Expanding Spherical Shell of Charge

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1 Problem

Discuss the electromagnetic fields and the radiation of an expanding (or contracting) spherical shell of electric charge.

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2 Solution

The electrical charges move purely radially as the shell expands, assuming perfect spherical symmetry. Then, the electric and magnetic fields must also have spherical symmetry, which is only possible for radial vector fields. This is fine for the electric field,

$$\mathbf{E}(r > a) = \frac{Q}{r^2}\,\hat{\mathbf{r}} \qquad \mathbf{E}(r < a) = 0, \qquad i.e., \qquad \mathbf{E} = \frac{Q}{r^2}\,\theta(r, a(t))\,\hat{\mathbf{r}} \tag{1}$$

in Gaussian units and in spherical coordinates where $a(t)$ is the radius of the shell, Q is the total surface charge, and $\theta(x, x_0) = \int_{-\infty}^{x} \delta(x' - x_0)\,dx'$ is the Heaviside step function. However, magnetic charges do not exist, so we cannot have a radial magnetic field of the form (1). That is, the magnetic field is zero, $\mathbf{B} = 0$ everywhere, no matter what is the time dependence $a(t)$ of the radius of the shell.[1]

Since there is no magnetic field, there is no Poynting vector, $\mathbf{S} = (c/4\pi)\,\mathbf{E} \times \mathbf{B} = 0$, and there is no radiation. This result was first crisply noted by Ehrenfest [3],[2] and is an example where accelerated charges are not accompanied by electromagnetic radiation.[3]

[1]Maxwell's equations for the fields $\mathbf{E}$ and $\mathbf{B}$ can be combined into wave equations of the form,

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 4\pi\boldsymbol{\nabla}\rho + \frac{4\pi}{c^2}\frac{\partial\mathbf{J}}{\partial t}, \qquad \nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{\nabla} \times \mathbf{J}, \tag{2}$$

in terms of source charge and current densities ρ and $\mathbf{J}$, which for the present example are,

$$\rho = \frac{Q}{4\pi r^2}\,\delta(r - a(t)), \qquad \mathbf{J} = \frac{Q\dot{a}}{4\pi r^2}\,\delta(r - a(t))\,\hat{\mathbf{r}} \equiv J_r\,\hat{\mathbf{r}}. \tag{3}$$

Since $\boldsymbol{\nabla} \times \mathbf{J} = 0$ in this case, there is no source term for the $\mathbf{B}$ field, and so $\mathbf{B} = 0$. Also, the form (1) for $\mathbf{E} = E_r(r, t)\,\hat{\mathbf{r}}$ obeys $\partial^2 E_r/\partial t^2 = -4\pi\,\partial J_r/\partial t$ and $(\nabla^2\mathbf{E})_r = (1/r)\,\partial^2(rE_r)/\partial r^2 - 2E_r/r^2 = 4\pi\,\partial\rho/\partial r$.

Note that if a spherical region around the origin is source free, the only nonsingular, spherically symmetric solution to the vector wave equations (2) is that the fields are zero in this region. And, in a source-free, spherical region not containing the origin, the only spherically symmetric solution is the time-independent form $\hat{\mathbf{r}}/r^2$. See, for example, sec. 7.11 of [1] or sec. 9.7 of [2].

[2]Earlier discussions of (non)radiation by radially oscillating charges include [4]-[10].

[3]Another simple example of acceleration without radiation is a loop of steady (DC) electrical current in

The electric field (1) remains static as the shell expands, except that the field is nonzero only outside the shell. As such, the electrostatic field energy drops as the radius r increases,

$$U_E(r) = \int_r^\infty \frac{E^2}{8\pi}\, d\mathrm{Vol} = \frac{Q^2}{8\pi}\int_r^\infty \frac{4\pi r^2\, dr}{r^4} = \frac{Q^2}{2r}\,. \tag{4}$$

Energy is conserved because the electric field does work on the surface charge density $\sigma = Q/4\pi r^2 = E/4\pi$ as the shell expands. The (radial) surface force density is $f = \sigma E/2 = E^2/8\pi = Q^2/8\pi r^4$, as follows from the Maxwell stress tensor (Arts. 630-646 of [13], and from more elementary arguments as well). The work done as the shell expands from radius r_1 to r_2 is,

$$W = \int_{r_1}^{r_2} 4\pi r^2 f\, dr = \frac{Q^2}{2}\int_{r_1}^{r_2}\frac{dr}{r^2} = \frac{Q^2}{2r_1} - \frac{Q^2}{2r_2} = U_E(r_1) - U_E(r_2). \tag{5}$$

This confirms that no field energy is radiated away during the expansion of the spherical shell, for any time dependence of the expansion.

This result is often summarized by saying that there is no monopole electromagnetic radiation, and that the "simplest" form is (electric or magnetic) dipole radiation.[4]

Dec. 9, 2023. In general relativity there are no monopole or dipole gravitational waves, with quadrupole waves being the simplest possible. As such, any spherically symmetric configuration must have a static metric (i.e., the Schwarzschild metric) outside the region of nonzero mass/energy, as noted by Birkhoff in sec XV.7, p. 253 of [15]. In case of a shell of mass/energy, spacetime is flat inside the shell, which is a region of zero "gravity".[5]

A Appendix: Electromagnetic Potentials

This problem was solved without the use of electromagnetic potential V and $\mathbf{A}$, but we now give some examples of potentials in various gauges.

