Slepian's Electromagnetic Spaceship

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1 Problem

In the late 1940's Joseph Slepian, a senior engineer at Westinghouse, posed a series of delightful pedagogic puzzles in the popular journal *Electrical Engineering*. One of these concerned how the electric displacement current might or might not be used to provide a form of rocket propulsion [1].

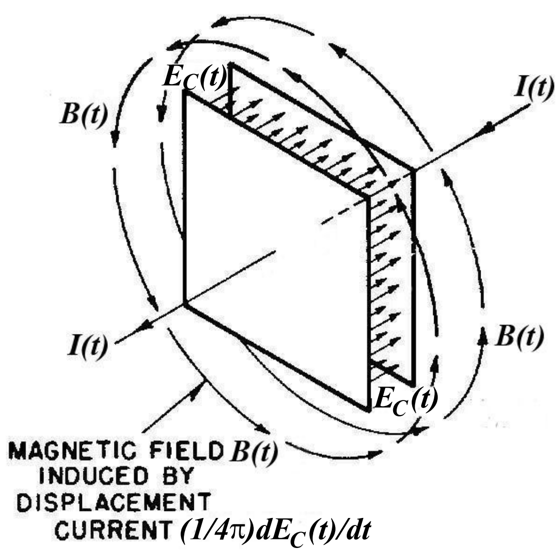
Slepian reminded us how the time-dependent electric field $\mathbf{E}_C(t)$ inside a parallel-plate capacitor that is fed by a current $I(t)$ leads to a magnetic field $\mathbf{B}(t)$ both inside and outside the capacitor. If retardation is neglected in the near zone, this magnetic field can be calculated using Ampère's law, as extend by Maxwell to include the displacement current (density),

$$\mathbf{J}_{\rm displacement} = \frac{1}{4\pi}\frac{d\mathbf{E}_C}{dt}\,, \tag{1}$$

in Gaussian units, where for simplicity we suppose that no dielectric media are present. For a capacitor whose plates have area A, the electric field inside is $E_C = 4\pi Q/A$, so the displacement current density is $J_{\rm displacement} = (1/A)dQ/dt = I/A$. Then, for a loop whose area lies in a plane between the capacitor plates, the induced magnetic field obeys,

$$\oint \mathbf{B}\cdot d\mathbf{l} = \frac{4\pi}{c}\int \mathbf{J}_{\rm displacement}\cdot d\mathbf{Area}_{\rm loop} = \frac{4\pi}{c} I \begin{cases} 1 & (A_{\rm loop} > A), \\ \frac{A_{\rm loop}}{A} & (A_{\rm loop} < A). \end{cases} \tag{2}$$

For a loop whose area is larger than that of the capacitor plates, the induced magnetic field (2) is the same as that if the capacitor were short circuited by a wire across its gap.

