Scattering of a Plane Electromagnetic Wave by a Small Conducting Cylinder

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1 Problem

Discuss the scattering of a plane electromagnetic wave of angular frequency ω that is normally incident on a perfectly conducting cylinder of radius a (in vacuum) when the wavelength obeys $\lambda \gg a$ ($ka \ll 1$). Comment also on the limit of large cylinders, $ka \gg 1$, via the optical theorem.

Extend the discussion to the case of small conducting elliptical cylinders, which have (small) flat strips as a limit. Relate the case of a small strip to that of a screen with a small slit using the electromagnetic version of Babinet's principle.

2 Solution

The solution for small cylinders follows secs. 361-368 of [1].

The scattering cross section in cylindrical coordinates (r, ϕ, z), with the cylinder along the z-axis, is given by,

$$\frac{d\sigma}{d\phi} = \frac{\text{power scattered into } d\phi}{\text{incident power per unit area}} = r\frac{\langle \mathbf{S}_{\rm scat}(\phi)\rangle}{\langle \mathbf{S}_{\rm incident}\rangle}. \tag{1}$$

where,

$$\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}, \tag{2}$$

is the Poynting vector (in Gaussian units) and c is the speed of light in vacuum.

Because the incident wavelength is large compared to the radius of the cylinder, and the wave is normally incident, the incident fields are essentially uniform over the cylinder, and we might suppose the induced fields near the cylinder are the same as the static fields of a conducting cylinder in an otherwise uniform electric and magnetic field. This approach was appropriate for the case of scatter by a small sphere [2], but there is no static solution for an external electric field along the axis of a conducting cylinder. Instead, we follow an approach perhaps first used by J.J. Thomson in sec. 359 of [1] in which we note that close to a small cylinder the incident plane wavefunction $e^{i(kx-\omega t)}$ can be approximated as $1 + ikx = 1 + ikr\cos\phi$.

We consider the total electric and magnetic fields to be the sum of the incident plane wave and a scattered wave,

$$\mathbf{E} = \mathbf{E}_0\, e^{i(kx-\omega t)} + \mathbf{E}_s(r,\phi)\, e^{-i\omega t}, \qquad \mathbf{B} = \hat{\mathbf{x}}\times\mathbf{E_0}\,\mathbf{e}^{\mathbf{i(kx-\omega t)}} + \mathbf{B_s}(\mathbf{r},\boldsymbol{\phi})\,\mathbf{e}^{-\mathbf{i}\boldsymbol{\omega}\mathbf{t}}, \tag{3}$$

The electric and magnetic fields $\mathbf{E}_s$ and $\mathbf{B}_s$ of angular frequency ω obey the vector Helmholtz equation (as do also the incident fields),

$$\nabla^2 \mathbf{E}_s + k^2 \mathbf{E_s} = 0 = \nabla^2 \mathbf{B}_s + k^2 \mathbf{B_s}, \tag{4}$$

but only the z-components $\psi = E_z$ or B_z obey the scalar Helmholtz equation with ∇^2 in cylindrical coordinates,

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial \psi}{\partial \phi^2} + k^2 \psi = 0, \tag{5}$$

noting that the fields in this problem do not depend on z. This equation is separable, permitting solutions that are sums of terms $R_n(r)\cos n\phi$, where the symmetry of the incident wave in x implies that terms in $\sin n\phi$ will not be present. The radial functions R_n obey Bessel's equation,

$$\frac{\partial^2 R_n}{\partial r^2} + \frac{1}{r}\frac{\partial R_n}{\partial r} + \left(k^2 - \frac{n^2}{r^2}\right) R_n = 0, \tag{6}$$

with solutions,

$$R_n = J_n(kr) + iN_n(kr) = H_n^{(1)}(kr) \approx \begin{cases} (-i)^{n+1}\sqrt{2i/\pi kr}\, e^{ikr} & (kr \text{ large}), \\ -i2^n(n-1)!/\pi(kr)^n & (kr \text{ small},\ n > 0). \end{cases} \tag{7}$$

The Hankel functions $H_n^{(1)}(kr)$ (rather than $H_n^{(2)}(kr)$) are appropriate in that for large r the solutions $R_n\, e^{-i\omega t}$ should be outgoing waves. A component ψ of $\mathbf{E}$ or $\mathbf{B}$ then has the form,

$$\psi\, e^{i\omega t} = \psi_0\, e^{ikx} + \sum_n A_n H_n^{(1)}(kr)\cos n\phi. \tag{8}$$

The Fourier coefficients A_n are to be determined by the conditions that the tangential component of the electric field, and the normal component of the magnetic field, vanish at the surface of the conducting cylinder.

