Some Mechanics of Toys

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Based on examples at

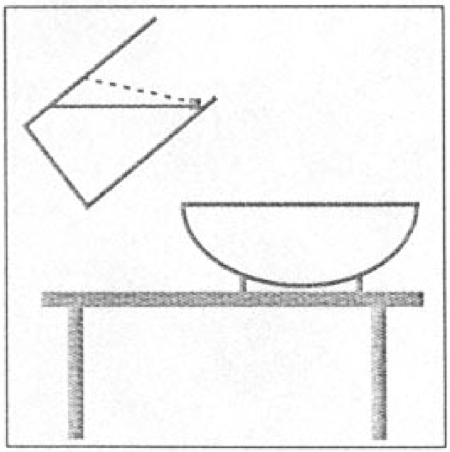
http:/kirkmcd.princeton.edu/examples/

Relative Horizontality

Psychology's famous water-level test—which involves drawing where the water level should be in a tilted container—has long intrigued researchers. It seems simple, but many grown-ups can't do it. And there's a pronounced sex difference: Men get it right much more often than do women.

Now researchers have chalked up another peculiarity of the test: Waitresses and bartenders, who spend a lot of time handling fluid in tilted glasses, do very badly on it. This finding, Heiko Hecht and Dennis Proffitt write in the March issue of *Psychological Science*, "is, to our knowledge, the only documented case in which performance declines with experience."

Proffitt, a psychologist at the University of Virginia, says the study first occurred to him 15 years ago when "I encountered the first male with a Ph.D. ever to get the problem wrong." He was a psychopharmacologist "who spends most of his day swishing things around in test tubes." Proffitt reasoned that the man's lab work had taught him to use an "object-relative" approach—relating the water level to the angle of the container—rather than an "environment-relative" approach that would have related it to a horizontal plane. The experiment finally took shape when



Draw the water level. Average error for waitresses was 27 degrees *(dotted line)*.

Hecht, Proffitt's graduate student, got a job in Munich, site of the Oktoberfest, where waiters carry a half-dozen beer steins in each hand.

Hecht gave the test to six groups: Oktoberfest waitresses, male bartenders, male truck drivers, housewives, and male and female graduate students. The results: Experience counted—badly. "Waitresses and bartenders taken together made larger errors than all other subjects," the authors report. The magnitude of the difference was similar to the difference between the sexes. Among the waitresses and bartenders, 32.5% gave correct answers, as opposed to 52.5% for the truck drivers and housewives. (Grad students did better.)

Proffitt says the results confirm that "the more likely you are to evaluate the situation relative to the container, the more likely there would be error." Psychologist Lynn Liben of Pennsylvania State University, who has been studying the water-level problem for 20 years, says the study helps show that it's "not a simple relationship between experience in the real world and what that means in terms of conceptual representation of that world."

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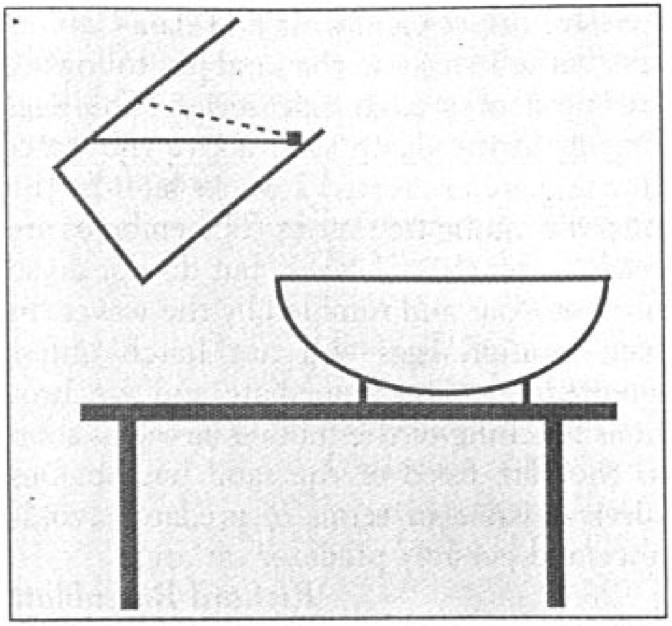

Water level test. Acceleration could account for subjects' "error" (dotted line).