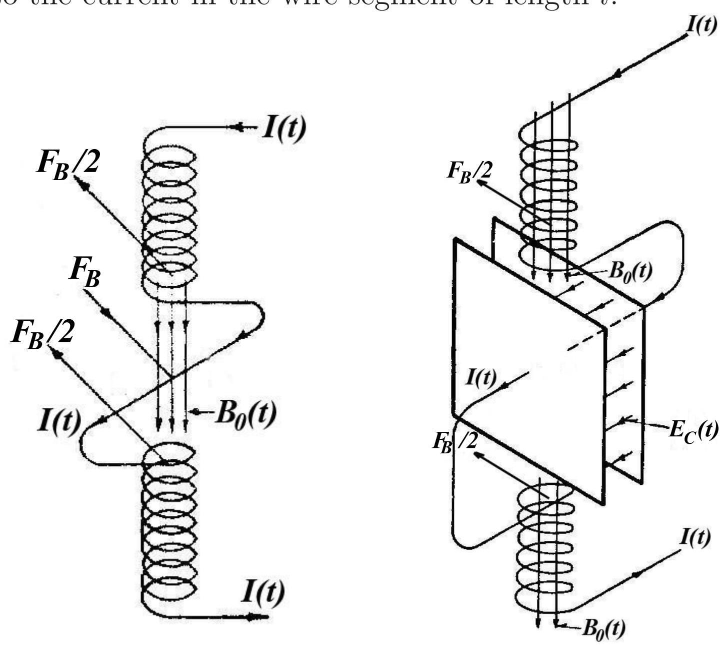


The displacement current (density) $\mathbf{J}_{\text{displacement}}$ is "real" in the sense that it leads to magnetic fields according to the same law (2) as holds for conduction currents. However, Slepian also reminded us that if the displacement current were placed in an external magnetic field, say $\mathbf{B}_0$, then it would not experience a force (density) that is the analog of the force (density) $\mathbf{J}/c \times \mathbf{B}_0$ on a conduction current (density) $\mathbf{J}$.

We now can appreciate Slepian's suggestion for rocket propulsion based on displacement current. First, consider a coil as shown on the left below, in which the solenoid winding creates a field $\mathbf{B}_0(t)$ when excited by an AC current $I(t)$. The solenoid winding is split into two halves, with a straight segment of wire joining the two halves by crossing the solenoid field $\mathbf{B}_0$ at a right angle. A length l of this wire experiences a sideways magnetic force,

$$\mathbf{F}_B = \frac{\mathbf{I}l \times \mathbf{B}_0}{c} . \tag{3}$$

The total force of the circuit on itself is zero, so each half of the solenoid experiences a force $-\mathbf{F}_B/2$, which is due to the interaction of the currents in the solenoid windings with the magnetic field due to the current in the wire segment of length l.



However, if the wire segment of length l is replaced by a parallel-plate capacitor of gap width l, as shown on the right above, there is no force on the displacement current of the capacitor, while the magnetic field created by that displacement current does exert force $-\mathbf{F}_B/2$ on each half of the solenoid winding. Thus, the total force on the circuit with the capacitor is unbalanced, *i.e.*, $-\mathbf{F}_B$, and it appears that the circuit will be propelled to the left.

Note that if the current I is reversed then the magnetic fields are also reversed, but the magnetic forces do NOT change sign, being the cross product of the current and the field. So, the unbalanced force is always to the left, and it appears that Slepian's AC circuit provides a new form a propulsion.

Explain why Slepian's rocket motor does not work.

2 Solution

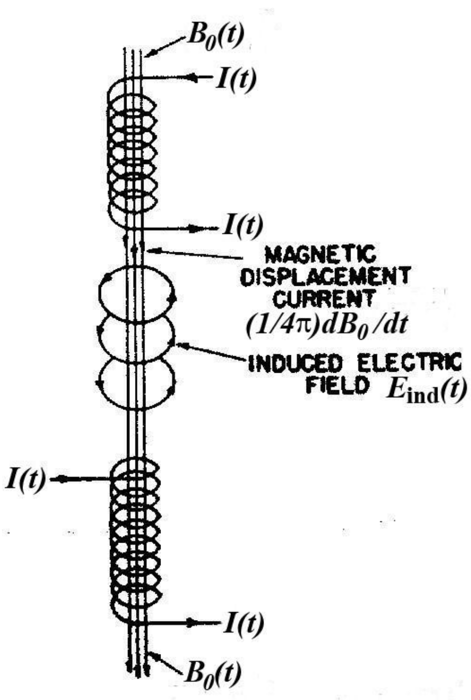
The solution follows Slepian [2], and expands upon a subtle point.[1]

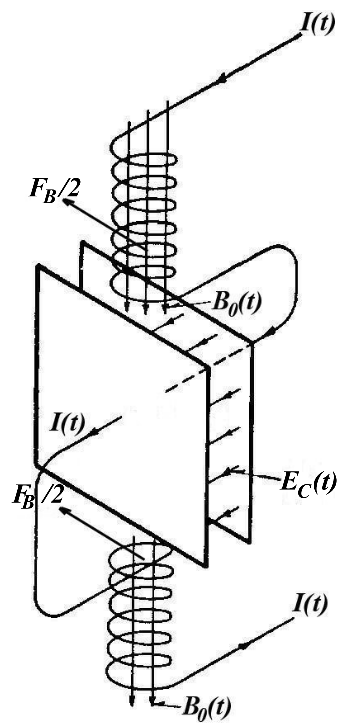
2.1 Slepian's Solution

Not only does a changing electric field create additional magnetic field, but also a changing magnetic field induces an additional electric field $\mathbf{E}_{\rm ind}$, as first noted by Faraday [4]. Indeed, Faraday's law can be put in a form similar to eq. (2) by defining a "magnetic displacement current" to be, $\mathbf{J}_{\rm magnetic\ displacement} = (1/4\pi)d\mathbf{B}_0/dt$, so that,

