Slab Rolling on a Rolling Cylinder

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(March 28, 2016; updated April 2, 2016)

1 Problem

Discuss the motion of a "slab" that rolls without slipping on a "cylinder", when the latter rolls without slipping on a horizontal plane.[1]

This problem was suggested by Alexandre Tort. For the related case of one cylinder on/inside another, see [1, 2]. For the case of a sphere on a fixed cylinder, see pp. 212-214 of [3].[2]

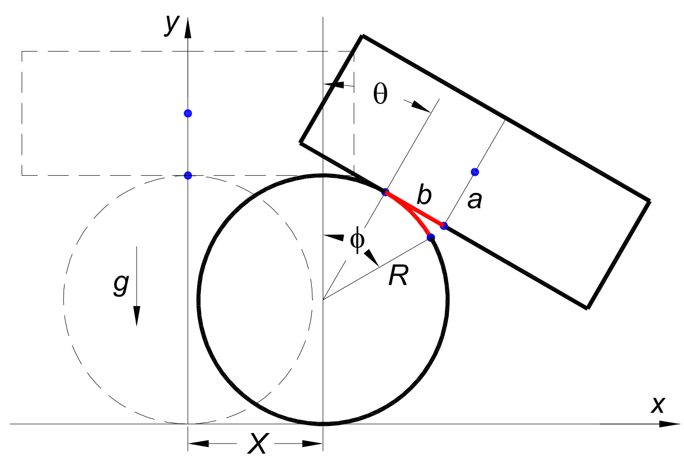
2 Solution

We will use a Lagrangian approach.

2.1 Coordinates and Constraints

When the slab, of thickness $2a$, mass m and moment of inertia kma^2, is directly above the cylinder, of radius R, mass M and moment of inertia KMR^2, and centered upon it, we define the line of contact of the cylinder with the horizontal plane to be the z-axis, at $x = y = 0$. Then, the condition of rolling without slipping for the cylinder is that when it has rolled (positive) distance X, the initial line of contact has rotated through angle $\phi = X/R$, clockwise with respect to the vertical, as shown in the figure below. This rolling constraint can be written as,

$$X = R\phi. \tag{1}$$



[1]Either the "slab" or the "cylinder" (but not both) could have a very large moment of inertia if it is in the form of a "bobbin" with flanges that extend beyond the supporting surface.

[2]The author's interest in such problems was inspired in part by Bob Dylan: "It balances on your head just like a mattress balances on a bottle of wine." (Leopardskin Pill-Box Hat, 1966).

If the slab rolls without slipping such that a line (in the x-y plane) from the center of the cylinder to the point of contact with the slab angle θ (positive clockwise) to the vertical, then the initial point of contact of the slab is at distance b from the original point, and the initial point of contact of the cylinder has rotated by angle ϕ. The second rolling constraint is that distance b on the slab equals arc length $R(\phi - \theta)$ on the cylinder,

$$b = R(\phi - \theta)\,. \tag{2}$$

The vertical center of the cylinder is at $Y = R$, and the center of the slab is at,

$$x = X + (a + R)\sin\theta + R(\phi - \theta)\cos\theta], \qquad y = R + (a + R)\cos\theta - R(\phi - \theta)\sin\theta. \tag{3}$$

Altogether there are 4 constraints on the 6 degree of freedom (x, y, X, Y, ϕ, θ), of the two-dimensional motion of the system, such that there are only two independent degrees of freedom, which we take to be the angles ϕ and θ.

2.2 Energy

The total energy $E = T + V$ is conserved, where the potential energy V (taken to be zero when $\phi = \phi_0$ and $\theta = \theta_0$),

$$V = mg(y - y_0) = mg\{(a + R)(\cos\theta - \cos\theta_0) - R[(\phi - \theta)\sin\theta - (\phi_0 - \theta_0)\sin\theta_0]\}, \tag{4}$$

depends on both coordinates ϕ and θ.[3]

The kinetic energy of cylinder, whose axis is at (X, Y), is,

$$T_{\text{cyl}} = \frac{M\dot{X}^2}{2} + \frac{I_{\text{cyl}}\,\dot{\phi}}{2} = \frac{1 + K}{2} MR^2\dot{\phi}^2, \tag{5}$$

using the rolling constraint (1) and the expression $I_{\text{cyl}} = KMR^2$ for the moment of inertia I_{cyl} in terms of parameter K.