2.1 Electric Field Polarized Parallel to the Wire

In this case we identify ψ as E_z, and the condition is that $E_z(a,\phi) = \psi(a,\phi) = 0$. Close to the cylinder eq. (8) has the form,

$$\psi\, e^{i\omega t} \approx E_0(1 + ikr\cos\phi) + \sum_n A_n H_n^{(1)}(kr)\cos n\phi. \tag{9}$$

Clearly, $A_n = 0$ unless $n = 0$ or 1. Then, the condition $\psi(a,\phi) = 0$ tells us that,

$$\frac{A_0}{E_0} = -\frac{1}{H_0^{(1)}(ka)} \approx \frac{i\pi}{2C}, \qquad \frac{A_1}{E_0} = -\frac{ika}{H_1^{(1)}(ka)} \approx \frac{\pi k^2 a^2}{2} \ll \frac{A_0}{E_0}, \tag{10}$$

where (referring, for example, to sec. 3.7 of [3]),

$$C = \ln(2/ka) - 0.5772 < 2/ka, \tag{11}$$

which becomes large for very small ka, but $1/C > ka$. The electric field for $r > a$ is, from eq. (8),

$$\mathbf{E} \approx E_0\, e^{-i\omega t} \left[e^{ikx} + \frac{i\pi}{2C} H_0^{(1)}(kr) + \frac{\pi k^2 a^2}{2} H_1^{(1)}(kr) \cos\phi \right] \hat{\mathbf{z}}, \tag{12}$$

and the magnetic field follows from Faraday's law as,

$$\begin{aligned} \mathbf{B} = -\frac{i}{k} \boldsymbol{\nabla} \times \mathbf{E} \;\; \approx \;\; & E_0\, e^{-i\omega t} \left\{ -e^{ikx}\, \hat{\mathbf{y}} - \frac{i\pi k a^2}{2r} H_1^{(1)}(kr) \sin\phi\, \hat{\mathbf{r}} \right. \qquad (13) \\ & \left. + \frac{\pi}{2} \left[\frac{1}{C} H_1^{(1)}(kr) - ik^2 a^2 \left(H_0^{(1)}(kr) - \frac{H_1^{(1)}(kr)}{kr} \right) \cos\phi \right] \hat{\boldsymbol{\phi}} \right\}. \end{aligned}$$

For large r the electric and magnetic field are, neglecting terms in $k^2 a^2$ compared to $1/C$, given by,

$$\mathbf{E}(kr \gg 1) \approx E_0\, e^{-i\omega t} \left[e^{ikx} + \frac{1}{C} \sqrt{\frac{i\pi}{2kr}}\, e^{ikr} \right] \hat{\mathbf{z}}, \tag{14}$$

recalling eq. (7),[1] and,

$$\mathbf{B}(kr \gg 1) \approx E_0\, e^{-i\omega t} \left[-e^{ikx}\, \hat{\mathbf{y}} - \frac{1}{C} \sqrt{\frac{i\pi}{2kr}}\, e^{ikr}\, \hat{\boldsymbol{\phi}} \right]. \tag{15}$$

The time-average Poynting vector in the far zone is,

$$\begin{aligned} \langle \mathbf{S}(kr \gg 1) \rangle &= \frac{c}{8\pi} Re(\mathbf{E} \times \mathbf{B}^\star) \\ \approx \;\; & \frac{cE_0^2}{8\pi} \left\{ \left[1 - \frac{\sin[kr(1-\cos\phi) - \pi/4]}{C} \sqrt{\frac{\pi}{2kr}} \right] \hat{\mathbf{x}} \right. \\ & \left. + \left[\frac{\pi}{2C^2 kr} - \frac{\sin[kr(1-\cos\phi) - \pi/4]}{C} \sqrt{\frac{\pi}{2kr}} \right] \hat{\mathbf{r}} \right\}. \end{aligned} \tag{16}$$

The sine functions with arguments $kr(1-\cos\phi) - \pi/4$ oscillate extremely rapidly in ϕ for large r, and average to zero. Hence,

$$\langle \mathbf{S}(kr \gg 1) \rangle \approx \frac{cE_0^2}{8\pi} \left[\hat{\mathbf{x}} + \frac{\pi}{2C^2 kr}\, \hat{\mathbf{r}} \right] \equiv \langle S_0 \rangle\; \hat{\mathbf{x}} + \langle \mathbf{S}_{\rm scat} \rangle\,. \tag{17}$$

At large r the cross terms in the Poynting vector involving the incident and scattered fields can be neglected, and the scattered Poynting vector $\langle \mathbf{S}_{\rm scat} \rangle$ is entirely due to the scattered fields. This appealing decomposition does not hold at small r.

The differential scattering cross section is, according to eq. (1),

$$\frac{d\sigma_\parallel}{d\phi} \approx \frac{\pi}{2C^2 k}\,, \tag{18}$$

[1]This result appears at the top of p. 432 of [1].

and the total scattering cross section is,

$$\sigma_\parallel \approx \frac{\pi^2}{C^2 k} \approx \frac{\pi^2}{k \ln^2(2/ka)}\,, \tag{19}$$

which is much smaller than the wavelength λ, with a logarithmic dependence on the wire radius a.

The result (19) gives only a faint hint of the fact that a grid of wires with $a \ll \lambda$ and spacing d such that $a \ll d \ll \lambda$ is essentially totally reflecting for polarization parallel to the wires, as if the scattering cross section of each wire equals the spacing $d \gg \lambda$ [4].

