The Sliding-Block Problem in Various Inertial Frames

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1 Problem

Discuss the classic problem of a block that slides without friction down a slope that makes angle θ to the horizontal at the surface of the Earth, considering the (inertial) lab frame, a frame with horizontal velocity u with respect to the lab frame, and a frame that has velocity $u = \sqrt{2gh}$ down the slope, where g is the (uniform) acceleration due to gravity as the Earth's surface and h is the vertical height of the slide.[1]

You may assume the all velocities are so small that effects of special and general relativity can be neglected.

2 Solution

This problem is "simple", but it is not "trivial", which justifies its ongoing presence in "elementary" physics texts.

A possibly disconcerting issue is that the mechanical energy KE + PE of the sliding block is conserved only if the normal force on the block from the slope does no work.[2] This is so in the lab frame, and in the frame with velocity parallel to the slope, but not in frames with other velocities relative to the lab frame.

We will consider this problem using Newtonian methods, as well as via Galilean transformations from the lab frame to the other inertial frames.

The motion is in a vertical plane, and we will use x and y axes in that plane, both horizontal and vertical axes, as well as axes parallel and perpendicular to the slope.

2.1 Galilean Transformations

The relevant Galilean transformations from a frame with unprimed axes to one with primed axes and (constant) velocity $\mathbf{u}$ with respect to the unprimed frame is,

$$t' = t, \qquad \mathbf{x}' = \mathbf{x} - \mathbf{u}t, \qquad \mathbf{v}' = \frac{d\mathbf{x}'}{dt'} = \frac{d\mathbf{x}'}{dt'} = \mathbf{v} - \mathbf{u}, \qquad \mathbf{a}' = \frac{d\mathbf{v}'}{dt'} = \mathbf{a}, \tag{1}$$

$$\mathbf{F}' = m\mathbf{a}' = m\mathbf{a} = \mathbf{F}, \tag{2}$$

where in eq. (2) we invoke Newton's $2^{\rm nd}$ law of motion, which relates the total (3-vector) force $\mathbf{F}$ on a mass m to its (3-vector) acceleration $\mathbf{a}$ (in an inertial frame).

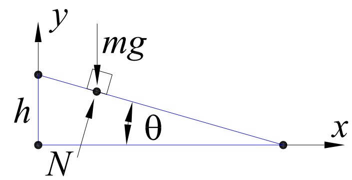
[1]See Appendix B below for discussion of the notion of a uniform gravitational field.

[2]This theme is discussed, for example, in [1]-[8].

2.2 Lab Frame

2.2.1 Horizontal and Vertical Axes

We first use horizontal and vertical axes, (x, y), such that the position of the block when at rest is $\mathbf{x}_0 = (0, h)$, as sketched in the figure below.



In the idealization of zero friction, the forces on the block (of mass m) are that due to gravity, $-mg\,\hat{\mathbf{y}}$, and the normal force $\mathbf{N}$ of the slope on the block, which latter force is perpendicular to the slope, which makes angle θ to the horizontal.

The gravitational force $-mg\,\hat{\mathbf{y}}$ can be related to a potential energy, PE, such that,

$$\mathbf{F}_{\rm gravity} = -\boldsymbol{\nabla}{\rm PE} = -mg\,\hat{\mathbf{y}}, \qquad \text{with} \qquad {\rm PE} = mgy, \tag{3}$$

defining the potential energy to be zero at $y = 0$.

In this frame, the normal force does no work on the block as it slides down the slope, and so does not contribute to the work-energy relation for the block. Hence, in this frame, in the absence of friction, we have conservation of mechanical energy of the sliding block,

$$E = {\rm KE} + {\rm PE} = \frac{mv^2}{2} + mgy = E_0 = mgh. \tag{4}$$

Thus, when the block has slid down to $y = 0$, its (final) velocity is related by

$$E_0 = mgh = E_f = \frac{mv_f^2}{2}, \qquad v_f = \sqrt{2gh}. \tag{5}$$

For later reference, we also display an analysis based on $\mathbf{F} = m\mathbf{a}$.