The kinetic energy of the slab, whose axis is at (x, y), is, using $I_{\text{slab}} = kma^2$,

$$T_{\text{slab}} = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} + \frac{I_{\text{slab}}\,\dot{\theta}^2}{2} = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} + \frac{kma^2\,\dot{\theta}^2}{2}\,. \tag{6}$$

From eq. (3) we have,

$$\begin{aligned} \dot{x} &= \quad (a + R)\cos\theta\,\dot{\theta} - R(\phi - \theta)\sin\theta\,\dot{\theta} + R(\dot{\phi} - \dot{\theta})\cos\theta, & (7)\\ \dot{y} &= -(a + R)\sin\theta\,\dot{\theta} - R(\phi - \theta)\cos\theta\,\dot{\theta} - R(\dot{\phi} - \dot{\theta})\sin\theta, & (8) \end{aligned}$$

so the kinetic energy of the slab can be written as,

$$\begin{aligned} T_{\text{slab}} &= \frac{m}{2}\left[(a + R)^2\,\dot{\theta}^2 + R^2(\phi - \theta)^2\,\dot{\theta}^2 + R^2(\dot{\phi} - \dot{\theta})^2 + 2R(a + R)\,\dot{\theta}(\dot{\phi} - \dot{\theta})\right] + \frac{kma^2\,\dot{\theta}^2}{2}\\ &= \frac{m}{2}\left[(1 + k)a^2\dot{\theta}^2 + R^2(\phi - \theta)^2\,\dot{\theta}^2 + 2aR\,\dot{\theta}\,\dot{\phi} + R^2\dot{\phi}^2\right]. \end{aligned} \tag{9}$$

The total kinetic energy $T_{\text{cyl}} + T_{\text{slab}}$ is,

$$T = \frac{[m + (1 + K)M]R^2}{2}\dot{\phi}^2 + maR\,\dot{\phi}\,\dot{\theta} + \frac{(1 + k)ma^2 + mR^2(\phi - \theta)^2}{2}\,\dot{\theta}^2. \tag{10}$$

[3]This contrasts with the case of a cylinder rolling on/inside another cylinder [1, 2], where the potential energy does not depend on ϕ, such that there is a second conserved quantity for the system.

2.3 Equations of Motion

2.3.1 ϕ

The ϕ-equation for the Lagrangian $\mathcal{L} = T - V$ is,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = [m + (1+K)M]R^2\,\ddot{\phi} + maR\,\ddot{\theta} = \frac{\partial \mathcal{L}}{\partial \phi} = mgR\sin\theta. \tag{11}$$

2.3.2 θ

The θ-equation can be written as,

$$\begin{aligned}
\frac{1}{m}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= aR\,\ddot{\phi} + \left[(1+k)a^2 + R^2(\phi-\theta)^2\right]\,\ddot{\theta} + 2R^2(\phi-\theta)(\dot{\phi}-\dot{\theta})\,\dot{\theta} \\
&= \frac{1}{m}\frac{\partial \mathcal{L}}{\partial \theta} = g\left[(a+R)\sin\theta + R(\phi-\theta)\cos\theta - R\sin\theta\right] \\
&= g\left[a\sin\theta + R(\phi-\theta)\cos\theta\right].
\end{aligned} \tag{12}$$

2.4 $\phi = \phi_0 =$ Constant

Before discussing the general case, we consider the special case that the cylinder is fixed, but the initial angle ϕ_0 in not necessarily zero.

Then, the ϕ-equation of motion (11) is to be ignored, and the θ-equation (12) becomes,

$$\left[(1+k)a^2 + R^2(\phi_0-\theta)^2\right]\,\ddot{\theta} - 2R^2(\phi_0-\theta)\,\dot{\theta}^2 = g\left[a\sin\theta + R(\phi_0-\theta)\cos\theta\right]. \tag{13}$$

The potential energy (4) becomes,

$$\frac{V}{m} = g\{(a+R)(\cos\theta - \cos\theta_0) - R[(\phi_0-\theta)\sin\theta - (\phi_0-\theta_0)\sin\theta_0]\}, \tag{14}$$

the kinetic energy (10) becomes,

$$\frac{T}{m} = \frac{(1+k)a^2 + R^2(\phi_0-\theta)^2}{2}\,\dot{\theta}^2, \tag{15}$$

and the total energy becomes,

$$\begin{aligned}
\frac{E}{m} &= \frac{(1+k)a^2 + R^2(\phi_0-\theta_0)^2}{2}\,\dot{\theta}_0^2 = \frac{(1+k)a^2 + R^2(\phi_0-\theta)^2}{2}\,\dot{\theta}^2 \\
&\quad + g\{(a+R)(\cos\theta - \cos\theta_0) - R[(\phi_0-\theta)\sin\theta - (\phi_0-\theta_0)\sin\theta_0]\}.
\end{aligned} \tag{16}$$

