a^\mu a_\mu = -\frac{2e^2}{3c^3}\frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau}\,, \tag{37}$$

and that the quantity,

$$\frac{dU^\mu_{\rm rad}}{d\tau} = \gamma\frac{dU^\mu_{\rm rad}}{dt} = \frac{P_{\rm rad}u^\mu}{c} = \gamma\left(P_{\rm rad}, c\frac{P_{\rm rad}\mathbf{v}}{c^2}\right) = \gamma\left(P_{\rm rad}, c\frac{d\mathbf{P}_{\rm rad}}{dt}\right), \tag{38}$$

is a 4-vector.

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The term "radiation reaction" was perhaps first used in English in [99] (1917, which is otherwise not particularly noteworthy). A review of various other terms associated with this topic is given in sec. 2.5 of [1].

# 21 Milner

In 1920, Milner [104] commented that the singular behavior on the plane $x = -ct$ for a uniformly accelerated charge along the $x$-axis is not present if the system consists of a pair of uniformly accelerated charges, $q$ and $-q$ at positions $x$ and $-x$ (without mention of Born [90]). Milner also noted that if the Poynting vector is integrated over a sphere surrounding a uniformly accelerated charge $q$, the power is exactly that given by the Larmor formula [46], $P = 2q^2a^{\star 2}/3c^3$, which *apparently verifies the presence of radiation in the solution*.

# 22 Pauli

In his treatise on the theory of relativity (1921) [103], Pauli (age 21) discussed the hyperbolic motion of a uniformly accelerated charge in sec. 32($\gamma$) (pp. 92-93 of the English edition) and the radiation reaction in sec. 32($\zeta$) (p. 99 of the English edition).

He noted in his eq. (250) of sec. 32($\gamma$) that the magnetic field is everywhere zero at the time when the charge is instantaneously at rest in the lab frame, which he considered as evidence that there is no radiation associated with this example.

In sec. 32($\zeta$) he presented what may be the first description of the radiation reaction in covariant notation, and gave a more compact version of Abraham's [86] and von Laue's [89] relativistic generalization of Lorentz' nonrelativistic form (5) [39]. He sought a 4-force $F^\mu_{\rm self} = \gamma(\mathbf{F}_{\rm self}\cdot\mathbf{v}/c, \mathbf{F}_{\rm self})$ (for which the relation $F^\mu_{\rm self}\,u_\mu = 0$ is an identity). Since,

$$\begin{aligned}\frac{d^2u^\mu}{d\tau^2} \;=\; & \left(\frac{\gamma^5\mathbf{v}\cdot\ddot{\mathbf{v}}}{c} + \frac{\gamma^5\dot{\mathbf{v}}^2}{c} + \frac{4\gamma^7(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^3}\,,\right. \\ & \left.\gamma^3\ddot{\mathbf{v}} + \frac{\gamma^5\mathbf{v}(\mathbf{v}\cdot\ddot{\mathbf{v}})}{c^2} + \frac{\gamma^5\mathbf{v}\dot{\mathbf{v}}^2}{c^2} + \frac{3\gamma^5\dot{\mathbf{v}}(\mathbf{v}\cdot\dot{\mathbf{v}})}{c^2} + \frac{4\gamma^7\mathbf{v}(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{c^4}\right),\end{aligned} \tag{39}$$

which reduces to $(0, \ddot{\mathbf{v}})$ for $\mathbf{v} = 0$, Pauli argued that the self 4-force could be written as,

$$F^\mu_{\rm self} = \frac{2e^2}{3c^3}\left(\frac{d^2u^\mu}{d\tau^2} + \alpha u^\mu\right). \tag{40}$$

The requirement that $F^\mu_{\rm self}\, u_\mu = 0$ implies that,

$$\alpha = -\frac{1}{c^2}u_\mu\frac{d^2u^\mu}{d\tau^2} = \frac{1}{c^2}\frac{du^\mu}{d\tau}\frac{du_\mu}{d\tau} = -\frac{P_{\rm rad}}{c^2}\,, \tag{41}$$

using eqs. (36), (37) and (39). Then, also using eq. (38),[55]

$$F^\mu_{\rm self} = \frac{2e^2}{3c^2}\frac{d^2u^\mu}{d\tau^2} - P_{\rm rad}\frac{u^\mu}{c^2} = -\frac{d}{d\tau}\left(-\frac{2e^2}{3c^2}\frac{du^\mu}{d\tau}\right) - P_{\rm rad}\frac{u^\mu}{c^2} = -\frac{dU^\mu_{\rm Schott}}{d\tau} - \frac{dU^\mu_{\rm rad}}{d\tau}\,. \tag{42}$$

The time component of this is eq. (29), and the spatial components are eq. (31).

Pauli's derivation starts from Lorentz' self force (5), originally deduced without consideration of radiation, and quickly arrives at the form (42) in which a relation to radiation can be seen. This reinforces the now-common description of the self force as the radiation-reaction force.

Pauli also noted that the self force (42) vanishes for uniformly accelerated motion, which he considered to confirm his conclusion in sec. 32($\gamma$) that there is no radiation in this case.

If Pauli had extended his very compact discussion to include the equation of motion for the electric charge, it would have read,

$$m\frac{du^\mu}{d\tau} = F^\mu_{\rm ext} + F^\mu_{\rm self} = eF^{\mu\nu}_{\rm ext}u_\nu - \frac{dU^\mu_{\rm Schott}}{d\tau} - \frac{dU^\mu_{\rm rad}}{d\tau} = eF^{\mu\nu}_{\rm ext}u_\nu + \frac{2e^2}{3c^2}\frac{d^2u^\mu}{d\tau^2} - \frac{P_{\rm rad}u^\mu}{c^2}\,, \tag{43}$$

where $F^{\mu\nu}_{\rm ext}$ is the electromagnetic field 4-tensor, supposing the external force to be electromagnetic.