$$F_x = N\sin\theta = ma_x, \qquad F_y = N\cos\theta - mg = ma_y, \tag{6}$$

and since the block slides down the slope, the acceleration $\mathbf{a}$ is parallel to the slope, and we also have the constraint relations that,

$$a_x = a\cos\theta, \qquad a_y = -a\sin\theta, \qquad a_x = -a_y\cot\theta. \tag{7}$$

From eqs. (6) and (7) we have that,

$$N = \frac{ma_x}{\sin\theta} = -\frac{ma_y\cos\theta}{\sin^2\theta}, \qquad ma_y = -\frac{ma_y\cos^2\theta}{\sin^2\theta} - mg \tag{8}$$

$$\frac{a_y}{\sin^2\theta} = -g, \qquad a_y = -g\sin^2\theta, \qquad N = mg\cos\theta, \tag{9}$$

$$v_y(t) = v_{y,0} + a_t\,t = -g\sin^2\theta\,t, \qquad y(t) = h - \frac{g\sin^2\theta\,t^2}{2}, \tag{10}$$

recalling that $y_0 = h$ and $v_{y,0} = 0$. Thus, the block reaches $y_f = 0$ at time t_f given by,

$$t_f = \frac{\sqrt{2h/g}}{\sin\theta}. \qquad (11)$$

The final y-velocity is,

$$v_{y,f} = a_y\, t_f = -g\sin^2\theta \frac{\sqrt{2h/g}}{\sin\theta} = -\sqrt{2gh}\sin\theta. \qquad (12)$$

The motion in x follows from eqs. (7) and (9) as, recalling that $x_0 = 0$ and $v_{x,0} = 0$,

$$a_x = -a_y\cot\theta = g\sin\theta\cos\theta, \qquad (13)$$

$$v_x(t) = v_{x,0} + a_x\, t = g\sin\theta\cos\theta\, t, \qquad v_{x,f} = g\sin\theta\cos\theta\, t_f = \sqrt{2gh}\cos\theta, \qquad (14)$$

$$x(t) = x_0 + v_{x,0}t + \frac{a_x\, t_f^2}{2} = \frac{g\sin\theta\cos\theta\, t^2}{2} \qquad x_f = \frac{g\sin\theta\cos\theta\, t_f^2}{2} = \frac{h}{\tan\theta}, \qquad (15)$$

confirming that $x_f = h/\tan\theta$ as follows from the geometry of the figure above.

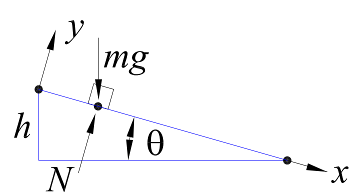
The final velocity is, from eqs. (12) and (14),

$$v_f = \sqrt{v_{x,f}^2 + v_{y,f}^2} = \sqrt{2gh}, \qquad (16)$$

in agreement with eq. (5) that was based on an energy method.

2.2.2 Axes Parallel and Perpendicular to the Slope

We now use axes (x, y) parallel and perpendicular to the slope such that the position of the block when at rest is $\mathbf{x}_0 = (0, 0)$, as sketched in the figure below.



In an analysis based on $\mathbf{F} = m\mathbf{a}$ we have,

$$F_x = mg\sin\theta = ma_x, \qquad F_y = N - mg\cos\theta = ma_y = 0, \qquad N = mg\cos\theta, \qquad (17)$$

since the block slides down the slope with $y = 0$ always. Thus,

$$a_x = g\sin\theta, \qquad x = \frac{g\sin\theta\, t^2}{2}, \qquad (18)$$

noting that $x_0 = h$ and $v_{x,0} = 0$. The final position of the block is $x_f = h/\sin\theta$ from the geometry of the problem, so the block reaches this position at time t_f given by,

$$t_f = \sqrt{\frac{h}{\sin\theta}\frac{2}{g\sin\theta}} = \frac{\sqrt{2h/g}}{\sin\theta}, \qquad (19)$$

as previously found in eq. (11). The final x-velocity, which is also the final total velocity, is,

$$v_{x,f} = v_f = a_x\, t_f = g \sin\theta \frac{\sqrt{2h/g}}{\sin\theta} = \sqrt{2gh}, \tag{20}$$

in agreement with eqs. (5) and (16).