2.2 Electric Field Polarized Perpendicular to the Wire

In this case the magnetic field is parallel to the z-axis, and we take $\psi = B_z$ and the condition that the tangential electric $E_\phi(a,\phi)$ field vanish at the surface of the wire becomes,

$$\begin{aligned} 0 &= \frac{1}{k}\frac{\partial B_z(a,\phi)}{\partial r} = \frac{1}{k}\frac{\partial \psi(a,\phi)}{\partial r} = iE_0\, e^{ika\cos\phi}\cos\phi + \sum_n A_n H_n^{(1)\prime}(ka)\cos n\phi \\ &\approx iE_0\cos\phi - kaE_0\cos^2\phi + \sum_n A_n H_n^{(1)\prime}(ka)\cos n\phi \\ &\approx -\frac{ka}{2}E_0 + iE_0\cos\phi - \frac{ka}{2}E_0\cos 2\phi + \sum_n A_n H_n^{(1)\prime}(ka)\cos n\phi. \end{aligned} \tag{20}$$

Thus, $A_n = 0$ for $n > 2$,

$$\frac{A_0}{E_0} \approx \frac{ka}{2H_0^{(1)\prime}(ka)} = -\frac{ka}{2H_1^{(1)}(ka)} \approx -\frac{i\pi k^2a^2}{4}\,, \tag{21}$$

$$\frac{A_1}{E_0} \approx -\frac{i}{H_1^{(1)\prime}(ka)} \approx -\frac{\pi k^2 a^2}{2}\,, \tag{22}$$

$$\frac{A_2}{E_0} \approx -\frac{i}{H_2^{(1)\prime}(ka)} \approx \frac{\pi k^3 a^3}{4} \approx 0\,, \tag{23}$$

noting that for $n > 0$ and small ka, $H_n^{(1)\prime}(ka) \approx iN_n'(ka) \approx i2^n n!/\pi(ka)^{n+1}$. The magnetic field for $r > a$ is, from eq. (8),[2]

$$\mathbf{B} \approx E_0\, e^{-i\omega t}\left[e^{ikx} - \frac{i\pi k^2a^2}{4}H_0^{(1)}(kr) - \frac{\pi k^2 a^2}{2}H_1^{(1)}(kr)\cos\phi\right]\hat{\mathbf{z}}, \tag{24}$$

and the electric field follows from the fourth Maxwell equation as,

$$\begin{aligned} \mathbf{E} = \frac{i}{k}\boldsymbol{\nabla}\times\mathbf{B} \;\approx\; & E_0\, e^{-i\omega t}\left\{e^{ikx}\,\hat{\mathbf{y}} + \frac{i\pi ka^2}{2r}H_1^{(1)}(kr)\sin\phi\,\hat{\mathbf{r}}\right. \\ & \left. + \frac{\pi k^2a^2}{2}\left[\frac{H_1^{(1)}(kr)}{2} + i\left(H_0^{(1)}(kr) - \frac{H_1^{(1)}(kr)}{kr}\right)\cos\phi\right]\hat{\boldsymbol{\phi}}\right\}. \end{aligned} \tag{25}$$

[2]This result appears near the bottom of p. 434 of [1].

For large r the electric and magnetic field are given by,

$$\mathbf{B}(kr \gg 1) \approx E_0\, e^{-i\omega t} \left[e^{ikx} - k^2 a^2 \sqrt{\frac{i\pi}{2kr}}\, e^{ikr} \left(\frac{1}{2} - \cos\phi \right) \right] \hat{\mathbf{z}}, \tag{26}$$

and,

$$\mathbf{E}(kr \gg 1) \approx E_0\, e^{-i\omega t} \left[e^{ikx}\, \hat{\mathbf{y}} - k^2 a^2 \sqrt{\frac{i\pi}{2kr}}\, e^{ikr} \left(\frac{1}{2} - \cos\phi \right) \hat{\boldsymbol{\phi}} \right]. \tag{27}$$

The time-average Poynting vector in the far zone is, again neglecting the rapidly oscillating cross terms,

$$\langle \mathbf{S}(kr \gg 1) \rangle \approx \frac{cE_0^2}{8\pi} \left[\hat{\mathbf{x}} + \frac{\pi k^3 a^4}{2r} \left(\frac{1}{4} - \cos\phi + \cos^2\phi \right) \hat{\mathbf{r}} \right] \equiv \langle S_0 \rangle\, \hat{\mathbf{x}} + \langle \mathbf{S}_{\rm scat} \rangle . \tag{28}$$

The differential scattering cross section is, according to eq. (1),

$$\frac{d\sigma_\perp}{d\phi} \approx \frac{\pi k^3 a^4}{2} \left(\frac{1}{4} - \cos\phi + \cos^2\phi \right), \tag{29}$$

which is much larger in the backward hemisphere than in the forward, and the total scattering cross section is,

$$\sigma_\perp \approx \frac{3\pi^2 k^3 a^4}{4}\,, \tag{30}$$

which is small compared to the geometric cross section $2a$.

The result (30) anticipates that a grid of fine wires is essentially transparent to fields polarized perpendicular to the wires [4].