Accelerating Fluid

In the Random Samples item "Relative horizontality" (28 Apr., p. 503), it is reported that people who frequently move liquids rapidly in open containers (waitpersons and bartenders) seem not to appreciate that the static surface of a liquid is "horizontal."

However, these people are paid not to spill the liquids, whose surfaces are often very near the rims of the containers. When one accelerates a liquid, its surface tends to be perpendicular to the effective gravity vector obtained by subtracting the acceleration vector from the ordinary downward gravity vector.

For example, a waitperson might accelerate a cup of coffee over its first meter of travel in 0.5 seconds, corresponding to an acceleration, a = 2(distance)/(time)^2, or 8 meters/(second)^2, which is nearly the pull of Earth's gravity (g = 9.8 meters/(second)^2. During this acceleration, the surface of the liquid would approach an angle, θ, where tangent $\theta = a/g$ = 8/9.8, or θ = 39 degrees.

To save his or her job, the waitperson would be well advised to tilt the cup during the initial acceleration, restoring it to the horizontal only during the steady walk to the table, and then giving it a reverse tilt as the cup is decelerated onto the table.

Thus, these workers might well respond to the psychologists' water level test by noting that in situations in which the surface of a liquid is not horizontal, the container has usually been tipped to keep the surface parallel to the rim.

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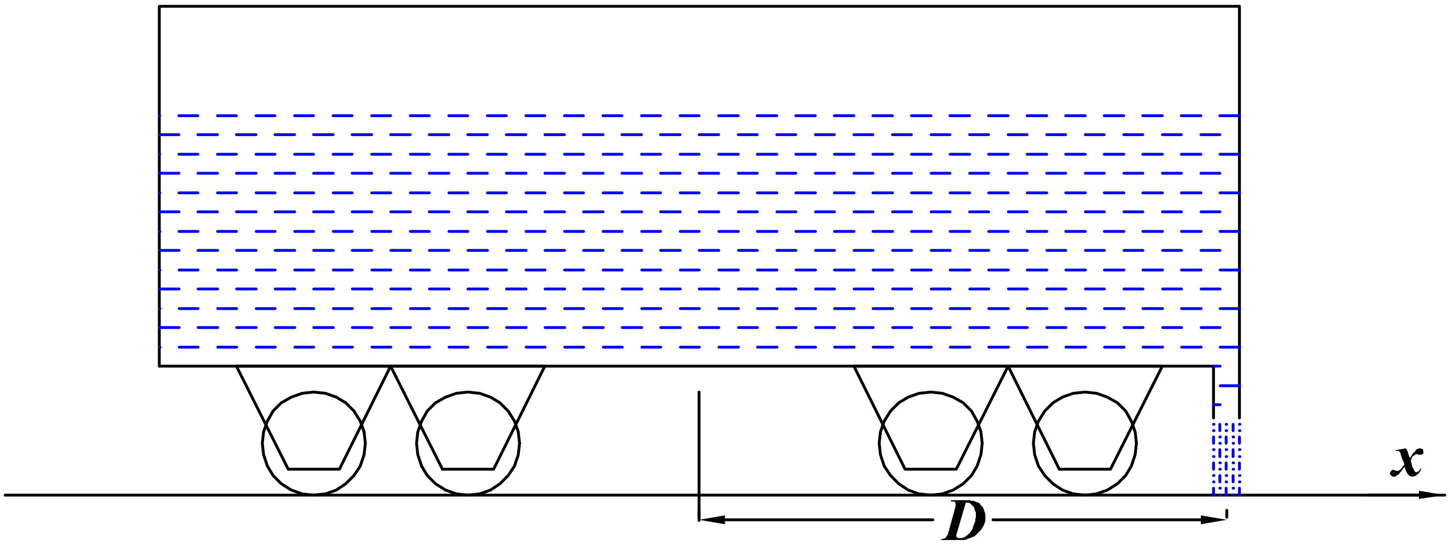
Motion of a Leaky Tank Car



The water exits the tank car through a drain at one end.