As is well known, detailed analysis of this problem in the lab frame via $\mathbf{F} = m\mathbf{a}$ *is easier with axes parallel and perpendicular to the slope than with horizontal and vertical axes (although the energy analysis is easier with horizontal and vertical axes).*

2.3 Inertial Frame with Horizontal velocity u

We use horizontal and vertical axes in both the lab frame and the moving ($'$) frame.

2.3.1 Analysis via Galilean Transformations

From eq. (1) we have that the time of the slide in the $'$ frame, which has horizontal velocity $\mathbf{u} = u\,\hat{\mathbf{x}}$ with respect to the lab frame, is that same as in the lab frame,

$$t'_f = t_f = \frac{\sqrt{2h/g}}{\sin\theta}\,. \tag{21}$$

The final velocity in the $'$ frame follows from eqs. (1). (10) and (14) as,

$$v'_{x,f} = v_{x,f} - u = \sqrt{2gh}\cos\theta - u, \qquad v'_{y,f} = v_{y,f} = \sqrt{2gh}\sin\theta, \tag{22}$$

so the final kinetic energy of the sliding block is,[3]

$$\mathrm{KE}'_f = \frac{mv'^2}{2} = mgh - mu\sqrt{2gh}\cos\theta + \frac{mu^2}{2}, \tag{23}$$

while the final gravitational potential energy is $\mathrm{PE}_0 = 0$. Hence, $\mathrm{KE}'_f + \mathrm{PE}'_f$ differs from the initial value, $\mathrm{KE}'_0 + \mathrm{PE}'_0 = mu^2/2 + mgh$, and the energy $\mathrm{KE}' + \mathrm{PE}'$ is not conserved.

However, the normal force $\mathbf{N}$ does work on the sliding block in the $'$ frame at rate,

$$\frac{dW'}{dt'} = \mathbf{N}\cdot\mathbf{v}' = \mathbf{N}\cdot(\mathbf{v}-\mathbf{u}) = -\mathbf{N}\cdot\mathbf{u} = -uN_x = -uN\sin\theta = -umg\sin\theta\cos\theta, \tag{24}$$

such that the work done at time t' is,

$$W'(t') = -umg\sin\theta\cos\theta\, t', \qquad W'_f = -umg\sin\theta\cos\theta\frac{\sqrt{2h/g}}{\sin\theta} = -mu\sqrt{2gh}\cos\theta, \tag{25}$$

which accounts for the difference between $\mathrm{KE}'_0 + \mathrm{PE}'_0$ and $\mathrm{KE}'_f + \mathrm{PE}'_f$.

In the larger picture, the work $W'(t')$ done on the sliding block in the $'$ frame is associated with an equal and opposite change in energy elsewhere in that frame, such that the total energy of the larger system is conserved.

[3] The final velocity v'_f is nonzero for any value of u as the quadratic equation $v'^2(u) = 0$ has no real solutions.

2.3.2 Analysis via $\mathbf{F}' = m\mathbf{a}'$

For completeness, we show that the results of sec. 2.3.1 can be obtained by Newtonian methods.

At any time t' the slope has angle θ to the x' axis, so the analysis is very similar to that in sec. 2.2.1, except that the initial condition for the sliding block is that $v'(x,0) = -u$, rather than zero.

The normal force has components with respect to the horizontal x' axis and the vertical y' axis,

$$F'_x = N\sin\theta = ma'_x, \qquad F'_y = N\cos\theta - mg = ma'_y, \tag{26}$$

and since the block slides down the slope, the acceleration $\mathbf{a}'$ is parallel to the slope, and we also have the constraint relations that,

$$a'_x = a'\cos\theta, \qquad a'_y = -a'\sin\theta, \qquad a'_x = -a'_y\cot\theta. \tag{27}$$