2.3 Surface Charge and Current on the Wire

The surface charge and current densities σ and $\mathbf{K}$ on the wires of radius a are given by,

$$\varsigma(\phi) = \frac{E_r(a,\phi)}{4\pi} = \frac{i}{4\pi ak} \frac{\partial B_z(a,\phi)}{\partial \phi}\,, \qquad \mathbf{K}(\phi) = \frac{c}{4\pi} \hat{\mathbf{r}} \times \mathbf{B}(a,\phi) = -\frac{ic}{4\pi k} \hat{\mathbf{r}} \times (\boldsymbol{\nabla} \times \mathbf{E}(a,\phi)). \tag{31}$$

When the electric field is polarized parallel to the wires, $\mathbf{E} = E_z\, \hat{\mathbf{z}}$ and,

$$\varsigma_\parallel = 0, \qquad K_{\parallel,z} = \frac{ic}{4\pi k} \frac{\partial E_z(a,\phi)}{\partial r}\,, \tag{32}$$

where E_z near the wires follows from eqs. (12),

$$\frac{E_z}{E_0}\, e^{i\omega t} \approx 1 + ikr\cos\phi + \frac{i\pi}{2C} H_0^{(1)}(kr), \tag{33}$$

with C given by eq. (11). Hence, the surface current is,

$$K_{\|,z} \approx \frac{c}{4\pi} E_0 \left(\frac{\pi}{2C} H_0^{(1)}(ka) - \cos\phi \right) e^{-i\omega t} \approx -\frac{c}{4\pi} \frac{iE_0}{Cka} e^{-i\omega t}. \tag{34}$$

since $1/C > ka$.

When the electric field is polarized perpendicular to the wires, the magnetic field is parallel to them, and,

$$\varsigma_\perp = \frac{i}{4\pi ak} \frac{\partial B_z(a,\phi)}{\partial \phi}, \qquad K_{\perp,\phi} = -\frac{c}{4\pi} B_z(a,\phi), \tag{35}$$

where B_z at $r = a$ follows from eqs. (24), neglecting the term of order k^2a^2/C,

$$\begin{aligned} \frac{B_z}{E_0} e^{i\omega t} &\approx 1 + ika\cos\phi + \frac{\pi k^2 a^2}{2} H_1^{(1)}(kr)\cos\phi \\ &\approx 1. \end{aligned} \tag{36}$$

Hence, the surface charge and current densities are,[3]

$$\varsigma_\perp \approx 0, \qquad K_{\perp,\phi} \approx -\frac{c}{4\pi} E_0 \, e^{-i\omega t}. \tag{37}$$

The current (34) for polarization parallel to the wire is much larger than that for the case of perpendicular polarization. The current density in a conducting mirror would be $K_{\text{mirror}} = cE_0/2\pi$, so the current in for polarization perpendicular to the wire roughly as expected if the wire were a piece of a mirror, while the current for polarization parallel to the wire is much larger than this expectation. *However, it has become popular to imply that the current for parallel polarization is as expected, and the current for perpendicular polarization is suppressed [5, 6].*

2.4 The Optical Theorem and Scattering by a Large Conducting Cylinder

The optical theorem (see, for example, the Appendix of [7]) states that the total scattering cross section is related to the imaginary part of the forward scattering amplitude according to,

$$\sigma = \frac{4\pi}{k} Im[f(0,0)], \tag{38}$$

where for scattering by a finite object of a plane wave the electric (or magnetic) field in the far zone is written, in spherical coordinates (r, θ, ϕ),

$$E(r \to \infty) = E_0 \, e^{i(kz-\omega t)} + f(\theta,\phi) \frac{e^{i(kr-\omega t)}}{r}. \tag{39}$$

[3]Note that the surface current circulates around the wire in this case, which behavior will also hold for a conducting elliptic cylinder.

Papas [8] showed that for scattering off a cylindrical object, with axis the z-axis and incident wave in the x-direction, the form (38) still holds provided we write the electric (or magnetic) field in the far zone (in cylindrical coordinates (r, ϕ, z)) as,[4]

$$E(r \to \infty) = E_0 \, e^{i(kx-\omega t)} + \sqrt{2\pi i} f(\phi) \frac{e^{i(kr-\omega t)}}{\sqrt{kr}} \,. \tag{40}$$

Comparing with the approximations (15) and (26) we obtain functions f that are purely real, and the optical theorem implies the cross sections are zero. To obtain small nonzero cross sections, we must make a better approximation, which does not ignore small imaginary parts.

We do this in a way that also permits analysis of cylinders of arbitrary radius, using a representation of plane waves in terms of Bessel functions due to Jacobi [9],

$$e^{ikx} = e^{ikr\cos\phi} = J_0(kr) + 2\sum_{n=1}^{\infty} i^n J_n(kr) \cos n\phi. \tag{41}$$

2.4.1 Electric Field Polarized Parallel to the Wire

Using eq. (41), we see that for the electric field polarized parallel to a conducting cylinder of radius a the wavefunction (8) will vanish at the surface of the cylinder if we set the Fourier coefficients to be [10],

$$A_0 = -E_0 \frac{J_0(ka)}{H_0^{(1)}(ka)} \,, \qquad A_n = -2i^n E_0 \frac{J_n(ka)}{H_n^{(1)}(ka)} \qquad (n > 0). \tag{42}$$

