Small Oscillations of a Mass That Slides inside a Cylinder Which Rotates about a Horizontal Axis

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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The mass m has coordinate θ measured with respect to a vertical diameter of the cylinder of radius r, and slides with friction coefficient μ as the cylinder rotates in, say, the counterclockwise sense about its horizontal axis.

The forces on the mass are gravity mg downwards, the normal force N radially inwards, and the counterclockwise friction force μN. The equations of motion of the mass are, in cylindrical coordinates (r, θ),

$$F_r = mg\cos\theta - N = ma_r = mr\dot{\theta}^2, \tag{1}$$

$$F_\theta = \mu N - mg\sin\theta = ma_\theta = mr\ddot{\theta}, \tag{2}$$

since r is a constant. The angular velocity of the cylinder does not appear in the equations of motion of the mass, but it must be nonzero for continuous sliding motion to occur.

There exists a static solution, $\ddot{\theta} = 0 = \dot{\theta}$, with equilibrium angle θ_0 of the mass given by,

$$\tan\theta_0 = \mu. \tag{3}$$

For small oscillations about this equilibrium we write $\theta = \theta_0 + \epsilon$ and ignore quantities of second order of smallness in eqs. (1)-(2) to find that,

$$N \approx mg(\cos\theta_0 - \epsilon\sin\theta_0), \tag{4}$$

$$\begin{aligned} mr\ddot{\epsilon} &\approx \mu mg(\cos\theta_0 - \epsilon\sin\theta_0) - mg(\sin\theta_0 + \epsilon\cos\theta_0) \\ &= -mg\epsilon(\mu\sin\theta_0 + \cos\theta_0) = -\frac{mg\epsilon}{\cos\theta_0} = -mg\epsilon\sqrt{1+\mu^2}. \end{aligned} \tag{5}$$

Hence, the frequency ω of small oscillations is,

$$\omega = \sqrt{\frac{g}{r\cos\theta_0}} = \sqrt{\frac{g\sqrt{1+\mu^2}}{r}}\,. \tag{6}$$

This can be used to determine μ from ω (and r),

$$\mu = \sqrt{\frac{\omega^4 r^2}{g^2} - 1}\,. \tag{7}$$

If $\mu = 0$ the problem is equivalent to a simple pendulum of length r. For nonzero friction the motion is not damped, being driven by the rotation of the cylinder, but unlike most driven oscillator problems that frequency of response is not related to that of the drive.