From eqs. (26) and (27) we have that,

$$N = \frac{ma'_x}{\sin\theta} = -\frac{ma'_y\cos\theta}{\sin^2\theta}, \qquad ma'_y = -\frac{ma'_y\cos^2\theta}{\sin^2\theta} - mg \tag{28}$$

$$\frac{a'_y}{\sin^2\theta} = -g, \qquad a'_y = -g\sin^2\theta, \qquad N = mg\cos\theta, \tag{29}$$

$$v'_y(t) = v'_{y,0} + a'_t\,t' = -g\sin^2\theta\,t', \qquad y'(t) = h - \frac{g\sin^2\theta\,t'^2}{2}, \tag{30}$$

recalling that $y_0 = h$ and $v'_{y,0} = 0$. Thus, the block reaches $y'_f = 0$ at time t'_f given by,

$$t'_f = \frac{\sqrt{2h/g}}{\sin\theta}, \tag{31}$$

which is the same time as that found in eq. (11). The final y'-velocity is,

$$v'_{y,f} = a'_y\,t'_f = -g\sin^2\theta\frac{\sqrt{2h/g}}{\sin\theta} = -\sqrt{2gh}\sin\theta. \tag{32}$$

The motion in x' follows from eqs. (26) and (29) as, recalling that $x'_0 = 0$ and $v'_{x,0} = -u$,

$$a'_x = -a'_y\cot\theta = g\sin\theta\cos\theta, \tag{33}$$

$$v'_x(t) = v'_{x,0} + a'_x\,t' = -u + g\sin\theta\cos\theta\,t', \tag{34}$$

$$v'_{x,f} = -u + g\sin\theta\cos\theta\,t_f = \sqrt{2gh}\cos\theta - u, \tag{35}$$

$$x'(t) = x'_0 + v'_{x,0}t' + \frac{a'_x\,t'2_f}{2} = -ut' + \frac{g\sin\theta\cos\theta\,t'^2}{2} \tag{36}$$

$$x'_f = -ut' + \frac{g\sin\theta\cos\theta\,t_f^2}{2} = \frac{h}{\tan\theta} - u\frac{\sqrt{2h/g}}{\sin\theta}. \tag{37}$$

The final velocity is, from eqs. (32) and (34),

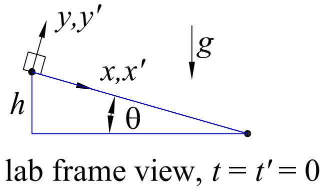
$$v'^2_f = v'^2_{x,f} + v'^2_{y,f} = 2gh - 2u\sqrt{2gh}\cos\theta + u^2, \tag{38}$$

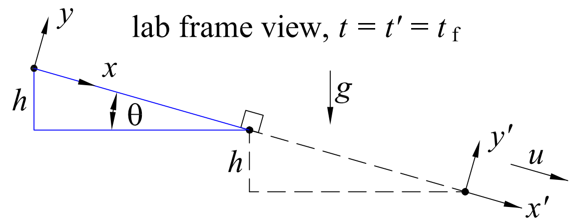
in agreement with eq. (23).

2.4 Inertial Frame with Velocity $u = \sqrt{2gh}$ Down the Slope

We also call this inertial frame the $'$ frame, but take the x' (and x) down the slope (parallel to it), with the y' (and y) axis perpendicular to the slope (upwards and to the right).

In the lab frame, the block starts at rest at time $t = 0$ with $(x_0, y_0) = (0, 0)$, as in sec. 2.2.2 above, and ends at $(x_f, y_f) = (h/\sin\theta, 0)$ at time $t = \sqrt{2h/g}\,/\sin\theta$, with velocity $(v_x, v_y) = \sqrt{2gh}(\sin\theta, -\cos\theta)$, *i.e.*, $u = \sqrt{2gh}$.





In the moving frame, the block is initially at $(x'_0, y'_0) = (0, 0)$, with initial velocity $(v'_{x,0}, v'_{y,0}) = (-\sqrt{2gh}, 0)$. The initial kinetic energy of the block is $\mathrm{KE}'_0 = mv'^2/2 = mgh$, and we declare its initial potential energy to be zero. That is, $\mathrm{KE}'_0 + \mathrm{PE}'_0 = mgh$.

At time $t' = t = \sqrt{2h/g}$, when the sliding block reaches the bottom of the slope, the origin of the $'$ coordinate system with respect to the lab frame is at $x = ut = \sqrt{(}2gh)\sqrt{(}2h/g)/\sin\theta = 2h/\sin\theta$, which is twice the length, $h/\sin\theta$, of the slope traveled by the sliding block. Of course, the origin of the $'$ coordinate system, with respect to the lab frame, is at $y = 0$ at this time.