The asymptotic electric field can now be written as,

$$E_z(r \to \infty) = E_0 \, e^{i(kx-\omega t)} + i\sqrt{\frac{2i}{\pi}} E_0 \left(\frac{J_0(ka)}{H_0^{(1)}(ka)} + 2\sum_{n=1}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)} \cos n\phi \right) \frac{e^{i(kr-\omega t)}}{\sqrt{kr}} \,, \tag{43}$$

noting that $e^{-i\pi/4} = \sqrt{1/i} = -i\sqrt{i}$. Comparing with eq. (40), the scattering amplitude is

$$f(\phi) = \frac{i}{\pi} \left(\frac{J_0(ka)}{H_0^{(1)}(ka)} + 2\sum_{n=1}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)} \cos n\phi \right) , \tag{44}$$

and the optical theorem (38) tells us that the total scattering cross section is,

$$\sigma_\| = \frac{4}{k} Im \left(i\frac{J_0(ka)}{H_0^{(1)}(ka)} + 2i\sum_{n=1}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)} \right) = \frac{4}{k} \left(\frac{J_0^2(ka)}{\left| H_0^{(1)}(ka) \right|^2} + 2\sum_{n=1}^{\infty} \frac{J_n^2(ka)}{\left| H_n^{(1)}(ka) \right|^2} \right) . \tag{45}$$

For small cylinders, $ka \ll 1$, $J_0(ka) \approx 1$, $N_0(ka) \approx 2C/\pi$, $J_n(ka) \approx 0$ for $n > 0$ and hence,

$$\sigma_\|(ka \ll 1) \approx \frac{4}{kN_0^2(0)} = \frac{\pi^2}{C^2 k} \,. \tag{46}$$

[4] Papas' version may be the earliest statement of the optical theorem.

as previously found in eq. (19).

For large cylinders, $ka \gg 1$, $J_n^2(ka)/\left|H_n^{(1)}(ka)\right|^2 \approx \cos^2(ka - n\pi/2 - \pi/4)$ and the resulting expression (45) is the sum of rapidly oscillating terms. The convergence of this sum is poor for large ka, although Papas [8] showed that one can transform it into an integral form which yields,

$$\sigma_\parallel(ka \gg 1) \approx 4a, \tag{47}$$

twice the diameter of the wire. Corrections for finite values of ka are reviewed in [11].

2.4.2 Electric Field Polarized Perpendicular to the Wire

In this case we write the magnetic field, B_z, in the form (8), and the condition that the radial derivative $\partial B_z(a,\phi)/\partial r$ vanish at the surface of the wire is satisfied with the aid of the derivative of eq. (41) with respect to r, such that,

$$A_0 = -E_0 \frac{J_0'(ka)}{H_0^{(1)\prime}(ka)}, \qquad A_n = -2i^n E_0 \frac{J_n'(ka)}{H_n^{(1)\prime}(ka)} \qquad (n > 0). \tag{48}$$

The asymptotic magnetic field can now be written as,

$$B_z(r \to \infty) = E_0\, e^{i(kx-\omega t)} + i\sqrt{\frac{2i}{\pi}} E_0 \left(\frac{J_0'(ka)}{H_0^{(1)\prime}(ka)} + 2\sum_{n=1}^{\infty} \frac{J_n'(ka)}{H_n^{(1)\prime}(ka)} \cos n\phi \right) \frac{e^{i(kr-\omega t)}}{\sqrt{kr}}. \tag{49}$$

Comparing with eq. (40), the scattering amplitude is,

$$f(\phi) = \frac{i}{\pi} \left(\frac{J_0'(ka)}{H_0^{(1)\prime}(ka)} + 2\sum_{n=1}^{\infty} \frac{J_n'(ka)}{H_n^{(1)\prime}(ka)} \cos n\phi \right), \tag{50}$$

and the optical theorem (38) tells us that the total scattering cross section is,

$$\sigma_\perp = \frac{4}{k} Im \left(i\frac{J_0'(ka)}{H_0^{(1)\prime}(ka)} + 2i\sum_{n=1}^{\infty} \frac{J_n'(ka)}{H_n^{(1)\prime}(ka)} \right) = \frac{4}{k} \left(\frac{J_0'^2(ka)}{\left|H_0^{(1)\prime}(ka)\right|^2} + 2\sum_{n=1}^{\infty} \frac{J_n'^2(ka)}{\left|H_n^{(1)\prime}[ka)\right|^2} \right). \tag{51}$$

For small cylinders, $ka \ll 1$, $J_0'(ka) = -J_1(ka) \approx -ka/2$, $N_0'(ka) = -N_1(ka) \approx 2/\pi ka$, $J_n'(ka) \approx (ka)^{n-1}/2^n(n-1)!$, $N_1'(ka) \approx 2^n n!/\pi(ka)^{n+1}$, for $n > 0$ and hence,

$$\sigma_\perp(ka \ll 1) \approx \frac{3\pi^2 k^3 a^4}{4}, \tag{52}$$

as previously found in eq. (30).

For large cylinders, $ka \gg 1$, $J_n'^2(ka)/\left|H_n^{(1)\prime}(ka)\right|^2 \approx \sin^2(ka - n\pi/2 - \pi/4)$ and the resulting expression (51) is the sum of rapidly oscillating terms. I believe this sum converges to,

$$\sigma_\perp(ka \gg 1) \approx 4a \approx \sigma_\parallel(ka \gg 1). \tag{53}$$

In the optical limit, $ka \gg 1$, we expect the cross section to be independent of polarization, but it remains somewhat surprising that it is twice the geometric cross section in the absence of absorption. This fact is likely related to the nonplanar character of a wire, for which currents on its "sides" play a significant role, as anticipated by Young [12].