# 23 Compton and a Reaction to Quantum Radiation

In 1922, Compton [105] provided evidence that the wavelength of energetic x-rays is decreased when they scatter off an electron initially at rest. This contrasts with classical scattering of (optical) light by electrons, perhaps first analyzed by J.J. Thomson in secs. 161-163 of [82], in which the frequency of the light in unchanged by scattering off a charge at rest.[56]

[55] To this author, Pauli's argument leading to eq. (42) is identical to that given in sec. 76 of [143] by Landau and Lifshifz, but the latter are sometimes credited with a new/independent derivation. Section 75 of [143] arrives at the nonrelativistic result (5) via an argument that is essentially the same as Planck's [47].

[56] Lord Rayleigh analyzed the scattering of light without change in frequency off particles in the sky starting in 1871, first in an æther theory of light [15, 16] and later in Maxwell's theory [26, 53], with a goal of explaining why the sky is blue. See also [153].

## 23.1 Thomson Scattering

We first quickly review Thomson scattering of low-intensity light.

The acceleration of a free electron of charge $e$ and mass $m$ in a plane electromagnetic wave with electric field $\mathbf{E} = E_0\, e^{i(kz-\omega t)}\, \hat{\mathbf{x}}$ is $\mathbf{a} = \dot{\mathbf{v}} = \mathbf{F}_E/m = e\mathbf{E}/m$, if the field strength is low enough that we can ignore the Lorentz force, $e\mathbf{v}/c \times \mathbf{B}$,[57] (and the radiation-reaction force). Then, the time derivative of the acceleration is $\ddot{\mathbf{v}} = -i\omega \mathbf{F}_E/m$, so the ratio of the radiation-reaction force (5) on an electron to that due to the incident wave has magnitude,

$$\frac{F_{\rm radreact}}{F_E} = \frac{2e^2\omega}{3mc^3} = \frac{2r_e}{3\lambda\!\!\!^-} \approx 10^{-9}, \tag{44}$$

where $r_e = e^2/mc^2 = 2.8 \times 10^{-15}$ m is the classical electron radius, and $\lambda\!\!\!^- = c/ = \lambda/\pi \approx 10^{-6}$m for optical light. Thus, the radiation reaction is negligible in "ordinary" Thomson scattering.[58]

## 23.2 Compton Scattering

In Compton scattering of a photon of angular frequency $\omega$ off an electron initially at rest, the scattered photon has the minimal angular frequency $\omega'_{\rm min}$, and the final-state electron has maximal kinetic energy $\mathrm{KE}'_{\rm max}$, for 180° backscattering,

$$\omega'_{\rm min} = \frac{\omega}{1 + 2\hbar\omega/mc^2}\,, \qquad \mathrm{KE}'_{\rm max} = \hbar\omega \frac{2\hbar\omega/mc^2}{1 + 2\hbar\omega/mc^2}\,. \tag{45}$$

The quantum scattering process can be thought of as due to acceleration of the electron by the incident photon, followed by radiation of the final-state photon. The gain of kinetic energy (and of momentum) by the electron during the scattering process can be thought of as a reaction to the radiation of the quantum of energy $\hbar\omega'$.

In this sense, there is a quantum radiation reaction that is very straightforward compared to the classical radiation reaction, which is associated with the self force on an accelerated charge.[59] This quantum radiation reaction is akin to the force $\mathbf{F}_{\rm rad}$ of eq. (17) above, as considered by Abraham [72].

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[57] Since $E = B$ for a plane wave, the Lorentz force is negligible if $v/c \ll 1$. If this is true, then the velocity has magnitude $a/\omega = eE_0/m\omega$, and $v/c = eE_0/m\omega c \equiv \eta \ll 1$, where $\eta$ is the dimensionless measure of the electric field strength of a wave. Only extremely high-power laser beams can lead to $v \approx c$ ($\eta \gtrsim 1$) when incident on electrons, so it is a very good approximation to ignore the Lorentz force in the Thomson scattering of "ordinary" light.

[58] For completeness, we note that the time-average power radiated by the electron in Thomson scattering is $\langle P_{\rm rad}\rangle = 2e^2 \langle a^2\rangle /3c^3 = e^4 E_0^2/3m^2c^3$, using the Larmor formula, so the scattering cross-section is $\sigma_{\rm Thomson} = \langle P_{\rm rad}\rangle / \langle S_E\rangle = P_{\rm rad}/(cE_0^2/8\pi) = 8\pi r_e^2/3$.

[59] There exists a literature on the "quantum radiation reaction", which seems (to this author) to consider the effect of quantum fluctuations in the emission of photons by charged particles on a quasiclassical trajectory. See, for example [183].

## 24 Dirac

In 1938, Dirac [111] published a paper that purported to arrive at the equation of motion of a classical electron with no assumptions as to its structure,[60] arriving at (p. 155) *the same as the equations of-motion obtained from the Lorentz theory of the extended electron by equating the total force on the electron to zero, if one neglects terms involving higher derivatives of* **v** *than the second. But whereas these equations, as derived from the Lorentz theory-, are only approximate, we now see that there is good reason for believing them to be exact, within the limits of the classical theory.*[61]

As we have seen, the derivation of these equations of motion by Abraham/von Laue/Pauli [72, 89, 103] followed the spirit of Planck [47], which also makes no assumptions as to the structure of the electron. Among Dirac's few references is the paper of Schott [96], that can be said to follow Lorentz [39] in using a model of an extended charge to deduce the self force thereon.

In any case, Dirac's derivation was somewhat different from its predecessors, such that one now often reads of eq. (43) above, Dirac's eq. (24), as the Lorentz-Dirac or Lorentz-Abraham-Dirac equation of motion.

Dirac did not use the terms "self force" or "radiation reaction" in his paper, although he did speak of "radiation damping" (Planck's term).[62]

## 25 Eliezer and Landau

It was mentioned at the end of sec. 12 above that Lorentz considered his self force (5) to be only an approximation, valid if the motion of the charge is not too abrupt. This theme was little considered by others until 1948 when independently Eliezer [115] and Landau (p. 235 of the second, Russian edition of [143]) considered Lorentz' *caveat* to mean that the self force must be small compared to the external force (due to electromagnetic fields).