2.4.1 Small Oscillations

We first seek an oscillatory solution, of the form,

$$\theta = \theta_0 + \alpha\, e^{i\omega t}, \qquad \dot{\theta} = i\alpha\omega\, e^{i\omega t}, \qquad \ddot{\theta} = -\alpha\omega^2\, e^{i\omega t}, \tag{17}$$

where α is small. Using this trial solution in eq. (13), and keeping terms only to order α, we have,

$$\sin\theta \approx \sin\theta_0 + \alpha\, e^{i\omega t}\, \cos\theta_0, \qquad \cos\theta \approx \cos\theta_0 - \alpha\, e^{i\omega t}\, \sin\theta_0, \tag{18}$$

$$\begin{aligned} & -\alpha\omega^2 \left[(1+k)a^2 + R^2(\phi_0-\theta_0)^2\right] e^{i\omega t} \\ \approx\ & g\left[a\left(\sin\theta_0 + \alpha\, e^{i\omega t}\, \cos\theta_0\right) + R(\phi_0 - \theta_0 - \alpha\, e^{i\omega t})\left(\cos\theta_0 - \alpha\, e^{i\omega t}\, \sin\theta_0\right)\right] \\ \approx\ & g\left[a\sin\theta_0 + R(\phi_0-\theta_0)\cos\theta_0\right] + \alpha g\left[a\cos\theta_0 - R(\phi_0-\theta_0)\sin\theta_0 - R\cos\theta_0\right] e^{i\omega t}. \end{aligned} \tag{19}$$

The constant term must be zero, which tells us that,

$$\tan\theta_0 = \frac{R}{a}(\theta_0 - \phi_0) = -\frac{b_0}{a}\,, \tag{20}$$

recalling eq. (2).

The terms in $\alpha\, e^{i\omega t}$ must be the same on both sides of eq. (19), which tells us that the angular frequency ω of small oscillations is related by,

$$\omega = \sqrt{\frac{g\{R[\cos\theta_0 - (\theta_0-\phi_0)\sin\theta_0] - a\cos\theta_0\}}{(1+k)a^2 + R^2(\phi_0-\theta_0)^2}} = \sqrt{\frac{g(R\cos\theta_0 - a/\cos\theta_0)}{a^2(1+k+\tan^2\theta_0)}}\,. \tag{21}$$

Oscillatory solutions exist only for,

$$R > \frac{a}{\cos^2\theta_0}\,, \qquad \cos\theta_0 < \sqrt{\frac{a}{R}}\,, \tag{22}$$

which always requires that $R > a$.

In particular, $\omega = \sqrt{g(R-a)/a^2(1+k)}$ for $\phi_0 = 0 = \theta_0$. For this case, the constant energy is,

$$\frac{E}{m} = \frac{(1+k)a^2}{2}\,\dot\theta_0^2 = \frac{(1+k)a^2}{2}\,\dot\theta^2 - g(a+R)(1-\cos\theta), \tag{23}$$

and we record the full equation of motion,

$$\left[(1+k)a^2 + R^2\,\theta^2\right]\ddot\theta + 2R^2\,\theta)\,\dot\theta^2 = g\left[a\sin\theta - R\,\theta\cos\theta\right]. \tag{24}$$

For reference, we also record that in this case,

$$\dot\theta^2 = \dot\theta_0^2 + \frac{2g(a+R)(1-\cos\theta)}{(1+k)a^2}\,, \tag{25}$$

$$\left[(1+k)a^2 + R^2\,\theta^2\right]\ddot\theta = g(a\sin\theta - R\,\theta\cos\theta) - 2R^2\,\theta\left(\dot\theta_0^2 + \frac{2g(a+R)(1-\cos\theta)}{(1+k)a^2}\right) \tag{26}$$

For a solid slab of half width c, $k = (1 + c^2/a^2)/3$, so for $\phi_0 = 0 = \theta_0$,

$$\omega = \sqrt{\frac{3g(R-a)}{4a^2+c^2}} \ \xrightarrow[\text{cube}]{\text{solid}}\ \sqrt{\frac{3g(R-a)}{5a^2}}\,. \tag{27}$$