3 Scattering by a Small Conducting Elliptical Cylinder

The method of sec. 2 (determining the Fourier coefficients of a solution to the Helmholtz wave equation using the good-conductor boundary condition at the surface of the cylinder) was applied to the case of an elliptic cylinder by Sieger [13], using elliptic cylindrical coordinates. However, the expansion functions appropriate for elliptic coordinates are not well known, and this approach has been used only infrequently [14, 15, 16, 17]. The notation of this section follows Brooker [17].

A flat conducting strip can be considered as a limit of a conducting elliptic cylinder as its minor axis goes to zero. In this limit the strip still has two sides that can, in general, support different charge and currents densities, and which can be singular along the edges of the strip. The analysis of elliptic cylinders in [17] assumed that the fields obey certain symmetry conditions (see, for example, sec. 11.2 of [18] and also [19]) that are valid for thin strips but not for general elliptic cylinders. The attempt here is to study elliptic cylinders in greater generality. *However, we will only obtain simple results for elliptic cylinders that are nearly circular. Brooker reports (private communication) that he has now obtained results for general elliptic cylinders which remain valid in the limit of thin strips.*

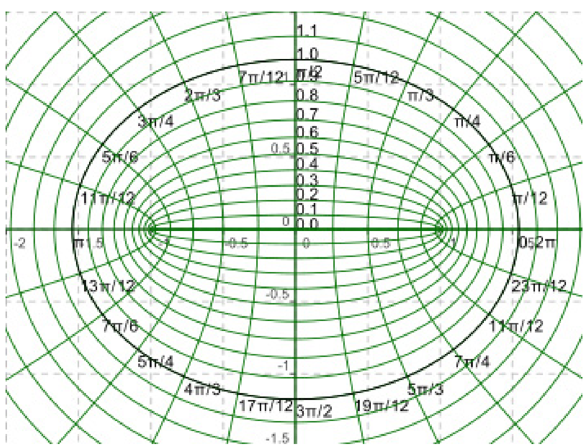
For elliptical cylinders with focal axes parallel to the z-axis in the plane $y = 0$, the relevant elliptic cylindrical coordinates (μ, φ, z) are related to rectangular coordinates by,

$$x = b\,\cosh\mu\cos\varphi, \qquad y = b\,\sinh\mu\sin\varphi. \tag{54}$$

Curves of constant μ are ellipses, and curves of constant φ are hyperbolae, with foci at $(x, y) = (0, \pm b)$. For large μ, cylindrical coordinates (r, ϕ) are related to (μ, φ) by $\mu \approx \ln(2r/b)$ and $\varphi \approx \phi$. The semimajor axis a of an ellipse of constant μ_a has value,

$$a = b\,\cosh\mu_a, \qquad \mu_a = \cosh^{-1}\frac{a}{b}\,, \qquad \sinh\mu_a = \sqrt{\frac{a^2}{b^2} - 1}. \tag{55}$$

In the figure below, the x-axis is to the right and the y-axis is up.



The Helmholtz equation is separable in elliptic cylindrical coordinates, and wavefunctions of the form $M(\mu)\Phi(\varphi)$ lead to versions of Mathieu's equation for both $M(\mu)$ and $\Phi(\varphi)$ in which the wavenumber k appears in a parameter,

$$q = \frac{k^2 b^2}{4}\,. \tag{56}$$

Since parameter b is less than the semimajor axis a of the elliptic cylinder, $q \ll 1$ for small cylinders with $ka \ll 1$. We will only consider small cylinders here, such that we can take $q \approx 0$.

The expansion functions M and Φ depend on q as well as on μ or φ, and can be labeled by an integer index n. Both cosinelike and sinelike functions exist, called $Mc_n(\mu, q)$ and $Ms_n(\mu, q)$ for M, and $ce_n(\varphi, q)$ and $se_n(\varphi, q)$ for Φ. Solutions to the Helmholtz equation involve products of cosinelike functions, or products of sinelike functions.

For small q,

$$\begin{aligned} ce_0(\varphi, q \ll 1) &\approx \frac{1}{\sqrt{2}}, &(57)\\ ce_n(\varphi, q \ll 1) &\approx \cos n\varphi \qquad (n > 0), &(58)\\ se_n(\varphi, q \ll 1) &\approx \sin n\varphi \qquad (n > 0), &(59)\end{aligned}$$

To represent the outgoing scattered wave we need "radial" functions M with asymptotic behavior $e^{ikr}/\sqrt{r}$, which are the so-called Mathieu functions of the third kind, $Mc_n^{(3)}(\mu, q)$ and $Ms_n^{(3)}(\mu, q)$. These functions have been defined so as to have asymptotic behavior very similar to that of the Hankel function $H_n^{(1)}$,

$$Mc_n^{(3)}(\mu \to \infty, q) \frac{e^{i[kr-(2n+1)\pi/4]}}{\sqrt{kr}}, \qquad (-1)^n Ms_n^{(3)}(\mu \to \infty, q) = \frac{e^{i[kr+(2n-1)\pi/4]}}{\sqrt{kr}}. \qquad (60)$$