The water leaves the drain vertically – from the point of view of the tank car.

Ignore rolling friction.

What is the motion of the leaky tank car?

Motion of a Leaky Tank Car

No horizontal force on the system $\Rightarrow$ center of mass remains fixed at $x = 0$.

Water initially leaks out at $x > 0, \Rightarrow$, c.m. of tank initially moves in negative x direction.

But if the tank keeps rolling in the $-x$ direction, the total momentum of tank + water will become negative.

So, the tank must reverse direction and move in the $+x$ direction after a while!

The water inside the tank moves relative to the tank in the $+x$ direction, and pushes the tank in this direction.

Eventually this push is sufficient to reverse the initial negative velocity of the tank.

Motion of a Leaky Tank Car

Let $x(t)$ be the horizontal coordinate of the center of the tank car.

$m =$ mass of tank car (without water).

$x_{\rm drain} = d$ relative to center of tank car.

$M(t)$ is the mass of the water remaining in the tank.

$dM(t') =$ amount of water that drained out in the interval dt' centered on an earlier time t'.

$X(t, t') =$ horizontal coordinate at time t of the water that drained out at time t'.

The center of mass of the entire system must remain at the origin,

$$0 = (m + M(t))x(t) + \int dM(t')X(t, t').$$

Motion of a Leaky Tank Car

In the interval dt' at an earlier time t', mass $-\dot{M}(t')dt'$ of water drains out with horizontal velocity $\dot{x}(t')$ in the lab frame.

At time t' the drain was at $x(t') + D$, so at time t the element dM is at $X(t, t') = x(t') + D + \dot{x}(t')(t - t')$.

Thus the c.m. of the whole system,

$$0 = (m+M(t))x(t)-t\int_0^t dt'\dot{M}(t')\dot{x}(t')-\int_0^t dt'\dot{M}(t')[x(t')+D-t'\dot{x}(t')].$$

Take time derivatives,

First derivative : $$0 = (m+M)\dot{x}-\int_0^t dt'\dot{M}(t')\dot{x}(t')-\dot{M}D,$$

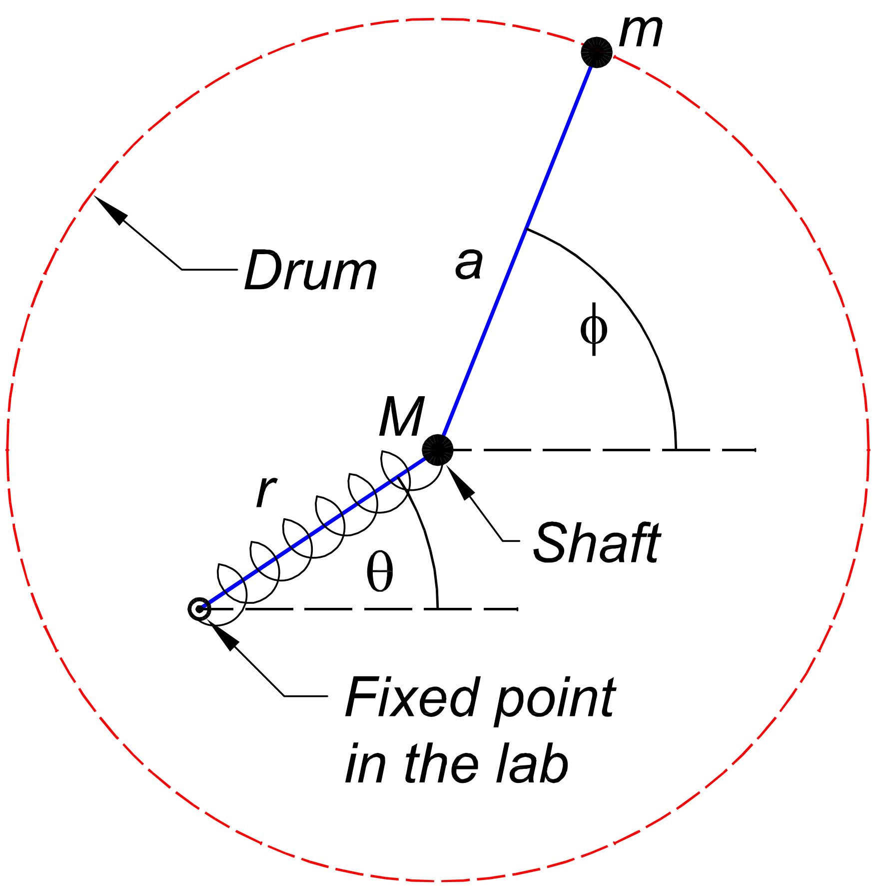
$\Rightarrow$ Total momentum of system is zero.