According to the Galilean transformations of sec. 2.1 above, at time $t'_f = t_f = \sqrt{2h/g}/\sin\theta$ the position and velocity of the block is,

$$x'_f = x - ut_f = \frac{h}{\sin\theta} - \frac{2h}{\sin\theta} = -\frac{h}{\sin\theta}\,, \qquad y'_f = y_f = 0, \tag{39}$$

$$v'_{x,f} = v_{x_f} - u = \sqrt{2gh} - \sqrt{2gh} = 0, \qquad v'_{y,f} = v_{y.f} = 0, \tag{40}$$

recalling the results of sec. 2.2.2 above.

We verify this via $\mathbf{F}' = m\mathbf{a}'$ in the moving frame, where $a'_x = g\sin\theta$ and $a'_y = 0$. The final x' position of the sliding block is,

$$x'_f = -u\,t'_f + \frac{g\sin\theta\,t'^2_f}{2} = -\sqrt{2gh}\sqrt{\frac{2h/g}{\sin\theta}} + \frac{g\sin\theta\,(2gh/\sin^2\theta)}{2} = -\frac{2h}{\sin\theta} + \frac{h}{\sin\theta} = -\frac{h}{\sin\theta}\,, \tag{41}$$

and the final x' velocity is,

$$v'_{x,f} = -u + g\sin\theta\,t'_f = -\sqrt{2gh} + g\sin\theta\frac{\sqrt{2h/g}}{\sin\theta} = 0, \tag{42}$$

in agreement with the Galilean transformation.

And, of course, since $a'_y = 0$ and the initial v'_y is zero, the final y' of the block is zero.

In the $'$ frame, the final kinetic energy of the block is zero, $\mathrm{KE}'_f = 0$.

What should we say about the final potential energy PE'_f in the $'$ frame?

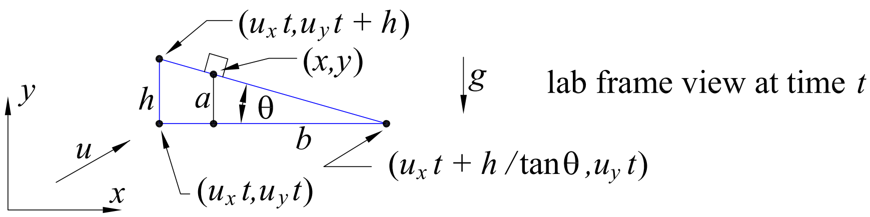
If we refer back to the lab frame, we see that the final position of the block is at vertical height h about the origin of the $'$ frame at time $t'_f = t_f$. So, it seems reasonable to say that the final potential energy of the block in the prime frame is $\mathrm{PE}'_f = mgh$ (relative to the origin of the $'$ frame). Hence, we have $\mathrm{KE}'_0 + \mathrm{PE}'_0 = mgh = \mathrm{KE}'_f + \mathrm{PE}'_f$. The total energy, $\mathrm{KE}' + \mathrm{PE}'$, is conserved in the $'$ frame (as well as in the lab frame).

This is to be expected as the constraint force (normal force) does no work in the $'$ frame.

A Appendix: A Lagrangian Approach

The Newtonian analysis in sec. 2 above is quite satisfactory, but perhaps is a bit convoluted. So it may be of interest to compare it to an analysis in the spirit of Lagrange.

For this, we pose the problem of a wedge that moves with velocity $\mathbf{u} = (u_x, u_y, 0)$ in rectangular coordinates with respect to an inertial lab frame at the surface of the Earth, where the x-axis is horizontal and the y axis is vertical. The wedge makes angle θ to the horizontal, and has vertical height h and horizontal extent $h/\sin\theta$ in x. The upper point of the wedge at time $t = 0$ is at $(x, y) = (0, h)$, when and where a block is released from rest in the frame of the moving wedge, and subsequently slides down the surface of the (moving) wedge without friction.



What is the velocity of the block in the lab frame when it reaches the bottom of the moving wedge?