For small $q = k^2b^2/2$, the functions $M_n^{(3)}(\mu, q \ll 1)$ can be related to Hankel functions $H_n^{(1)}(2\sqrt{q}\cosh\mu) = H_n^{(1)}(kb\cosh\mu)$ according to eqs. 20.6.12-13 of [20], noting that the only nonzero coefficients $A_n^m(q = 0)$ or $B_n^m(q = 0)$ are $A_0^0(0) = 1/\sqrt{2}$ and $A_n^n(0) = B_n^n(0) = 1$ for $n > 0$, according to eq. 20.2.29, and hence,

$$ce_n(0, 0) = A_n^n(0), \qquad se_n'(0, 0) = nB_n^n(0). \qquad (61)$$

Then,

$$\begin{aligned} Mc_n^{(3)}(\mu, q \ll 1) &\approx H_n^{(1)}(kb\cosh\mu), &(62)\\ Ms_n^{(3)}(\mu, q \ll 1) &\approx \tanh\mu\, H_n^{(1)}(kb\cosh\mu), &(63)\end{aligned}$$

but $Ms_0^{(3)}(\mu, q) = 0$. It is important to note that the simple forms (62)-(63) hold only if $kb\cosh\mu$ is not small, *i.e.*, for large μ in which case the elliptic cylinder is very close in form to a circular cylinder.[5]

The general form of a solution ψ to the scalar Helmholtz equation is, for $q = 0$ as appropriate for small cylinders, and for incident waves whose direction makes angle ϕ_0 to the x-axis (in the x-y plane),

$$\begin{aligned} \psi\, e^{i\omega t} &= E_0\, e^{ik(x\cos\phi_0 + y\sin\phi_0)} \\ &\quad + \sum_{n=0}^{\infty} \left(A_n Mc_n^{(3)}(\mu, q \ll 1)\, ce_n(\mu, q \ll 1) + B_n Ms_n^{(3)}(\mu, q \ll 1)\, se_n(\mu, q \ll 1) \right) \\ &\approx E_0\, e^{ik(x\cos\phi_0 + y\sin\phi_0)} \\ &\quad + \sum_{n=0}^{\infty} \left(A_n A_n^n(0)\cos n\varphi + B_n B_n^n(0)\tanh\mu\sin n\varphi \right) H_n^{(1)}(kb\cosh\mu). \qquad (64)\end{aligned}$$

[5]Thanks to Geoff Brooker for pointing this out.

In the far field, where $\tanh\mu \approx 1$ and $\varphi \approx \phi$, this becomes,

$$\begin{aligned}\psi(kr \gg 1) \quad &\approx \quad E_0\, e^{i[k(x\cos\phi_0 + y\sin\phi_0) - \omega t]} \\ &\quad + \sqrt{\frac{2i}{\pi kr}}\, e^{i(kr-\omega t)} \sum_{n=0}^{\infty} (-i)^{n+1} \left(A_n A_n^n(0)\cos n\phi + (-1)^n B_n B_n^n(0)\sin n\phi\right).\end{aligned} \tag{65}$$

3.1 Electric Field Polarized Parallel to the Cylinder

We again consider the scalar function $\psi = E_z$, now subject to the condition that $\psi(\mu,\varphi) = 0$ on the elliptical surface $\mu = \mu_a$ of semimajor axis $a \ll \lambda$ given by eq. (55).

For small x the incident waveform is approximately,

$$E_0[1 + ik(x\cos\phi_0 + y\sin\phi_0)] = E_0[1 + ikb(\cosh\mu\cos\varphi\cos\phi_0 + \sinh\mu\sin\varphi\sin\phi_0)]. \tag{66}$$

The condition that $\psi(\mu_a,\varphi) = 0$ implies that the only nonzero Fourier coefficients are,

$$\frac{A_0}{E_0} \approx -\frac{1}{A_0^0(0)H_0^{(1)}(kb\cosh\mu_a)} \approx \frac{i\sqrt{2}\pi}{2\ln(ka/2)} = \frac{i\sqrt{2}\pi}{2C}, \tag{67}$$

$$\frac{A_1}{E_0} \approx -\frac{ikb\cosh\mu_a\cos\phi_0}{B_1^1(0)H_1^{(1)}(kb\cosh\mu_a)} \approx -\frac{ka\cos\phi_0}{N_1(ka)} \approx \frac{\pi k^2a^2\cos\phi_0}{2} \ll \frac{A_0}{E_0}, \tag{68}$$

$$\frac{B_1}{E_0} \approx -\frac{ikb\sinh\mu_a\sin\phi_0}{B_1^1(0)H_1^{(1)}(kb\cosh\mu_a)} \approx -\frac{kb\sinh\mu_a\sin\phi_0}{N_1(ka)} \approx \frac{\pi k^2a^2\sqrt{1-\frac{b^2}{a^2}}\sin\phi_0}{2} \ll \frac{A_0}{E_0} \tag{69}$$

The far-field (65) of E_z is the same as that given in eq. (14), and hence the scattering cross section for a near-circular elliptical cylinder of semimajor axis $a \ll \lambda$ is the same as that for a circular cylinder of the same radius, eqs. (18)-(19), for any angle of incidence ϕ_0.[6]

3.2 Electric Field Polarized Perpendicular to the Cylinder

In this case we take $\psi = B_z$, and the condition is that the tangential electric field $E_\varphi \propto \partial B_z/\partial\mu = \partial\psi/\partial\mu$ vanish on the surface of the elliptical cylinder $\mu = \mu_a$.