Second derivative: $$0 = (m + M)\ddot{x} - \ddot{M}D,$$

$\Rightarrow$ The force on the tank + water is just the reaction force $\ddot{M}D$ of the acceleration of the water relative to the tank.

This can be integrated for simple hypotheses as to the velocity of the water as it leaves the drain....

Mechanics of a Washing Machine



The drum and symmetrical part of the load have mass M.

The shaft of the drum is at (r, θ) and is connected to the origin by a zero-length spring of constant k (and damping factor γ).

An unbalanced load of mass m lies at distance a from the center of the drum and at angle ϕ with respect to a fixed direction in the laundromat.

The washer motor turns the drum with angular velocity $\dot{\phi} = \Omega$.

Mechanics of a Washing Machine

Equations of motion:

$$M\ddot{\mathbf{r}} + m\ddot{\mathbf{r}}_m = -k\,\mathbf{r} - \gamma\,\dot{\mathbf{r}},$$

where $\mathbf{r}_m = (r + a\cos(\phi - \theta))\,\hat{\mathbf{r}} + a\sin(\phi - \theta)\,\hat{\theta}$,

$$r \text{ component}: \qquad \ddot{r} = r\,\dot{\theta}^2 + b\,\Omega^2\cos(\phi - \theta) - \omega_0^2\,r - \Gamma\,\dot{r},$$

$$\theta \text{ component}: \qquad r\,\ddot{\theta} = -2\dot{r}\,\dot{\theta} + b\,\Omega^2\sin(\phi - \theta) - \Gamma\,r\,\dot{\theta},$$

where $\qquad \omega_0 = \sqrt{\dfrac{k}{m+M}}, \qquad b = \dfrac{m}{m+M}a, \qquad \Gamma = \dfrac{\gamma}{m+M}.$

$b =$ distance from shaft of drum to c.m.

$b\,\Omega^2 =$ centrifugal force.

$-2\dot{r}\dot{\theta} =$ Coriolis force.

Mechanics of a Washing Machine

Steady Motion: $\dot{r} = 0$, $\ddot{r} = 0$ and $\ddot{\theta} = 0$.

The shaft of the drum moves in a circle of radius r_0 and the mass m is at constant azimuth $\phi_0 = \phi - \theta$ relative to the azimuth of the shaft.

$$r_0 = \frac{b\,\Omega^2}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \Gamma^2\Omega^2}}, \qquad \tan\phi_0 = \frac{\Gamma\Omega}{\omega_0^2 - \Omega^2}.$$

Balanced load $\Rightarrow m = 0, \quad \Rightarrow \quad b = r_0 = 0.$

Unbalanced load:

Low spin, $\quad \Omega \ll \omega_0, \quad \Rightarrow \quad \phi_0 \approx 0,$

High spin, $\quad \Omega \gg \omega_0, \quad \Rightarrow \quad \phi_0 \approx \pi.$

A kind of inverted pendulum!

$$r_{\rm cm} = \frac{b\,\omega_0^2}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \Gamma^2\Omega^2}}.$$

At high spin, the center of mass of the system approaches the origin, although the shaft of the drum is off center in the lab.