$$\oint \mathbf{E}_{\rm ind} \cdot d\mathbf{l} = -\frac{1}{c}\frac{d}{dt}\int \mathbf{B}_0 \cdot d\mathbf{Area}_{\rm loop} = -\frac{4\pi}{c}\int \mathbf{J}_{\rm magnetic\ displacement} \cdot d\mathbf{Area}_{\rm loop}. \tag{4}$$

Taking the loop to be in a plane perpendicular to the field $\mathbf{B}_0$ of the solenoid winding, we see that the "magnetic displacement current" induces an electric field $\mathbf{E}_{\rm ind}$ that circulates around $\mathbf{B}_0$ as shown in the figure below. This induced electric field interacts with the charges on the capacitor plates and exerts a sideways force on them.





We adopt a coordinate system in which the z-axis is the axis of the solenoid. The capacitor is centered on the origin, with its plates parallel to the x-z plane and its electric field along the y-axis. Then, for a loop of radius ρ in the x-y plane with its center on the z-axis, and which lies within the region of the uniform magnetic field of the solenoid, we find using Faraday's law that the induced, azimuthal electric field is,

$$E_{{\rm ind},\phi}(x, \pm l/2, z) = -\frac{\rho}{2c}\frac{dB_0}{dt}\,. \tag{5}$$

[1]For discussion of related examples, see [3].

The component $E_{\mathrm{ind},x}$ of the induced electric field that is parallel to the capacitor plates, whose separation is l, is,

$$E_{\mathrm{ind},x}(x, \pm l/2, z) = \pm \frac{l/2}{\rho} E_{\mathrm{ind},\phi} \approx \mp \frac{l}{4c} \frac{dB_0}{dt}\,, \tag{6}$$

independent of position on the plates. The charge $\pm Q$ on each plate is conveniently expressed in terms of the electric field $E_C = 4\pi Q/A$ that is perpendicular to the plates. Thus, the sideways electric force on the two plates is,

$$\mathbf{F}_E = 2Q E_{\mathrm{ind},x}(0, l/2, 0)\,\hat{\mathbf{x}} = 2\frac{\mathbf{E}_C A}{4\pi} \times \frac{-l}{4c}\frac{d\mathbf{B}_0}{dt} \approx -\frac{1}{8\pi c}\int \mathbf{E} \times \frac{d\mathbf{B}_0}{dt}\, d\mathrm{Vol}, \tag{7}$$

noting that $\mathbf{E} = \mathbf{E}_C + \mathbf{E}_{\mathrm{ind}}$, and that $\int \mathbf{E}_{\mathrm{ind}} \times \mathbf{B}_0\, d\mathrm{Vol} = 0$.

It is useful to rewrite the magnetic force (3) in a manner similar to the last form of eq. (7). Recalling that $I = dQ/dt = (A/4\pi) dE_C/dt$, we have,

$$\mathbf{F}_B = \frac{\mathbf{I}l \times \mathbf{B}_0}{c} = \frac{Al}{4\pi c}\frac{d\mathbf{E}_C}{dt} \times \mathbf{B}_0 \approx \frac{1}{4\pi c}\int \frac{d\mathbf{E}}{dt} \times \mathbf{B}_0\, d\mathrm{Vol}. \tag{8}$$