From eq. (64) with $x\cos\phi_0 + y\sin\phi_0 = b(\cosh\mu\cos\varphi\cos\phi_0 + \sinh\mu\sin\varphi\sin\phi_0)$,

$$\begin{aligned}\frac{\partial\psi(x \ll \lambda)}{\partial\mu}\, e^{i\omega t} \quad &\approx \quad E_0\, ikb(\sinh\mu\cos\varphi\cos\phi_0 + \cosh\mu\sin\varphi\sin\phi_0) \\ &\quad [1 + ikb(\cosh\mu\cos\varphi\cos\phi_0 + \sinh\mu\sin\varphi\sin\phi_0)] \\ &\quad + \sum_{n=0}^{\infty} \Bigg(kb\sinh\mu A_n A_n^n(0) H_n^{(1)\prime}(kb\cosh\mu)\cos n\varphi \\ &\quad + \frac{B_n B_n^n(0)}{\cosh\mu}\left[\frac{H_n^{(1)}(kb\cosh\mu)}{\cosh\mu} + kb\sinh^2\mu\, H_n^{(1)\prime}(kb\cosh\mu)\right]\sin n\varphi\Bigg).\end{aligned} \tag{70}$$

[6]This result is implicit in [14], but was not made explicit there perhaps because of lack of knowledge of the Mathieu functions for large and small arguments.

The term in E_0 can be rewritten as,

$$
\begin{aligned}
E_0 \Bigg(& -\frac{k^2 b^2}{2} \cosh\mu \sinh\mu \\
& + ikb(\sinh\mu\cos\varphi\cos\phi_0 + \cosh\mu\sin\varphi\sin\phi_0) \\
& -\frac{k^2 b^2}{2}\left[\cosh\mu\sinh\mu\cos 2\varphi\cos 2\phi_0 + \frac{\cosh^2\mu + \sinh^2\mu}{2}\sin 2\varphi \sin 2\phi_0\right]\Bigg) . \qquad (71)
\end{aligned}
$$

The condition that $\partial\psi(\mu_a)/\partial\mu = 0$ on the ellipse μ_a (where $kb\cosh\mu_a = ka$) implies that the only nonzero Fourier coefficients are,

$$
\begin{aligned}
\frac{A_0}{E_0} &\approx \frac{kb\cosh\mu_a}{2A_0^0(0)H_0^{(1)\prime}(ka)} = -\frac{ka}{\sqrt{2}H_1^{(1)}(ka)} \approx -\frac{i\pi k^2 a^2}{2\sqrt{2}}\,, && (72)\\
\frac{A_1}{E_0} &\approx -\frac{ikb\cosh\mu_a\cos\phi_0}{A_1^1(0)H_1^{(1)}(ka)} \approx \frac{\pi k^2 a^2\cos\phi_0}{2}\,, && (73)\\
\frac{B_1}{E_0} &\approx -\frac{ikb\cosh^2\mu_a\sin\phi_0}{B_1^1(0)\left[H_1^{(1)}(ka)/\cosh\mu_a + kb\sinh^2\mu_a H_1^{(1)\prime}(ka)\right]} \approx \frac{\pi k^2 a^2 \sin\phi_0}{2}\,, && (74)\\
\frac{A_2}{E_0} &\approx -\frac{kb\cosh\mu_a\cos 2\phi_0}{2A_2^2(0)H_2^{(1)\prime}(ka)} \approx \frac{i\pi k^4 a^4\cos 2\phi_0}{16} \ll \frac{A_0}{E_0}\,, && (75)\\
\frac{B_2}{E_0} &\approx -\frac{kb\cosh\mu_a(\cosh^2\mu_a + \sinh^2\mu_2)\sin 2\phi_0}{4B_2^2(0)\left[H_2^{(1)}(ka)/\cosh\mu_a + kb\sinh^2\mu_a H_2^{(1)\prime}(ka)\right]} \approx \frac{i\pi k^4 a^4\sin 2\phi_0}{16} \ll \frac{A_0}{E_0} && (76)
\end{aligned}
$$

For large r the magnetic field now follows from eq. (65) as,

$$
\mathbf{B}(kr \gg 1) \approx E_0\, e^{-i\omega t}\left[e^{ik(x\cos\phi_0 + y\sin\phi_0)} - k^2 a^2\sqrt{\frac{i\pi}{2kr}}\,e^{ikr}\left(\frac{1}{2} - \cos(\phi-\phi_0)\right)\right]\hat{\mathbf{z}}, \qquad (77)
$$

which is the same as eq. (26) if we measure angle ϕ with respect to the angle of incidence ϕ_0. Hence the scattering cross section for a near-circular elliptical cylinder of semimajor axis $a \ll \lambda$, for any angle of incidence in the x-y plane, is the same as that for a circular cylinder of the same radius, eqs. (18)-(19).

3.3 Large Elliptical Cylinders

Scattering by an elliptical cylinder of arbitrary size can be discussed using the equivalent of Jacobi's identity (41) for the expansion of a plane wave in Mathieu functions. See, for example, eqs. (38) and (41) of [21]. However, we do not pursue this further here.

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