In consequence, the self force (42) is slightly different from that found by Abraham [72] (and affirmed by von Laue [89], by Pauli [103], and in eq. (76.2) of [143]), and is better approximated by eq. (52) of [115] or eq. (76.3) of [143],

$$F^\mu_{\rm self} = \frac{2e^2}{3mc^3} u_\nu u^\lambda \frac{\partial F^{\mu\nu}_{\rm ext}}{\partial x^\mu} - \frac{2e^4}{3mc^5} F^{\mu\nu}_{\rm ext} F_{{\rm ext},\lambda\nu} u^\lambda + \frac{2e^4}{3mc^5} u^\mu F^{\nu\lambda}_{\rm ext} u_\lambda F_{{\rm ext},\lambda\kappa} u^\kappa. \qquad \text{(Landau)}. \quad (46)$$

This self force, which as anticipated by Lorentz cannot be too strong, avoids the mathematical "runaway" solutions that can be associated with the Abraham-Pauli-Dirac equation of motion.[63]

[60] Dirac did assume that the Poincaré stresses could be ignored, which issue is reviewed in Appendix B.

[61] Dirac expanded on this theme (in French) [112], which acknowledged an influence by Wentzel [109].

[62] In 1941, Dirac noted [114] that difficulties with the self energy of an electron remain in quantum electrodynamics, and he considered how negative-energy photon states might mitigate this issue.

[63] Neither Eliezer nor Landau (nor most subsequent authors) seem to have considered what is meant by the self force/radiation reaction not being too strong. So, we transcribe in Appendix C some discussion by the author on this theme, also given in [1].

The prescription (46) for modifying the self force (42) is not unique, and papers continue to be written advocating variants of eq. (46). See, for example, [138, 146, 165, 171, 178]; for a review, see [186].

*Among the vast number of papers on the theme of the "radiation reaction" since 1950, I restrict my further comments to those of five (sets of) authors.*[64]

# 26 Fulton and Rohrlich

## 26.1 Uniform Acceleration and the Equivalence Principle

In 1960, Fulton and Rohrlich published a paper [121] in which they affirmed Schott's view that a uniformly accelerated charge does "radiate," despite the fact that the self force/radiation reaction is zero. Then, they addressed an interesting issue raised by Bondi and Gold, sec. 6 of [118]), as to whether a charge at rest in a gravitational field radiates, as would be inferred from the principle of equivalence.

To clarify this conundrum, Fulton and Rohrlich noted that one should distinguish four configurations of charge and observer in the case of zero gravity, and four equivalent configurations in the case of, say, an idealized uniform gravitational field. For zero gravity, these cases are:

1. The charge is accelerating and the observer is at rest (in the inertial lab frame).
2. The charge is at rest and the observer is accelerating.
3. Both the charge and the observer are at rest.
4. Both the charge and the observer are accelerating, in the same manner.

Fulton and Rohrlich argued that the observer detects "radiation" in cases 1 and 2, but not in cases 3 and 4.

When considering gravity, one should recall that a free-falling observer is the equivalent to an observer in an inertial frame with zero gravity. Then, the four cases with gravity are:

- $1'$. The charge is at rest (with respect to the source of the static gravitational field) and the observer is free falling.
- $2'$. The charge is free falling and the observer is at rest.
- $3'$. Both the charge and the observer are free falling.
- $4'$. Both the charge and the observer are at rest.

[64] A very recent result [197] is that the self force of a (classical) point charge in rectilinear motion has been verified to be eq. (21), by computing the negative of the time rate of change of the momentum of its retarded self-field.

Case $n'$ is the gravitational equivalent of case $n$ in zero gravity.

The observer detects radiation in cases $1'$ and $2'$, but not in cases $3'$ or $4'$.

A lesson is that while there is an invariant rate of radiation, eq. (37), for observers in different inertial frames, that rate is not invariant for transformations to accelerated frames.[65] In the language of quantum physics, virtual photons in an inertial frame can be "real" to accelerated observers, as later discussed by Hawking [139] and Unruh [145].[66]

## 26.2 Where Does the Radiated Energy Come From?

In 1961, Rohrlich [123] followed Milner [104] and Schild [120] in arguing that the integral of the Poynting vector, $\int \mathbf{S}(t + r/c) \cdot d\mathbf{Area}$, over a large surface surrounding an accelerated charge, where $r$ is the distance from the present position of the charge to a surface element, is the same as that over any sphere centered on a retarded position of the center of the charge, and which completely encloses the charge at the present time $t$. Rohrlich concluded that if we call the integral of a very large surface the "radiation", in accordance with the Sommerfeld criterion [95], then we are also justified in saying that this "radiation" existed at the surface of any of the smaller spheres, considered above, as well.

## 26.3 Should the Schott Energy-Momentum Be "Renormalized" Away

In 1961, Rohrlich did not make (but could have) the inference that the result of sec. 26.2 implies the Schott energy resides inside these smaller spheres, and in particular, inside the smallest sphere centered on a retarded position of the center of the charge, and which completely encloses the charge at the present time $t$. Rather, this conclusion was only made in 2000, in paper III of [172], as reviewed in sec. 30 below.

If the charge is considered to be a point, then the Schott energy-momentum also resides at that point; only for a charge of nonzero spatial extent does the Schott energy-momentum occupy a (slightly) different volume from that of the charge itself.

In the case of an electron, the Schott energy-momentum will never be identified by experiment as distinct from the energy-momentum of the charge. Hence, it seems reasonable to follow Thomson [25] and Lorentz [39] and rewrite the equation of motion (43) as,

$$\frac{d}{d\tau}(m\,u^\mu + U^\mu_{\text{Schott}}) = \frac{d}{d\tau}\left(m\,u^\mu - \frac{2e^2}{3c^2}\frac{du^\mu}{d\tau}\right) = F^\mu_{\text{ext}} + F^\mu_{\text{rad}} = eF^{\mu\nu}_{\text{ext}}u_\nu - \frac{P_{\text{rad}}u^\mu}{c^2}\,, \tag{47}$$

which suggest that we "renormalize" the 4-vector $m\,u^\mu + U^\mu_{\text{Schott}}$ into the "mechanical" energy-momentum $p^\mu$ of an electron. If we did so, the (classical) differential equation of motion of the electron would be second order, and all debate over possible mathematically ill-behaved solutions to eq. (43) would be moot.