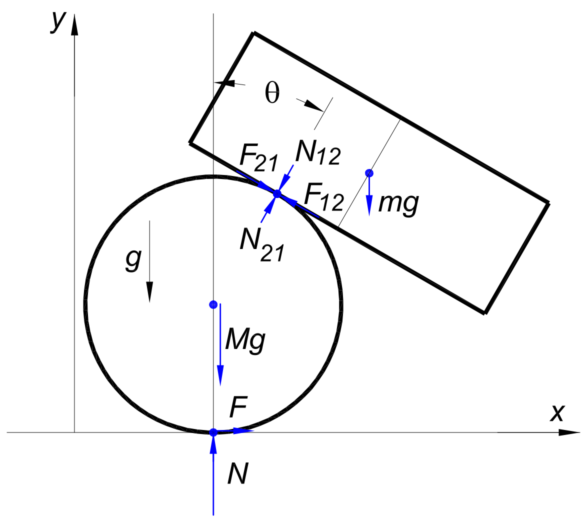
Note that it is possible to have small oscillations about a nonzero value of θ_0, if the slab is appropriately off center with respect to the initial point of contact with the cylinder. For example, a cube of half length $a = R/2$ would oscillate about an initial configuration with $b_0 = -\sqrt{3}R/6$, $\theta_0 = 30^\circ$ and $\phi_0 = 15^\circ$ at angular frequency $\omega = \sqrt{2g/\sqrt{3}R} = 1.075\sqrt{g/R}$. This is about 2% less than the angular frequency $\omega = \sqrt{6g/5R} = 1.095\sqrt{g/R}$ for $\theta_0 = 0$.

2.4.2 Angle θ_s at which the Slab Falls off the Cylinder

As the slab rotates it may lose contact with (separate from) the cylinder, say at angle θ_s.

This happens when the normal force N_{12} of the cylinder on the slab vanishes, which occurs when the component of the gravitational force mg equals the component of $m\mathbf{a}$ along an axis that makes angle θ_s to the vertical,

$$mg\cos\theta_s = m(-\ddot{x}_s \sin\theta_s - \ddot{y}_s \cos\theta_s). \tag{28}$$



From eqs. (7)-(8), we find,

$$\begin{aligned}
\ddot{x} &= (a+R)(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2) - R[(\phi-\theta)(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2) + \sin\theta\,\dot{\theta}(\dot{\phi}-\dot{\theta})] \\
&\quad + R[\cos\theta(\ddot{\phi}-\ddot{\theta}) - \sin\theta\,\dot{\theta}(\dot{\phi}-\dot{\theta})], && (29) \\
\ddot{y} &= -(a+R)(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2) - R[(\phi-\theta)(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2) + \cos\theta\,\dot{\theta}(\dot{\phi}-\dot{\theta})] \\
&\quad - R[\sin\theta(\ddot{\phi}-\ddot{\theta}) + \cos\theta\,\dot{\theta}(\dot{\phi}-\dot{\theta})], && (30)
\end{aligned}$$

$$\begin{aligned}
\ddot{x}\sin\theta + \ddot{y}\cos\theta &= -(a+R)\,\dot{\theta}^2 - R[(\phi-\theta)\,\ddot{\theta} + \dot{\theta}(\dot{\phi}-\dot{\theta})] - R\,\dot{\theta}(\dot{\phi}-\dot{\theta}) \\
&= -R(\phi-\theta)\,\ddot{\theta} + (R-a)\,\dot{\theta}^2 - 2R\,\dot{\phi}\,\dot{\theta}. && (31)
\end{aligned}$$

For constant angle ϕ_0, the relation (28) becomes,

$$g\cos\theta_s = R(\phi_0 - \theta_s)\,\ddot{\theta}_s - (R-a)\,\dot{\theta}_s^2. \tag{32}$$

Equations (13) and (16) can be used in eq. (32) to determine $\ddot{\theta}_s$ and $\dot{\theta}_s$, but the resulting expression is lengthy. Even for the particular case that $\phi_0 = 0 = \theta_0$, the resulting expression for θ_s is complicated,

$$\begin{aligned}
g\cos\theta_s &= \frac{R\,\theta_s}{(1+k)a^2 + R^2\,\theta_s^2}\left[2R^2\,\theta_s\left(\dot{\theta}_0^2 + \frac{2g(a+R)(1-\cos\theta_s)}{(1+k)a^2} + g(R\,\theta_s\cos\theta_s - a\sin\theta_s)\right)\right] \\
&\quad -(R-a)\left(\dot{\theta}_0^2 + \frac{2g(a+R)(1-\cos\theta_s)}{(1+k)a^2}\right). && (33)
\end{aligned}$$

One conclusion that can be drawn is that for $R \gg a$ the (thin) slab will not fall off the (large) cylinder (which is perhaps obvious without the preceding analysis).

2.5 Both ϕ and θ Vary

2.5.1 Coupled Oscillations

We first consider the possibility of small, coupled oscillations in both ϕ and θ, with equilibrium angles ϕ_0 and θ_0.