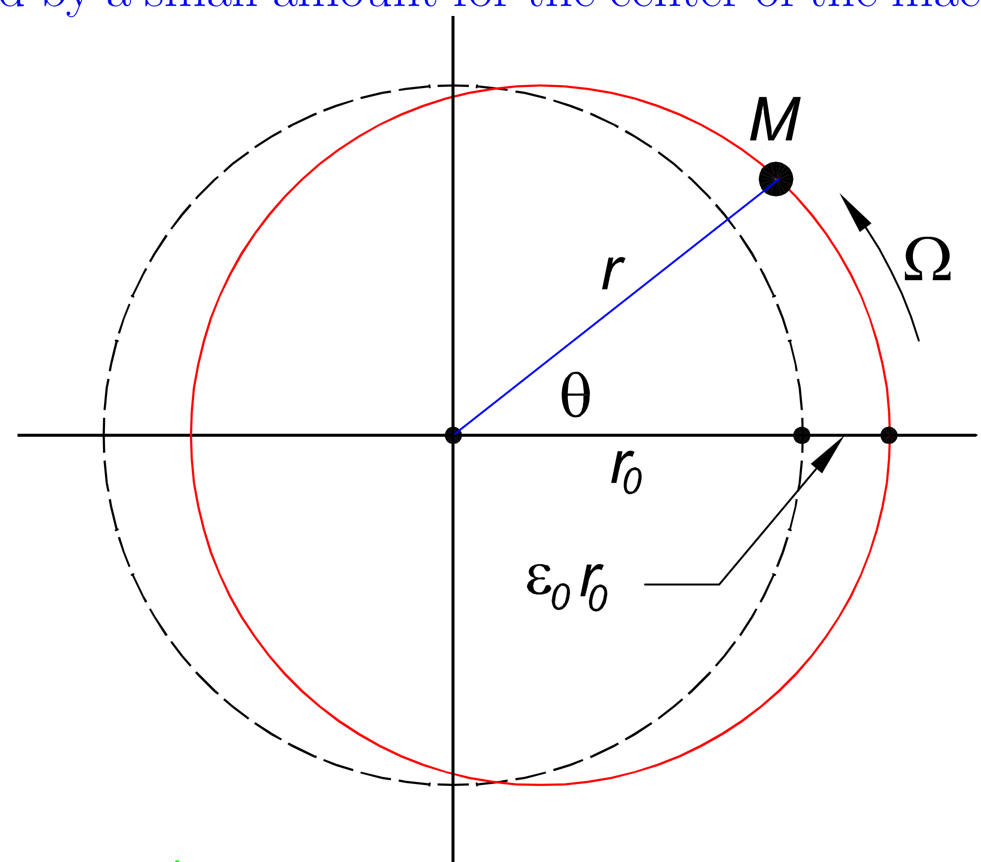
Mechanics of a Washing Machine

An unbalanced load would be unstable for $\Omega > \omega_0$ if only radial motion of the shaft were possible.

$$\phi - \theta = \phi_0 \quad \Rightarrow \quad \dot{\theta} = \Omega \quad \Rightarrow \quad \ddot{r} = (\Omega^2 - \omega_0^2)r + b\,\Omega^2 \cos\phi_0 - \Gamma\dot{r}.$$

Stability against perturbations is due to the Coriolis force!

At high spin, the perturbed motion is a circle of radius r_0 whose center is displaced by a small amount for the center of the machine in the lab frame.



Try it yourself sometime!

A Toy with a Gravitational Critical Radius

A particle on a surface of revolution $r = -k/z$ $(z < 0)$ about a vertical axis experiences an inward horizontal force kg/r^2.