Advocates of a classical electron theory seem not to take this step, perhaps because $\mathbf{F}_{\text{rad}} \cdot \mathbf{v}$ does not equal the radiated power $P_{\text{rad}}$, as noted by Abraham [72]. In any case, while Rohrlich considered doing this, in eq. (6.7) of [122], he did not recommend it. For example,

[65]Among many subsequent commentaries on this theme, see, for example, [150, 175].

[66]For comments by the author on Hawking-Unruh radiation, see [168].

in his review of 2000 [170], he reverted to advocacy of the Abraham-Pauli equation of motion (43), calling this the LAD (Lorentz-Abraham-Dirac) equation.[67]

Another reason not to "renormalize" the Schott energy-momentum into the "mechanical" energy-momentum $p^\mu$ is that doing so would imply that $p^2 = p^\mu p_\mu$ is not an invariant, but depends on the (invariant) acceleration $a = \sqrt{a^\mu a_\mu}$ of the electron.

## 27 Coleman

In 1961, Coleman wrote an unpublished note [124] for the Rand Corporation titled *Classical Electron Theory from a Modern Standpoint*. He took the view (shared by this author) that attempts to discuss the structure of the electron within the context of classical electrodynamics are "pointless". However, he did salute Lorentz [39] for "renormalizing" the self-field energy of an electron into its mass, while remarking that it is better to think of the "bare" mass of an electron as $-\infty$ rather than zero as was argued by Lorentz in [81].

## 28 Does the Quantum Radiation Reaction "Cause" Spontaneous Emission?

Beginning in 1973, several authors [135]-[162] have argued that spontaneous emission of light by excited atoms is "caused" partly by quantum vacuum fluctuations and partly by the quantum "radiation reaction". This view is an extension of an argument by Weisskopf and others [110, 116, 152] that spontaneous emission is not spontaneous, but is "caused" by vacuum fluctuations. Such views follow Einstein's comment that "God does not play dice", which denies that random phenomena occur in the quantum realm.

The notion that the "radiation reaction" "causes" radiation seems at odds with the usual meaning of "radiation reaction", and such usage is not widespread. Indeed, the interpretation of spontaneous emission as partly due to the "radiation reaction" depends on the choice of ordering of certain commuting operators [157, 161].[68]

## 29 Gordeyev

In 1975, Gordeyev [142] deduced the damping/self force (26) from an expansion of the fields of an accelerated charge in terms of present, rather than retarded, quantities.

[67] In papers that followed Rohrlich's 1961 argument, Teitelboim [151, 131, 132] also could not bring himself to recommend "renormalizing" the Schott energy-momentum into the "mechanical" energy-momentum.

[68] An argument in favor of this interpretation is that a famous question, *Why isn't an atom in its ground state (in an inertial frame) excited by vacuum fluctuations*, has an answer that for this case the effect of the quantum fluctuations is canceled by the effect of the "radiation reaction" [155]. In this view, the quantum "radiation reaction" can either "cause" or "prevent" radiation, which reinforces that this quantum usage is somewhat removed from its classical origin.

## 30 Eriksen and Grøn

Eriksen and Grøn discussed uniformly accelerated motion in great detail in a series of papers [172] from 2000 to 2004. Here, we emphasize their discussion of the fields and Poynting vector of an charge in arbitrary motion, section III of paper III [172].

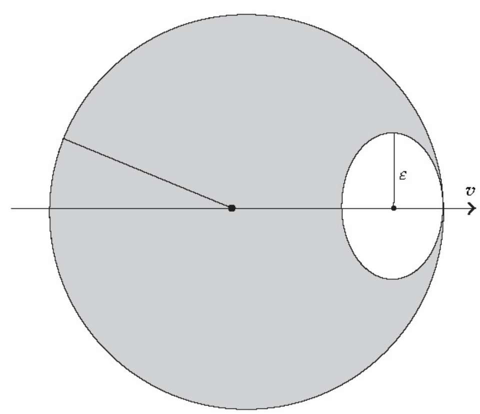
Writing the fields as $\mathbf{E} = \mathbf{E}_{\rm nonrad} + \mathbf{E}_{\rm rad}$, where the "radiation" fields fall off with (retarded) distance from the charge as $1/r$, the field energy and momentum can each be written with three integrals as,

$$U_{E,\rm field} = \int \frac{\mathbf{E}_{\rm nonrad}^2}{8\pi}\,d{\rm Vol} + \int \frac{\mathbf{E}_{\rm rad}^2}{8\pi}\,d{\rm Vol} + \int \frac{\mathbf{E}_{\rm nonrad}\cdot\mathbf{E}_{\rm rad}}{4\pi}\,d{\rm Vol}, \qquad (48)$$

$$\begin{aligned} \mathbf{P}_{\rm field} &= \int \frac{\mathbf{E}_{\rm nonrad}\times\mathbf{B}_{\rm nonrad}}{4\pi c}\,d{\rm Vol} + \int \frac{\mathbf{E}_{\rm rad}\times\mathbf{B}_{\rm rad}}{4\pi c}\,d{\rm Vol} \\ &\quad + \int \frac{\mathbf{E}_{\rm nonrad}\times\mathbf{B}_{\rm rad} + \mathbf{E}_{\rm rad}\times\mathbf{B}_{\rm nonradrad}}{4\pi c}\,d{\rm Vol}. \end{aligned} \qquad (49)$$

Erikson and Grøn followed the lead of Lorentz in absorbing ("renormalizing") the terms that involve only the "nonradiation" fields into the "mechanical" energy and momentum of the charge. The terms involving only the "radiation" fields form a 4-vector (as previously noted by Fulton and Rohrlich [121] among others). And, the cross terms involving both "radiation" and "nonradiation" fields turn out to be exactly the Schott energy-momentum of eq. (33).

This last result validates the insight of Schott [96] that there exists an "acceleration energy" in the electromagnetic fields (other than the radiated energy, which also depends on the acceleration). Furthermore, Eriksen and Grøn were able to show, sec. 6, paper IV of [172], that for a model of the charge as a small sphere in its rest frame, which appears as a Lorentz-contracted ellipsoid in the lab frame, the Schott energy resides inside the smallest sphere centered on the various retarded positions of the center of the charge such that the sphere completely encloses the charge at its present position.