We seek an oscillatory solution of the form (17)-(18) for θ, and,

$$\phi = \phi_0 + \beta\, e^{i\omega t}, \tag{34}$$

where β is small, for ϕ. To use these trial solutions in eqs. (11)-(12), we have in addition to eq. (18) that,

$$\sin\phi \approx \sin\phi_0 + \beta\, e^{i\omega t}\, \cos\phi_0, \qquad \cos\phi \approx \cos\phi_0 - \beta\, e^{i\omega t}\, \sin\phi_0. \tag{35}$$

Then, keeping terms only that are constant or proportional to $e^{i\omega t}$,[4] the θ-equation of motion (12) becomes,

$$\begin{aligned} & -\omega^2 \beta a R\, e^{i\omega t} - \omega^2 \alpha\, e^{i\omega t} \left[(1+k)a^2 + R^2(\phi_0 - \theta_0)^2\right] \\ \approx\ & g\left\{a(\sin\theta_0 + \alpha\, e^{i\omega t}\cos\theta_0) + R[\phi_0 - \theta_0 + (\beta - \alpha)\, e^{i\omega t}](\cos\theta_0 - \alpha\, e^{i\omega t}\sin\theta_0)\right\}. \end{aligned} \tag{36}$$

The constant term in eq. (36) must be zero, which again tells us that,

$$\tan\theta_0 = \frac{R}{a}(\theta_0 - \phi_0) = -\frac{b_0}{a}\,. \tag{37}$$

The terms in $e^{i\omega t}$ must be the same on both sides of eq. (36), which tells us that,

$$\begin{aligned} & \omega^2\left[\beta a R + \alpha(1+k)a^2 + \alpha R^2(\phi_0 - \theta_0)^2\right] \\ =\ & g\left\{R[(\alpha - \beta)\cos\theta_0 - \alpha(\theta_0 - \phi_0)\sin\theta_0] - \alpha a\cos\theta_0\right\} \\ =\ & g\left[R(\alpha - \beta)\cos\theta_0 - \alpha a/\cos\theta_0\right]. \end{aligned} \tag{38}$$

To go further, we now consider the ϕ-equation (11),

$$-\omega^2\, e^{i\omega t}\left\{\beta[m + (1+K)M]R^2 + \alpha m a R\right\} \approx mgR(\sin\theta_0 + \alpha\, e^{i\omega t}\cos\theta_0). \tag{39}$$

The constant term in eq. (39) must vanish, which implies that coupled oscillations are only possible for $\theta_0 = 0$. Then, from eq. (37) we have that $\phi_0 = b_0$ also, and eq. (38) becomes

$$\omega^2\left[\beta a R + \alpha(1+k)a^2\right] = g\left[R(\alpha - \beta) - \alpha a\right]. \tag{40}$$

In addition, the terms in $e^{i\omega t}$ on the left and right sides of eq. (39) must be the same, which implies that,

$$-\omega^2\left\{\beta[m + (1+K)M]R^2 + \alpha m a R\right\} = \alpha m g R. \tag{41}$$

This condition cannot be satisfied, so **there is no coupled oscillatory motion when the cylinder is free to roll**; it will always roll out from under the slab, which rotates until it falls off the cylinder at some angle θ_s of separation. An analysis of angle θ_s could be given via an extension of the discussion in sec. 2.4.2, but we will not pursue this here.

[4] We will not consider terms in $e^{2i\omega t}$ since the approximations (18) and (35) have omitted terms of this type.

2.5.2 Small Oscillations of the Slab

We next consider the possibility that as the cylinder rolls the slab executes small oscillatory motion in θ, with an angular frequency that varies "slowly" with time. In the "instantaneous" approximation, the angular frequency $\omega(t)$ is just that associated with that found in sec. 2.4 for $\phi_0 = \phi(t)$.

It does not appear that analytic techniques are especially helpful in the next approximation, such that it is best to use numerical integration of the equations of motion (11)-(12) to carry the discussion further.

References

[1] K.T. McDonald, *Cylinder Rolling on a Rolling Cylinder*, (Oct. 2, 2014), http://kirkmcd.princeton.edu/examples/2cylinders.pdf

[2] K.T. McDonald, *Cylinder Rolling inside Another Rolling Cylinder*, (Oct. 21, 2014), http://kirkmcd.princeton.edu/examples/2cylinders_in.pdf

[3] K.T. McDonald, *Motion of a Rigid Body Which is Rolling without Slipping*, Princeton U. Ph205 Lecture 20 (1980), http://kirkmcd.princeton.edu/examples/Ph205/ph205l20.pdf