While the motion is 2-dimensional, because the block is assumed to lie always on the slope of the wedge, there is actually only one degree of freedom, which we take to be the x-coordinate of the (pointlike) block.

Lagrange advises us to consider the kinetic and potential energies of the block (in the lab frame), and to form the Lagrangian $\mathcal{L} = \mathrm{KE} - \mathrm{PE}$. To do this, we need to know the $y(t)$ of the block at time t in terms of $x(t)$ for the block. Although Lagrange avoided the use of figures, we resort to use of that above.

The y coordinate of the block is,

$$y = u_y t + a = u_y t + b\tan\theta = u_y t + \left(u_x t + \frac{h}{\tan\theta} - x\right)\tan\theta, \tag{43}$$

$$v_y = u_y + (u_x - v_x)\tan\theta. \tag{44}$$

The kinetic energy is, writing the x-velocity of the block as $v_x = dx/dt = \dot{x}$,

$$\mathrm{KE} = \frac{m(v_x^2 + v_y^2)}{2} = \frac{m(\dot{x}^2 + (u_y + (u_x - \dot{x})\tan\theta)^2)}{2}, \tag{45}$$

and the gravitational potential energy, relative to the origin is,

$$\mathrm{PE} = mgy = mg\left(u_y t + \left(u_x t + \frac{h}{\tan\theta} - x\right)\tan\theta\right), \tag{46}$$

which depends on time t in general.

However, if $u_y/u_x = -\tan\theta$, there is no time explicit time dependence in the Lagrangian. This holds if the velocity of the wedge with respect to the lab frame is parallel to the slope of the wedge. In this interesting special case, the kinetic energy is,

$$\mathrm{KE} = \frac{m(\dot{x}^2 + (\dot{x}\tan\theta)^2)}{2} = \frac{m\dot{x}^2}{2\cos^2\theta} \qquad (u_y/u_x = -\tan\theta), \tag{47}$$

and the potential energy is,

$$\mathrm{PE} = mg(h - x\tan\theta) \qquad (u_y/u_x = -\tan\theta), \tag{48}$$

Then, the Hamiltonian of the system is a conserved quantity,

$$H = \dot{x}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \mathcal{L} = \frac{m\dot{x}^2}{\cos^2\theta} - \frac{m\dot{x}^2}{2\cos^2\theta} + \mathrm{PE} = \mathrm{KE} + \mathrm{PE} \qquad (u_y/u_x = -\tan\theta), \tag{49}$$

which is the mechanical energy of the system.

But, in general the energy KE + PE is not conserved in this problem.

We can continue with the Lagrangian approach to find the equation of motion in coordinate x for the general case, using $\mathcal{L}$ = KE - PE of eqs. (45) and (46),

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = m\ddot{x}(1+\tan^2\theta) = \frac{m\ddot{x}}{\cos^2\theta} = \frac{\partial\mathcal{L}}{\partial x} = mg\tan\theta, \qquad \ddot{x} = g\sin\theta\cos\theta, \tag{50}$$

which is independent of the velocity $\mathbf{u}$ of the wedge with respect to the lab frame. This agrees with eq. (13), found in sec. 2.2.1 for a wedge at rest in the lab frame.

The equations of motion in x are,

$$v_x(t) = u_x + g\sin\theta\cos\theta\, t, \qquad x(t) = u_x\, t + \frac{g\sin\theta\cos\theta\, t^2}{2}. \tag{51}$$

The block reaches the bottom of the wedge at time t_f at x-position,

$$x_f = u_x\, t_f + \frac{h}{\tan\theta}, \tag{52}$$

which implies that the travel time is given by,

$$\frac{h}{\tan\theta} = \frac{g\sin\theta\cos\theta\, t_f^2}{2}, \qquad t_f = \frac{\sqrt{2h/g}}{\sin\theta}, \tag{53}$$

as found in eq. (10). The final x-velocity is,

$$v_{x,f} = u_x + \sqrt{2gh}\cos\theta. \tag{54}$$

From eq. (44), the final y-velocity is,

$$v_{y,f} = u_y + (u_x - v_{x,f})\tan\theta = u_y - \sqrt{2gh}\sin\theta, \tag{55}$$

and the final velocity is given by,

$$v_f^2 = u_x^2 + u_y^2 + 2gh + 2u_x\sqrt{2gh}\cos\theta - 2u_y\sqrt{2gh}\sin\theta. \tag{56}$$