Such toys appear in science museums to illustrate orbital motion under the influence of gravity.

Is there a surface of revolution, $r = r(z) \geq 0$, such that circular orbits are unstable for $r < r_{\text{critical}}$?

The surface may have a nonzero minimum radius R at which the slope dr/dz is infinite. Then the motion of a particle with $r < r_{\text{critical}}$ rapidly leads to excursions to the minimum radius R, after which the particle falls off the surface.

A Toy with a Gravitational Critical Radius

A stability analysis shows that the frequency ω of small oscillations about a circular orbit of radius r_0 on a surface of revolution $r(z)$ is related by

$$\omega^2 = \Omega^2 \frac{3r_0'^2 - r_0 r_0''}{1 + r_0'^2},$$

where Ω is the angular frequency of the circular motion at r_0.

The orbit is unstable when $\omega^2 < 0$, *i.e.*, when $r_0 r_0'' > 3r_0'^2$.

Equivalently, the orbit is unstable wherever $(1/r^2)'' < 0$, *i.e.*, where the function $1/r^2$ is concave inwards.

Examples:

$$\text{Hyperboloid of revolution}: \qquad r^2 - z^2 = R^2,$$

$$r_{\rm min} = R, \qquad r_{\rm critical} = \frac{2\sqrt{3}R}{3} = 1.15 r_{\rm min}.$$

$$\text{Modified } \frac{k}{z}: \qquad r = -\frac{k}{z\sqrt{1-z^2}}, \qquad (-1 < z < 0),$$

$$r_{\rm min} = 2k, \qquad r_{\rm critical} = \frac{6k}{\sqrt{5}} = 1.34 r_{\rm min}.$$

Static Equilibrium in a Force Field

Impossible if $\mathbf{F} = \nabla V$, and $\nabla^2 V = 0$.

On the Nature of the Molecular Forces which regulate the Constitution of the Luminiferous Ether. By S. Earnshaw, M.A. *of St. John's College, Cambridge.*

[Read *March* 18, 1839.]

In order that V may be a maximum, we must have fulfilled the following conditions, viz.

$$d_f V = 0, \quad d_g V = 0, \quad d_h V = 0 \quad \ldots\ldots (1),$$

$$d_f^2 V, \quad d_g^2 V, \quad d_h^2 V \text{ all negative} \quad \ldots\ldots (2),$$

and

$$\left.\begin{array}{l} d_f^2 V . d_g^2 V > (d_f d_g V)^2 \\ d_g^2 V . d_h^2 V > (d_g d_h V)^2 \\ d_f^2 V . d_h^2 V > (d_f d_h V)^2 \end{array}\right\} \quad \ldots\ldots (3).$$

The most curious and perhaps least expected result of this assumption is, *that the molecular forces which regulate the vibrations of the ether do not vary according to Newton's law of universal gravitation:* and it is not a little remarkable, that a force, whether attractive or repulsive, varying according to this law, is the only one which *cannot possibly actuate* the particles of a *vibrating* medium.

Trans. Camb. Phil. Soc. **7**, 97 (1842).

ON THE SUSPENSION OF A BALL BY A JET OF WATER.

[*From the Fourth Volume of the Third Series of* "Memoirs of the Literary and Philosophical Society of Manchester." *Session* 1869–70.]

(*Read* March 8, 1870.)

When a ball made of cork, or any very light material, is placed in a concave basin, from the middle of which a jet of water rises to the height of four or five feet, the jet maintains the ball in suspension; that is to say, it takes and keeps it out of the basin. The ball is not kept in one position, it oscillates up and down the jet; nor is its centre kept exactly in a line with the jet, it often remains for a long time on one side of it. In fact, the ball appears to be in equilibrium when it is struck by the jet in a point about 45° below the horizontal circle. In this way, for some seconds at a time, the ball appears as though it were hanging to the jet, and then oscillates in an irregular manner about this position. If its oscillations become so great that it leaves the jet, it instantly drops, but in descending it generally comes back into the jet before it reaches the basin. The friction of the water causes the ball to spin rapidly; and as it moves about the jet, it spins sometimes in one direction, sometimes in another, always about a horizontal axis. Of the water which strikes the ball, part is immediately splashed off in all directions, part is deflected off at the tangent, and part adheres to the ball, and is carried round with it, until it is thrown off by centrifugal force.