The present example is spherically symmetric, so we expect the vector potential $\mathbf{A}$ in any gauge, like the electric field $\mathbf{E}$, to have only a radial component, which insures that $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A} = 0$.

a resistive wire. The Poynting vector is nonzero, but flows from the battery to the resistive wire [11], and there is no flow of electromagnetic field energy away from the circuit, so it is usual to say that there is no radiation here (although the author prefers to identify any nonzero Poynting vector with "radiation" [12].

A closed loop of steady current of a single sign of charge also does not radiate energy to infinity, in the limit of a continuous current [14].

[4]In a multipole expansion of the potentials there is no monopole term for the vector potential $\mathbf{A}$ (since $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ implies that magnetic monopoles don't exist), while the monopole term of the (retarded) scalar potential for a charge density with fixed total charge Q at all times is just $V_{\mathrm{monopole}} = Q/r$, where r is the distance to the observer to the origin (which typically is somewhere inside the charge distribution). That is V_{monopole} has no time dependence, so the electric field $\mathbf{E}_{\mathrm{monopole}} = -\boldsymbol{\nabla}V_{\mathrm{monopole}} - \partial\mathbf{A}_{\mathrm{monopole}}/\partial ct$ has no time dependence, and there is no radiation associated with monopole potentials.

[5]Birkhoff's "theorem" is often characterized as implying that the metric of a spherically symmetric configuration is static, but this is not accurate, since the region of nonzero mass/energy can be time dependent, as for a collapsing/expanding spherically symmetric configuration.

A.1 Coulomb Gauge

In the Coulomb gauge, where $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\mathrm{C})} = 0$,[6] the scalar potential is the instantaneous Coulomb potential,

$$V^{(\mathrm{C})}(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\, d\mathrm{Vol}' = Q \begin{cases} 1/a & (r < a) \\ 1/r & (r > a) \end{cases} \qquad \text{(Coulomb).} \tag{6}$$

And, we can set the vector potential to zero,[7]

$$\mathbf{A}^{(\mathrm{C})} = 0 \qquad \text{(Coulomb).} \tag{9}$$

A.2 Gibbs Gauge

In the Gibbs gauge [18, 19] (also called the Hamiltonian or temporal gauge [20]), the scalar potential is identically zero,

$$V^{(\mathrm{G})} = 0 \qquad \text{(Gibbs),} \tag{10}$$

so the electric field is related to the vector potential according to,

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}^{(\mathrm{G})}}{\partial t}\,. \tag{11}$$

Thus,

$$\mathbf{A}^{(\mathrm{G})}(\mathbf{r},t) = -c\int_{t_0}^{t} \mathbf{E}(\mathbf{r},t')\, dt' \qquad \text{(Gibbs).} \tag{12}$$

In the present example, the electric field, and the Gibbs-gauge vector potential, have only a radial component (in spherical coordinates). To complete the integral, we consider the example that $t_0 = 0$, and,

$$a(t) = \begin{cases} a_0 & (t < 0), \\ a_0 + vt & (t > 0). \end{cases} \tag{13}$$

[6] In sec. 98 of [16], Maxwell noted that if potentials V_0, $\mathbf{A_0}$ do not obey $\boldsymbol{\nabla}\cdot\mathbf{A}_0 = 0$, then a function χ can be found such that the potentials $V = V_0 - (1/c)\,\partial\chi/\partial t$, and $\mathbf{A} = \mathbf{A}_0 + \boldsymbol{\nabla}\chi$ obey $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$ (Coulomb gauge), which he thereafter considered to be the proper type of potentials.

[7] One justification follows from eq. (12) of [17], that,

$$\mathbf{A}^{(\mathrm{C})}(\mathbf{r},t) = \boldsymbol{\nabla}\times\int \frac{\mathbf{B}(r',t)}{4\pi\,|\mathbf{r}-\mathbf{r}'|}\, d\mathrm{Vol}' = 0 \qquad \text{(Coulomb),} \tag{7}$$

since $\mathbf{B} = 0$ everywhere. More tediously, we can use the classic prescription involving the transverse current density $\mathbf{J}_t$ (see, for example, sec. 6.3 of [2]),

$$\mathbf{A}^{(\mathrm{C})}(\mathbf{r},t) = \int \frac{[\mathbf{J}_t]}{c\,|\mathbf{r}-\mathbf{r}'|}\, d\mathrm{Vol}', \qquad \mathbf{J}_t(\mathbf{r},t) = \frac{1}{4\pi}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\times\int \frac{\mathbf{J}(\mathbf{r}',t)}{c\,|\mathbf{r}-\mathbf{r}'|}\, d\mathrm{Vol}'\right) \qquad \text{(Coulomb).} \tag{8}$$

In the present example, $\mathbf{J}$ is spherically symmetric and radial, so $\int[\mathbf{J}(\mathbf{r}',t)/c\,|\mathbf{r}-\mathbf{r}'|]\, d\mathrm{Vol}'$ is also spherically symmetric and radial, such that its curl is zero, $\mathbf{J}_t$ is zero, and finally $\mathbf{A}^{(\mathrm{C})} = 0$.