For the special case that $u_x = -\sqrt{2gh}\cos\theta$, and $u_y = \sqrt{2gh}\sin\theta$, we have that $v_f = 0$. Here, the block has velocity $\sqrt{2gh}$ up the slope with respect to the lab frame, which is equivalent to the block at rest in the lab frame as observed in a frame with velocity $\sqrt{2gh}$ down the slope, as considered in sec. 2.4 above.

B Appendix: Uniform Gravitational Field[4]

The notion of a uniform gravitational field is somewhat elusive. If one associates gravitational fields with sources of mass/energy, then physical gravitational fields are typically associated with distortions of spacetime.[5] On the other hand, the equivalence principle implies that a uniformly accelerated reference frame in flat spacetime should be equivalent to a uniform gravitational field. Of course, a uniform field over all spacetime is a mathematical idealization, such that there is room for discussion as to the relevant physical approximation to this concept. Lengthy debate on this topic may or may not have converged, but present wisdom seems to be that reasonably physical assumptions as to the sources of a uniform gravitational field are consistent with it being associated with flat spacetime [12, 15]-[27].

Often a weak, uniform gravitational is taken to be described by the metric,[6]

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2 \left(1 + \frac{gz}{c^2}\right)^2 dt^2, \qquad (|z| < c^2/g), \tag{57}$$

where $g = 2\pi G\rho$, G is Newton's gravitational constant and ρ is the density of mass/energy. See, for example, sec. 97 of [17].

For spacetime described by the static metric (57), electrodynamics obey Maxwell's equation with the alterations that the vacuum has relative permittivity and permeability given by,

$$\epsilon = \mu = \frac{1}{1 + gz/c^2}\,, \tag{58}$$

as discussed, for example, in sec. 90 of [30]. A consequence is that the speed, u, of light emitted at $z = 0$ is a function of z according to,[7,8]

$$u(z) = c\,(1 + gz/c^2). \tag{59}$$

[4]This Appendix is transcribed from secs. secs. 2.2-3 of [9].

[5]These distortions are often called "curvature", but in the case of hypothetical "cosmic strings" and "domain walls" [10, 11] spacetime is flat with topological defects. Vacuum "domain walls" are not physically viable, but remain an interesting theoretical construct.

[6]The metric (17) may have been first used by Kottler (1914) [28], and more clearly in sec. VII of [29].

[7]Equation (59) appears near the end of Einstein's 1907 paper [12].

[8]Our brief discussion avoids the issue of variation with z of the rate of clocks in a uniform gravitational field. However, the metric (57) indicates that a clock (that reads time t) at position z has proper time interval $d\tau = (1 + gz/c^2)dt$, such clock at $z > 0$ runs slower compared to proper time than a clock at $z = 0$. Hence, reporting the speed of light at position $z > 0$ as $u(z) = dz/dt = (dz/d\tau)(d\tau/dt) = c\,(1 + gz/c^2)$ gives a value larger than c. If light is emitted in the $+z$-direction at $z = -c^2/g$ its initial speed is zero according to eq. (59), such that it takes an infinite time interval Δt to reach $z = 0$, and we speak of $z = -c^2/g$ as the "event horizon" for the observer at $z = 0$. However, an observer at $z = -c^2/g$ could consider that the light has local speed c, and the metric to be eq. (57) with z replaced by $z + c^2/g$, such that the speed of light varies with z according to $u(z) = c\,(1 + g(z + c^2/g)/c^2) = c\,(2 + gz/c^2)$, and the event horizon for this observer is $z = -2c^2/g$. Similarly, an observer at $z = c^2/g$ who considers the local speed of light to be c concludes that light emitted at $z = 0$ takes an infinite time to reach him, so that in effect an observer at $z = 0$ cannot communicate with one at $z = c^2/g$. Hence, we say that the metric (57) is valid only for $|z| < c^2/g$.