OSBORNE REYNOLDS.

What provides the horizontal stability?

Levitation in a Fluid Jet

Consider the example of styrofoam balls in an air jet.

Three forces:

- Gravity: $\mathbf{F}_g = -mg\hat{z} = -\frac{4}{3}\pi a^3 \rho_{\rm ball} g\hat{\mathbf{z}}$, $\quad (\nabla^2 \phi_{\rm gravity} = 0)$.
- High-speed air drag: $\mathbf{F}_{\rm drag} = \frac{C_D}{2}\rho_{\rm air}\pi a^2 v^2 \hat{\mathbf{v}}$.
- Pressure-gradient effects: $F_{\nabla P,u} \approx -\frac{4}{3}\pi a^3 P' = \frac{2}{3}\pi a^3 \rho_{\rm air}\frac{\partial v^2}{\partial u}$, using Bernoulli's law, $P + \rho_{\rm air} v^2/2 = P_0$.

Only the third force can provide horizontal stability.

Velocity of the jet:

$$v_z \approx \frac{A}{z} e^{-r^2/2\beta^2 z^2} \approx \frac{A}{z}\left(1 - \frac{r^2}{2\beta^2 z^2}\right),$$

$$\nabla \cdot \mathbf{v} = 0 \Rightarrow v_r \approx -\frac{r}{2}\frac{\partial v_z}{\partial z} \approx \frac{Ar}{2z^2},$$

where β = cone angle of jet. [Need $\nabla \times \mathbf{v} \neq 0$, so $\nabla^2 \mathbf{v} \neq 0$.]

[This is a viscous flow pattern (Schlichting, Landau).]

Levitation in a Fluid Jet

Vertical equilibrium: $0 = F_g + F_{\text{drag},z} + F_{\nabla P,z}$

$$= -\frac{4}{3}\pi a^3 \rho_{\text{ball}} g + \pi a^2 \rho_{\text{air}} v^2 + \frac{2}{3}\pi a^3 \rho_{\text{air}} \frac{dv^2}{dz}$$

$$\Rightarrow \qquad \frac{1}{z_0^2}\left(1 - \frac{4a}{3z_0}\right) = \frac{4a\rho_{\text{ball}} g}{3A^2 \rho_{\text{air}}}.$$

Vertical stability : $\omega_{\text{vert}} = \sqrt{\frac{-F'(z_0)}{m}} = \frac{2\pi a^2 A^2 \rho_{\text{air}}}{m z_0^3}\left(1 - \frac{2a}{z_0}\right),$

so stable when $z_0 > 2a$ = diameter of ball.

Horizontal stability: $F_r(r, z_0) = F_{\text{drag},r} + F_{\nabla P,r}$

$$= \pi a^2 \rho_{\text{air}} v v_r + \frac{2}{3}\pi a^3 \rho_{\text{air}} \frac{\partial v^2}{\partial r}$$

$$\approx -\frac{4\pi a^3 A^2 \rho_{\text{air}} r}{3\beta^2 z_0^4}\left(1 - \frac{\beta^2}{4} - \frac{3\beta^2 z_0}{8a}\right) = -m\omega_{\text{horiz}}^2 r.$$

$\beta < 1$ is the cone angle.

For $\beta z_0 \approx 2a$, $\quad (...) \approx 1 - \beta^2 > 0$, $\quad \Rightarrow$ stable oscillations.

$z_0 \gg a, \quad \beta z_0 = 2a, \quad \Rightarrow \quad \frac{\omega_{\text{horiz}}}{\omega_{\text{vert}}} \approx \sqrt{\frac{z_0}{6a}} \approx 1 \quad \Rightarrow$ "jumpy".