The electric field is zero for $r < a_0$ and $t < 0$, while for $t > 0$ it vanishes for $t < (r - a_0)/v$; otherwise $E_r = Q/r^2$. Hence,

$$A_r^{(\mathrm{G})}(r,t) = \begin{cases} 0 & (r < a_0, t < 0) \text{ and } (r < a_0 + vt, t > 0) \\ -\frac{cQt}{r^2} & (r > a_0, t < 0) \text{ and } (r > a_0 + vt, t > 0) \end{cases} \qquad \text{(Gibbs)}. \tag{14}$$

A.3 Lorenz Gauge

The argument in this section is due to Vladimir Onoochin.

The electromagnetic potentials in the Lorenz Gauge [21] obey the condition,

$$\boldsymbol{\nabla} \cdot \mathbf{A}^{(\mathrm{L})} + \frac{1}{c}\frac{\partial V^{(\mathrm{L})}}{\partial t} = 0 \qquad \text{(Lorenz)}, \tag{15}$$

and the wave equations,

$$\nabla^2 V^{(\mathrm{L})} - \frac{1}{c^2}\frac{\partial^2 V^{(\mathrm{L})}}{\partial t^2} = -4\pi\rho, \qquad \nabla^2 \mathbf{A}^{(\mathrm{L})} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}^{(\mathrm{L})}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} \qquad \text{(Lorenz)}. \tag{16}$$

The wave equations (16) have formal solutions as the famous retarded potentials, where $t' = t - |\mathbf{r} - \mathbf{r}'|/c$,

$$V^{(\mathrm{L})}(\mathbf{r},t) = \int \frac{\rho(\mathbf{r}',t')}{|\mathbf{r} - \mathbf{r}'|}\, d\mathrm{Vol}', \qquad \mathbf{A}^{(\mathrm{L})}(\mathbf{r},t) = \int \frac{\mathbf{J}(\mathbf{r}',t')}{c\,|\mathbf{r} - \mathbf{r}'|}\, d\mathrm{Vol}' \qquad \text{(Lorenz)}. \tag{17}$$

The retarded vector potential is spherically symmetric, $\mathbf{A}^{(\mathrm{L})}(\mathbf{r},t) = A^{(\mathrm{L})}(r,t)\,\hat{\mathbf{r}}$, such that the magnetic field is zero, $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}^{(\mathrm{L})} = 0$.

Likewise, the retarded scalar potential is spherically symmetric, $V^{(\mathrm{L})}(\mathbf{r},t) = V^{(\mathrm{L})}(r,t)$, so the electric field is,

$$\mathbf{E} = -\boldsymbol{\nabla} V^{(\mathrm{L})} - \frac{1}{c}\frac{\partial \mathbf{A}^{(\mathrm{L})}}{\partial t} = -\frac{\partial V^{(\mathrm{L})}}{\partial r}\,\hat{\mathbf{r}} - \frac{1}{c}\frac{\partial A^{(\mathrm{L})}}{\partial t}\,\hat{\mathbf{r}}. \tag{18}$$

The functions $V^{(\mathrm{L})}(r,t)$ and $A^{(\mathrm{L})}(r,t)$ are complicated, and it is not immediately evident that eq. (18) actually has the simple form of eq. (1).

A.4 Poincaré Gauge

In cases where the fields $\mathbf{E}$ and $\mathbf{B}$ are known, we can compute the potentials in the so-called Poincaré gauge[8] (see sec. 9A of [20] and [22, 23, 24]),

$$V^{(\mathrm{P})}(\mathbf{r},t) = -\mathbf{r} \cdot \int_0^1 du\, \mathbf{E}(u\mathbf{r},t), \qquad \mathbf{A}^{(\mathrm{P})}(\mathbf{r},t) = -\mathbf{r} \times \int_0^1 u\, du\, \mathbf{B}(u\mathbf{r},t) \qquad \text{(Poincaré)}. \tag{19}$$

[8]The Poincaré gauge is also called the multipolar gauge [25].

These forms are remarkable in that they depend on the instantaneous value of the fields only along a line between the origin and the point of observation.[9]

The Poincaré-gauge condition can be stated as,

$$\mathbf{r} \cdot \mathbf{A}^{(\mathrm{P})} = 0 \qquad \text{(Poincaré)}. \tag{20}$$

In the present example the magnetic field is everywhere zero, so the Poincaré-gauge vector potential is also zero, $\mathbf{A}^{(\mathrm{P})} = 0$. The scalar potential is given by,

$$V^{(\mathrm{P})} = -r \int_0^1 du\, E_r(ur) = -Q \int_0^r \frac{d(ur)}{(ur)^2}\, \theta(ur, a) = \left(\frac{Q}{r} - \frac{Q}{a}\right) \theta(r, a(t)) \quad \text{(Poincaré)}. \tag{21}$$

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