Another way to see this is to note that the gravitational redshift brings the energy of any photon emitted at $z = 0$ to zero at $z = c^2/g$ [13], so there is no meaningful physical interaction possible between an observer at $z = 0$ and one at $z > c^2/g$.

A universe with a uniform gravitational field is effectively partitioned into regions of extent $\Delta z = \pm c^2/g$ around any observer. Each observer cannot know about the rest of the universe outside this domain. That is, early cosmological visions that assumed a flat Earth and "turtles all the way down" were actually consistent with general relativity.

If we approximate a uniform gravitational field by that at the surface of the Earth, then the symbol g in eq. (59) becomes, approximately, $g_0(1 - z^2/2R_E^2)$ where $g_0 = GM_E/R_E^2$, G is Newton's gravitational constant, M_E and R_E are the mass and radius of the Earth, respectively.

B.1 Does a Uniform Gravitational Field Have a Source?

Using coordinates $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$, the metric tensors g_{ij} and g^{ij} corresponding to eq. (57) have nonzero components,[9]

$$g_{00} = \frac{1}{g^{00}} = f^2(z) = \left(1 + \frac{gz}{c^2}\right)^2, \qquad g_{11} = g_{22} = g_{33} = g^{11} = g^{22} = g^{33} = -1, \tag{60}$$

such that $g_{ik}\, g^{jk} = \delta_i^j$. The nonzero Christoffel symbols are,

$$\Gamma_{i,jk} = \Gamma_{i,kj} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right), \qquad \Gamma_{0,03} = \Gamma_{0,30} = -\Gamma_{3,00} = f\frac{df}{dz} \equiv ff'. \tag{61}$$

The Riemann curvature tensor has nonzero components,

$$R_{ijkl} = \frac{\partial \Gamma_{i,jl}}{\partial x^k} - \frac{\partial \Gamma_{i,jk}}{\partial x^l} + g^{mn}\Gamma_{i,mk}\Gamma_{n,jl} - g^{mn}\Gamma_{i,ml}\Gamma_{n,jk}, \tag{62}$$

$$R_{0330} = R_{3003} = -R_{0303} = -R_{3030} = ff''. \tag{63}$$

The Ricci tensor has nonzero components,

$$R_{ij} = g^{kl}R_{kilj}, \qquad R_{00} = ff'', \qquad R_{33} = -\frac{f''}{f}. \tag{64}$$

The Ricci curvature scalar is,[10]

$$R = g^{ij}R_{ij} = \frac{2f''}{f}. \tag{65}$$

Einstein's gravitational equations are,

$$\frac{8\pi G}{c^4}T_{ij} = R_{ij} - g_{ij}R, \tag{66}$$

$$T_{00} = -\frac{c^4}{8\pi G}ff'', \qquad T_{11} = \frac{c^4}{4\pi G}\frac{f''}{f} \qquad T_{22} = \frac{c^4}{4\pi G}\frac{f''}{f} \qquad T_{33} = \frac{c^4}{8\pi G}\frac{f''}{f}. \tag{67}$$

Hence, the choice $f(z) = 1 + gz/c^2$, for which $f'' = 0$, implies that the stress-energy tensor T_{ij} is everywhere zero. The "uniform gravitational field" corresponding to the metric (60) has no source, or spacetime curvature, and is only a kind of "coordinate force" akin to the centrifugal force and the Coriolis force.[11,12]

[9] For the general case of symmetric metric tensors, see prob. 2, sec. 92 of [30].

[10] Probably, $R = f''/f$, such that $T_{00} = T_{33} = 0$, and I have errors somewhere.

[11] The metric (60) is valid only for $z > z_0 = -c^2/g$, which leaves open the possibility of sources at $z < z_0$, and in particular a plane sheet of mass at $z = z_0$.

[12] As the nonphysical, mathematical idealization of a "uniform gravitational field" is associated with flat spacetime (zero Ricci scalar), many people (including this author) consider it not to be an actual gravitational field. However, others consider that even flat spacetime is a kind of gravitational field.

Requiring a uniform gravitational field to have an infinite planar source and flat spacetime apparently leads to metrics with spatial anisotropy. See, for example, [10, 15, 22]-[27].

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