That is, an observer outside this sphere could/should consider that the "radiation" which he detects flows from inside this sphere, but not directly off the charge itself.

Since the Schott energy-momentum is localized so close to the charge (inside its Compton wavelength), it would see reasonable to "renormalize" the Schott energy-momentum into the "mechanical" energy-momentum of the charge. Then, the Abraham-Pauli equation of motion

(43) would simplify to,

$$\frac{d}{d\tau}(m_{\rm eff}\, u^\mu) = F^\mu_{\rm ext} + F^\mu_{\rm rad} = eF^{\mu\nu}_{\rm ext} u_\nu - \frac{P_{\rm rad} u^\mu}{c^2}\,, \qquad (50)$$

and a century of debate over eq. (43) would become largely irrelevant.

These results were summarized in a pedagogic paper by Grøn [179], but he also could not bring himself to recommend "renormalizing" the Schott energy-momentum into the "mechanical" energy-momentum, as in eq. (50).

# 31 A Recent Experiment

We noted earlier that while the radiation reaction for oscillating currents has clear manifestation in the so-called radiation resistance of antennas, there is no experimental evidence for the classical radiation reaction of an individual electric charge. A recent proposal [189] is that such evidence might be obtained in the spectrum of radiation emitted by electrons "channeling" through a crystal. This spectrum is to be reconstructed from measurements of individual photons of GeV energy, so the proposed experiment is on the border between the classical and quantum regimes.

A recent report [192] claims evidence for deviations in the photon spectrum from predictions based on classical analysis without the radiation reaction, but the results are not in crisp agreement with a calculation that includes the classical radiation reaction, perhaps indicating that the experiment is more in the quantum realm. See also [193, 195].[69]

# 32 Summary

1. The notion of a reaction to the emission of light was discussed qualitatively by Stewart in 1871 [17], who did not relate light to electromagnetic waves.

2. The force on oscillating electric charge in reaction to its emission of electromagnetic radiation was first discussed qualitatively by Poincaré in 1891 [38], and quantitatively by Planck in 1896 [47]. Planck seemed unaware that his result was identical to the self force on a (slow-moving) extended electric charge deduced by Lorentz in 1892 [39] without mention of radiation.

3. Planck's argument was applied to antennas by Abraham in 1898 [49], which eventually led to the concept of the "radiation resistance" of an antenna, which for the case of an oscillatory current equals twice the time-average radiated power divided by the square of the peak current at the antenna terminals. This remains the most "practical" application of the concept of a "radiation reaction".

4. Planck's argument was also extended by Abraham in 1905 [72] for a single electric charge of arbitrary velocity that has nonzero acceleration over only a finite time interval. This argument assumed that the electron was a "point" charge in that it used the

[69]For reviews of the radiation reaction in quantum electrodynamics, see, for example, [199, 200].

Liénard-Wiechert fields [50, 57] to deduce that total power radiated (into the far zone) by an accelerated charge. Abraham's result, eq. (21), for the "radiation reaction" force contains a term that is the reaction to the momentum radiated by the charge, plus an additional term (often called the "Schott term") that has become the subject of much debate (confusion) in subsequent discussions.

5. Separately, Abraham [66, 70, 72] discussed models of extended electric charges with low velocity, but he made no attempt to deduce the self force on such a charge with arbitrary velocity in the manner of Lorentz [39]. Of course, his generalization of Planck's argument [47] on the radiation damping/reaction force can be also considered as a generalization of Lorentz' result for the self force (since the results of Lorentz and Planck were the same, although their arguments were quite different).

6. In 1908, Abraham [86] and Von Laue [89] used a Lorentz transformation of the form (5) of Lorentz [39] and Planck [47] for the self/radiation-reaction force of a charge with arbitrary velocity, arriving at eq. (21) as previously found by Abraham [72] via a "prerelativistic" argument.

7. In 1912, Schott [94] extended Lorentz' deduction of the self force on an extended charge to the case of arbitrary velocity, arriving at the forms previously displayed by Abraham [72] and von Laue [89] for the radiation-reaction force.[70] In 1915, Schott [96] identified the term, other than the back reaction to the radiated momentum, in Abraham's result (21) as the time derivative on an energy, called the "acceleration energy" by Schott (and the "Schott energy" in subsequent discussions).

8. In 1921, Pauli [103] gave a covariant version of Abraham/von Laue's argument (which can be regarded as an extension of both Lorentz' argument [39] and that of Planck [47] for eq. (5), and identified an energy-momentum 4-vector now called the Schott 4-vector. Pauli's argument was later transcribed by Landau and Lifshitz [143], who are often given credit for it.

9. In 1938, Dirac [111] gave an argument, nominally independent of any assumption as to the structure of a charge, for the "relativistic radiation-reaction" force first deduced by Abraham [72] (whose argument, following that of Planck [47] can be regarded as applying to a "point" charge).

10. Lorentz' remark in sec. 37 of [81] that (in effect) the self force must not be stronger than the external force leads, together with an awareness of Planck's quantum constant $\hbar$, to an understanding of limits to the domain of applicability of classical models of charged particles.

11. In 1960, Fulton and Rohrlich [121] resolved a paradox as to the relation between the radiation by (and the "radiation reaction" on) an accelerated charge and one at rest in a gravitational field.

[70] Abraham, rather than Schott, is sometimes credited with having so extended Lorentz' argument.

12. In 1961, Coleman [124] wrote a report on "Classical Electron Theory from a Modern Standpoint", that remains more "modern" than most subsequent "classical" discussions of this topic.

13. In 2000-2002, Eriksen and Grøn [172] showed that the Schott energy-momentum 4-vector equals the cross terms between the "radiation" and "nonradiation" fields in the energy and momentum of an accelerated charge, and demonstrated that this 4-vector is localized in a small sphere that surrounds the charge.