The total force on the solenoid + capacitor is,

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_E - \mathbf{F}_B \approx -\frac{1}{4\pi c}\int \mathbf{E} \times \frac{d\mathbf{B}_0}{dt}\, d\mathrm{Vol} - \frac{1}{4\pi c}\int \frac{d\mathbf{E}}{dt} \times \mathbf{B}_0\, d\mathrm{Vol} = -\frac{d}{dt}\int \frac{\mathbf{E} \times \mathbf{B}_0}{4\pi c}\, d\mathrm{Vol} \\ &= -\frac{d\mathbf{P}_{\mathrm{field}}}{dt}\,, \end{aligned} \tag{9}$$

where,

$$\mathbf{P}_{\mathrm{field}} = \int \frac{\mathbf{E} \times \mathbf{B}_0}{4\pi c}\, d\mathrm{Vol} \tag{10}$$

is the electromagnetic field momentum introduced by J.J. Thomson [5, 6, 7], Heaviside [8], and by Poincaré [10], building on the Poynting vector description of energy flow in electromagnetic fields [11]. Since the total force $\mathbf{F}$ equals the rate of change of mechanical momentum of the system, according to Newton, we have that,

$$0 = \mathbf{F} + \frac{d\mathbf{P}_{\mathrm{field}}}{dt} = \frac{d\mathbf{P}_{\mathrm{mech}}}{dt} + \frac{d\mathbf{P}_{\mathrm{field}}}{dt} = \frac{d\mathbf{P}_{\mathrm{total}}}{dt}\,. \tag{11}$$

Hence the total momentum of the system is constant, which means that the center of mass/energy of the system is either at rest or in a state of uniform motion. The electromagnetic forces $\mathbf{F}_E$ and $\mathbf{F}_B$ cause small oscillatory motion of the mechanical parts of the system relative to its center of mass energy, but there is no net propulsion.[2]

References

[1] J. Slepian, *Electromagnetic Space-Ship*, Electrical Engineering **68**, 145 (1949), http://kirkmcd.princeton.edu/examples/EM/slepian_ee_68_145_49

[2]For a related paradox, see [12].

[2] J. Slepian, *Answer to Previous Essay*, Electrical Engineering **68**, 245 (1949),
http://kirkmcd.princeton.edu/examples/EM/slepian_ee_68_245_49

[3] K.T. McDonald, *No Bootstrap Spaceships* (Aug. 9, 2018),
http://kirkmcd.princeton.edu/examples/bootstrap.pdf

[4] M. Faraday, *Experimental Researches in Electricity*, Phil. Trans. Roy. Soc. London **122**, 125 (1832), kirkmcd.princeton.edu/examples/EM/faraday_ptrsl_122_163_32.pdf

[5] J.J. Thomson, *On the Illustration of the Properties of the Electric Field by Means of Tubes of Electrostatic Induction*, Phil. Mag. **31**, 149 (1891),
http://kirkmcd.princeton.edu/examples/EM/thomson_pm_31_149_91.pdf

[6] J.J. Thomson, *Recent Researches in Electricity and Magnetism* (Clarendon Press, 1893),
http://kirkmcd.princeton.edu/examples/EM/thomson_recent_researches_in_electricity.pdf

[7] K.T. McDonald, *J.J. Thomson and "Hidden" Momentum* (Apr. 30, 2014),
http://kirkmcd.princeton.edu/examples/thomson.pdf

[8] O. Heaviside, *Electromagnetic Theory–XV*, Electrician **27**, 493 (1891),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_27_493_91.pdf
See also pp. 107-108 of [9].

[9] O. Heaviside, *Electromagnetic Theory*, Vol. 1 (Electrician Publishing Co., 1893),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electromagnetic_theory_1.pdf

[10] H. Poincaré, *La Théorie de Lorentz et la Principe de Réaction*, Arch. Neer. **5**, 252 (1900), http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00.pdf
Translation: *The Theory of Lorentz and the Principle of Reaction*,
http://kirkmcd.princeton.edu/examples/EM/poincare_an_5_252_00_english.pdf

[11] J.H. Poynting, *On the Transfer of Energy in the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **175**, 343 (1884),
http://kirkmcd.princeton.edu/examples/EM/poynting_ptrsl_175_343_84.pdf

[12] K.T. McDonald, *Onoochin's Paradox* (Jan. 1, 2006),
http://kirkmcd.princeton.edu/examples/onoochin.pdf