    *If this energy-momentum were "renormalized" into the "mechanical" energy momentum of the charge, as in eq. (50), then the topic of the "radiation reaction" would cease to be a "perpetual problem". That is, after "renormalizing" away the Schott energy-momentum, (almost) all that is left of the story is Planck's view (inspired by Poincaré, and before he performed his famous integration by parts) that there is a force on an accelerated charge in reaction to its radiated momentum.*[71]

## 32.1 Summary of the Summary

The most practical aspect of the radiation reaction is the radiation damping of sources (antennas) of electromagnetic radiation, as first anticipated by Poincaré in 1891 [38, 42], shortly after Hertz' experimental generation of electromagnetic waves [31]. In the circuit analysis of an antenna, one should include the radiation resistance of the antenna, in addition to its Ohmic resistance, to account for the radiation of energy by the circuit.[72] For this, one simply relates the time-average radiated power $\langle P \rangle$ to the peak current $I_0$ at the antenna terminals according to $\langle P \rangle = I_0^2 R_{\rm rad}/2$, to deduce the radiation resistance $R_{\rm rad}$.

Concurrently with the thoughts of Poincaré and others about antennas, a theory of charged particles/electrons was being developed, that is now called "classical electron theory". In this theory, charged particles had small but finite size, although it was often convenient to consider them to be point particles, as in the famous deductions by Liénard [50] and Wiechert [57] of the electromagnetic fields of an accelerated point charge. Independent arguments by Lorentz [39] and Planck [47] led to the notion of a self force/radiation reaction on an accelerated charge proportional to the time derivative of its acceleration. Such a force does not fit well into Newton's vision that $\mathbf{F} = m\mathbf{a}$, nominally a second-order differential equation, and leads to difficulties that have exercised the proponents of classical electron theory ever since.

[71] There remains the entertaining details of the case of a charge in a uniform external electric field, for which the equation of motion can be solved exactly if one supposes that Abraham's self force appears in the equation of motion, since this happens to be zero. Whereas, in the view that the Schott terms are "renormalized" away, there would be a nonzero "radiation reaction" force which spoils the analytic integrability of the equations of motion (with little practical impact on the results, as noted by Schott in his 1912 book [94] where he argued that the radiation reaction is negligible, if not identically zero).

[72] Heaviside, whose career began with analyses of telegraphy, considered such radiation to be a "waste" [64].

# Appendix A: From Lorentz' Self Force to the Radiation Reaction

Historically, Lorentz [39] deduced eq. (1),

$$\mathbf{F}_{\text{self}} = \frac{2e^3\ddot{\mathbf{v}}}{3c^3} \tag{1}$$

by consideration of the self force on an extended, accelerated charge distribution with low velocity. Then, apparently independently, Planck [47] considered a radiation damping/reaction force, and also arrived at eq. (5) after a clever integration by parts of the power supplied by the damping force. Here, we indicate how one might go from Lorentz' self force to a radiation-reaction force by inverting the logic of Planck's (and Abraham's [72]) argument.

We suppose that the acceleration $\dot{\mathbf{v}}$ is nonzero only for $t_1 < t < t_2$, and consider the time integral of the power supplied to the charge by Lorentz' self force, and integrate this by parts,

$$W = \int_{t_1}^{t_2} \mathbf{F}_{\text{self}} \cdot \mathbf{v}\, dt = \frac{2e^3}{3c^3} \int_{t_1}^{t_2} \ddot{\mathbf{v}} \cdot \mathbf{v}\, dt = \frac{2e^3}{3c^3} \dot{\mathbf{v}} \cdot \mathbf{v} \Big|_{t_1}^{t_2} - \frac{2e^3}{3c^3} \int_{t_1}^{t_2} \dot{\mathbf{v}}^2\, dt = - \int_{t_1}^{t_2} \frac{dU_{\text{rad}}}{dt}\, dt, \tag{51}$$

where,

$$\frac{dU_{\text{rad}}}{dt} = \frac{2e^3\dot{\mathbf{v}}^2}{3c^3} \tag{52}$$

is the power radiated by a (low velocity) accelerated charge, according to Larmor [46].

This establishes that the self force (1) as being a kind of reaction to radiation, but does not yet identify a "radiation reaction" force. For this, we might follow Einstein [78] and identify $U_{\text{rad}}/c^2$ as a kind of effective mass, with the implication that an accelerated charge with velocity $\mathbf{v}$ radiates momentum at the rate,

$$\frac{d\mathbf{P}_{\text{rad}}}{dt} = \frac{dU_{\text{rad}}}{dt} \frac{\mathbf{v}}{c^2}\,. \tag{53}$$

As mentioned in sec. 15 above, this result was first deduced by Abraham [72] via a prerelativistic argument, and extended by him to charges with arbitrary velocity with the result given in eq. (16). Then, we arrive at the radiation-reaction force as having the form,

$$\mathbf{F}_{\text{rad}} = -\frac{d\mathbf{P}_{\text{rad}}}{dt} = -\frac{dU_{\text{rad}}}{dt} \frac{\mathbf{v}}{c^2}\,. \tag{54}$$

This form actually holds for arbitrary velocity if we use the general expression (13) for $dU_{\text{rad}}/dt$, rather than the low-velocity Larmor formula (52).

While the above argument is suggestive, it suffers from the defect that the power supplied to the charge by $\mathbf{F}_{\text{rad}}$ is $\mathbf{F}_{\text{rad}} \cdot \mathbf{v} = -(\mathbf{v}^2/c^2)\, dU_{\text{rad}}/dt$ and not $-dU_{\text{rad}}/dt$ as desired. This defect likely motivated Planck and Abraham to pursue the paths that they did, in which $\mathbf{F}_{\text{rad}}$ alone was not identified with the force in reaction to the radiation.

# Appendix B: Can the Poincaré Stresses Be Ignored in the Equation of Motion of an Electron?

*Written Jan. 9, 2020, this Appendix was inspired by sec. 1 of [197].*

Much of the literature of the radiation reaction is in the context of the vision that electromagnetism is the only relevant interaction, with gravity being neglected, despite the evidence of additional basic interactions that emerged in the late 1890's, characterized by Rutherford [52] as associated with $\alpha$ and $\beta$ rays. An exception was Poincaré's comment that a charged particle cannot be stable unless it involves some nonelectromagnetic interaction [79, 80]. Perhaps because the Poincaré stresses were interpreted by many as a "mechanical" effect, which would not exert a self force, the possibility of nonelectromagnetic effects in considerations of the radiation reaction have been largely ignored.

Following a discussion in [197] of the self force based on field momentum (which built on considerations of Abraham [72]) we present an argument that there must be some nonelectromagnetic contribution to the self force on an accelerated, charged particle.

We begin by supposing that the momentum of a particle of electric charge $e$ can be expressed as the sum of "mechanical" and "field" momenta, and consider the interaction of the particle with external electromagnetic fields $\mathbf{E}_{\rm ext}$ and $\mathbf{B}_{\rm ext}$. In the spirit of Poincaré, we do not preclude that the charge is associated with a nonelectromagnetic field that generates the stresses required for stability of the charge.[73] Then, conservation of momentum of this system can be expressed as,

$$\frac{d\mathbf{P}_{\rm mech,ext}}{dt} + \frac{d\mathbf{P}_{{\rm mech},e}}{dt} + \frac{d\mathbf{P}_{\rm EM}}{dt} + \frac{d\mathbf{P}_{\rm other}}{dt} = 0, \tag{55}$$

where $\mathbf{P}_{\rm other}$ is associated only with the charge $e$. The electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ consist of the external fields, and the (self) fields of the charge,

$$\mathbf{E} = \mathbf{E}_{\rm ext} + \mathbf{E}_e, \qquad \mathbf{B} = \mathbf{B}_{\rm ext} + \mathbf{B}_e, \tag{56}$$

such that the electromagnetic-field momentum can be written as,

$$\mathbf{P}_{\rm EM} = \frac{c}{4\pi}\int \mathbf{E}\times\mathbf{B}\, d{\rm Vol} = \mathbf{P}_{\rm EM,ext} + \mathbf{P}_{{\rm EM},e} + \mathbf{P}_{\rm EM,int}, \tag{57}$$

where,

$$\mathbf{P}_{\rm EM,ext} = \frac{c}{4\pi}\int \mathbf{E}_{\rm ext}\times\mathbf{B}_{\rm ext}\, d{\rm Vol}, \qquad \mathbf{P}_{{\rm EM},e} = \frac{c}{4\pi}\int \mathbf{E}_e\times\mathbf{B}_e\, d{\rm Vol}, \tag{58}$$

and the interaction electromagnetic-field momentum is,

$$\mathbf{P}_{\rm EM,int} = \frac{c}{4\pi}\int \mathbf{E}_{\rm ext}\times\mathbf{B}_e\, d{\rm Vol} + \frac{c}{4\pi}\int \mathbf{E}_e\times\mathbf{B}_{\rm ext}\, d{\rm Vol}. \tag{59}$$

[73] We ignore the possibility that the sources of the external field are associated with a nonelectromagnetic field as well.

Then, eq. (55) can be written as,

$$\frac{d\mathbf{P}_{\text{mech,ext}}}{dt}+\frac{d\mathbf{P}_{\text{mech},e}}{dt}=-\frac{d\mathbf{P}_{\text{EM,ext}}}{dt}-\frac{d\mathbf{P}_{\text{EM,int}}}{dt}-\frac{d\mathbf{P}_{\text{EM},e}}{dt}-\frac{d\mathbf{P}_{\text{other}}}{dt}\,. \tag{60}$$

If the sources of the external electromagnetic field are very far from the accelerated charge, the external field will not be affected by the charge (during some finite time interval of interest), then $-d\mathbf{P}_{\text{EM,ext}}/dt$ is equal to $d\mathbf{P}_{\text{mech,ext}}/dt$, and eq. (60) simplifies to,

$$\frac{d\mathbf{P}_{\text{mech},e}}{dt}=-\frac{d\mathbf{P}_{\text{EM,int}}}{dt}-\frac{d\mathbf{P}_{\text{EM},e}}{dt}-\frac{d\mathbf{P}_{\text{other}}}{dt}\,. \tag{61}$$

The impressive result of [197] is that, independent of whether the charge is finite or pointlike,

$$-\frac{d\mathbf{P}_{\text{EM},e}}{dt}=\mathbf{F}_{\text{damping}}, \tag{62}$$

where $\mathbf{F}_{\text{damping}}$ is the electromagnetic radiation-reaction force of our eq. (21), as first found by Abraham [72], and later found by Dirac [111]. This leads us to infer that,

$$-\frac{d\mathbf{P}_{\text{EM},e}}{dt}=\mathbf{F}_{\text{S,EM}}. \tag{63}$$

For the case of a uniform, static, external electric field (with zero external magnetic field), both the damping force (21) and the interaction electromagnetic-field momentum (59) vanish, the acceleration is constant in the instantaneous rest frame of the charge,[74] and eq. (61) reduces to,

$$\frac{d\mathbf{P}_{\text{mech},e}}{dt}=-\frac{d\mathbf{P}_{\text{other}}}{dt}\qquad\text{(constant acceleration)}. \tag{64}$$

This suggests that in general the Poincaré stresses should not be ignored in discussions of the classical equation of motion of a charged particle.

We can also relate the rate of change of mechanical momentum of the charge $e$, with electric charge density $\rho_e$, to the sum of the forces on it, which consist of the Lorentz forces,

$$\mathbf{F}_{\text{ext}}=\int\rho_e\left(\mathbf{E}_{\text{ext}}+\frac{\mathbf{v}}{c}\times\mathbf{B}_{\text{ext}}\right)d\text{Vol},\qquad \mathbf{F}_{\text{S,EM}}=\int\rho_e\left(\mathbf{E}_e+\frac{\mathbf{v}}{c}\times\mathbf{B}_e\right)d\text{Vol}, \tag{65}$$

due to the external and self electromagnetic fields, as well as the possible self force $\mathbf{F}_{\text{S,other}}$ associated with the nonelectromagnetic field. That is,

$$\frac{d\mathbf{P}_{\text{mech},e}}{dt}=\mathbf{F}_{\text{ext}}+\mathbf{F}_{\text{S,EM}}+\mathbf{F}_{\text{S,other}}. \tag{66}$$

If we accept the relation (63), then eqs. (61) and (65) together imply that

$$\mathbf{F}_{\text{ext}}+\mathbf{F}_{\text{S,other}}=-\frac{d\mathbf{P}_{\text{EM,int}}}{dt}-\frac{d\mathbf{P}_{\text{other}}}{dt}\,. \tag{67}$$

For the case of uniform acceleration, where the electromagnetic self force and the electromagnetic interaction momentum both vanish, we arrive at the peculiar relation,

$$\frac{d\mathbf{P}_{\text{mech},e}}{dt}=\mathbf{F}_{\text{ext}}+\mathbf{F}_{\text{S,other}}=-\frac{d\mathbf{P}_{\text{other}}}{dt}\qquad\text{(constant acceleration)}. \tag{68}$$

[74] See, for example, sec. 2.1 of [187]. The interaction field momentum vanishes due to the high symmetry of this case.

# Appendix C: Limits on the Applicability of Classical Electromagnetic Fields as Inferred from the Radiation Reaction

Neither Eliezer nor Landau, sec. 25 above, (nor most subsequent authors) seem to have considered what is meant by the self force/radiation reaction not being too strong. So, we transcribe here some discussion by the author on this theme, given in [1].

It suffices to consider only the low-velocity limit (as used by Lorentz [39] and Planck [47]), in which the equation of motion is simply,

$$m\dot{\mathbf{v}} \approx \gamma \dot{m}\mathbf{v} = \mathbf{F}_{\rm ext} + \mathbf{F}_{\rm self} \approx e\mathbf{E} + e\frac{\mathbf{v}}{c} \times \mathbf{B} + \frac{2e^2}{3c^3}\ddot{\mathbf{v}}\,, \tag{69}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the external electromagnetic fields. From this we learn that in the first approximation, the acceleration of the charge is just $\dot{\mathbf{v}} \approx e\mathbf{E}/m$. Further, if the second time derivative of the velocity is small we estimate it by taking the derivative of (69),

$$\ddot{\mathbf{v}} \approx \frac{e\dot{\mathbf{E}}}{m} + \frac{e}{m}\frac{\dot{\mathbf{v}}}{c} \times \mathbf{B} + \frac{e}{m}\frac{\mathbf{v}}{c} \times \dot{\mathbf{B}} \approx \frac{e\dot{\mathbf{E}}}{m} + \frac{e^2}{mc^2}\mathbf{E} \times \mathbf{B}, \tag{70}$$

where we have ignore the term proportional to $\mathbf{v}/c$. Thus, in these approximations, the self force is given by,

$$\mathbf{F}_{\rm self} \approx \frac{2e^2}{3c^3}\left(\frac{e\dot{\mathbf{E}}}{m} + \frac{e^2}{m^2c}\mathbf{E} \times \mathbf{B}\right). \tag{71}$$

The first term in (71) contributes only for time-varying fields, which we now take to have frequency $\omega$ and reduced wavelength $\lambda\!\!\!^{-}$; hence, $\dot{\mathbf{E}} \propto \omega\mathbf{E}$. The second term contributes only when $\mathbf{E} \times \mathbf{B} \neq 0$, which is most likely to be in a wave (with $E = B$) if the fields are large. So, for a charge in an external wave field, the magnitude of the self force is,

$$F_{\rm resist} \approx \frac{2}{3}eE\sqrt{\left(\frac{e^2}{mc^2}\frac{\omega}{c}\right)^2 + \left(\frac{e^3E}{m^2c^4}\right)^2} \approx F_{\rm ext}\sqrt{\left(\frac{r_0}{\lambda\!\!\!^{-}}\right)^2 + \left(\frac{E}{e/r_0^2}\right)^2}, \tag{72}$$

where $r_0 = e^2/mc^2 = 2.8 \times 10^{-13}$ cm is the classical electron radius.

Following the spirit of Lorentz comment in 1906 [81], the premise of this discussion is that the notion of the self force makes physical sense only when it is small compared to the external force. Here we don't explore whether the length $r_0$ describes a physical electron; we simply consider it to be a length that arises from the charge and mass of an electron.[75] Rather, we concentrate on the implication of eq. (72) for the electromagnetic field. Then, we infer that a classical description becomes implausible for fields whose wavelength is small compared to length $r_0$, or whose strength is large compared to $e/r_0^2$.

Already in 1900, Planck [55] had introduced to quantum of action, $\hbar$.

[75] This theme is pursued in sec. 10.2 of [151].

In this context, it is suggestive to multiply and divide eq. (72) by Planck's constant $\hbar$,

$$F_{\rm self} \approx F_{\rm ext} \sqrt{\left( \frac{e^2}{\hbar c} \frac{\hbar}{mc} \frac{\omega}{c} \right)^2 + \left( \frac{e^2}{\hbar c} \frac{e\hbar}{m^2 c^3} E \right)^2} = \alpha F_{\rm ext} \sqrt{\left( \frac{\lambda\!\!\!^{-}_C}{\lambda\!\!\!^{-}} \right)^2 + \left( \frac{E}{E_{\rm crit}} \right)^2}, \qquad (73)$$

where $\alpha = e^2/\hbar c$ is the QED fine structure constant, $\lambda\!\!\!^{-}_C = \hbar/mc$ is the (reduced) Compton wavelength of an electron and,

$$E_{\rm crit} = \frac{m^2 c^3}{e\hbar} = 1.6 \times 10^{16}\ {\rm V/cm} = 3.3 \times 10^{13}\ {\rm gauss}, \qquad (74)$$

is the QED critical field strength. The quantity $E_{\rm crit}$ was introduced by Sauter in 1931 [108] when he resolved Klein's paradox [107] by arguing that for $E > E_{\rm crit}$ the electric field spontaneously breaks down into $e^+e^-$ pairs, so it should not be surprising that the reflection coefficient of a charge energy such a strong-field region could exceed unity.[76]

Thus, our naïve quantum theory (classical electromagnetism plus $\hbar$) leads us to expect important departures from classical electromagnetism for waves of wavelength much shorter than the Compton wavelength of the electron, and for fields of strength larger than the QED critical field strength.[77]

*While this argument (without the interpretation of the QED critical field strength) could have been given in 1906, it has never appeared in the literature.*

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