The Levitron™

A magnetic dipole $\vec{\mu}$ of mass m in a gravtiational and a magnetic field has energy

$$U(r, z) = mgz - \vec{\mu} \cdot \mathbf{B}(r, z).$$

For a stable equilibrium we need

$$F_z = -\frac{\partial U(0, z_0)}{\partial z} = 0 = -mg + \vec{\mu} \cdot \frac{\partial \mathbf{B}(0, z_0)}{\partial z},$$

$$F_r = -\frac{\partial U(0, z_0)}{\partial r} = 0 = \vec{\mu} \cdot \frac{\partial \mathbf{B}(0, z_0)}{\partial r}.$$

For an equilibrium above a source with $B_z > 0$, we have $\partial B_z / \partial z < 0$, so $\vec{\mu}$ must be opposite to **B**.

But in magnetostatics, $\vec{\mu}$ will align with **B**.

$\Rightarrow$ Need a dynamic mechanism to keep $\vec{\mu}$ opposite to **B**.

Mechanical spin of the dipole provides the mechanism to defeat Earnshaw's theorem.

`http://www.physics/ucla.edu/marty/levitron/`

The Levitron™

$$\text{Torque equation :} \qquad \frac{d\mathbf{L}}{dt} = \vec{\mu} \times \mathbf{B} \, ,$$

$$\text{“Large” spin } \omega \qquad \Rightarrow \qquad \mathbf{L} = I\vec{\omega} = I\omega\frac{\vec{\mu}}{\mu} \, ,$$

$$\Rightarrow \qquad \frac{d\vec{\mu}}{dt} = -\frac{\mu\mathbf{B}}{I\omega} \times \vec{\mu} \, .$$

$\Rightarrow$ **L** precesses about **B** with angular velocity

$$\Omega = -\frac{\mu B}{I\omega} \, ,$$

$$\Rightarrow \qquad \vec{\mu} \cdot \mathbf{B} = \text{ const } = \mu B \cos\theta_0 \, ,$$

where θ_0 is the (constant) angle $\approx 180^\circ$ between $\vec{\mu}$ and **B**.

If ω too small, the dipole “falls over”;

If ω too big, then $\Omega < \omega_{\rm osc}$, $\quad \Rightarrow \quad$ $\vec{\mu}$ can’t stay aligned with **B**.

$$\frac{1}{r_{\rm gyration}} \left(\frac{\mu B_0}{m}\right)^{1/2} \lesssim \omega \lesssim \frac{1}{g r_{\rm gyration}^2} \left(\frac{\mu B_0}{m}\right)^{3/2} , \qquad \frac{\omega_{\rm max}}{\omega_{\rm min}} \approx 3.$$

The Levitron$^{\mathrm{TM}}$

Stability requires (in addition to $\omega_{\min} < \omega < \omega_{\max}$)

$$\frac{\partial^2 U(0,z_0)}{\partial z^2} = -\vec{\mu}\cdot\frac{\partial^2 \mathbf{B}(0,z_0)}{\partial z^2} > 0,$$

$$\frac{\partial^2 U(0,z_0)}{\partial r^2} = -\vec{\mu}\cdot\frac{\partial^2 \mathbf{B}(0,z_0)}{\partial r^2} > 0.$$

$\vec{\mu}$ opposite to local $\mathbf{B} \Rightarrow$ need

$$\frac{\partial^2 B(0,z_0)}{\partial z^2} = \frac{\partial^2 B_z(0,z_0)}{\partial z^2} > 0\,, \qquad \text{using} \qquad \frac{\partial B_r(0,z)}{\partial z} = 0,$$

$$\text{and} \qquad \frac{\partial^2 B(0,z_0)}{\partial r^2} = \frac{\partial^2 B_z(0,z_0)}{\partial r^2} + \frac{1}{B_0}\left(\frac{\partial B_r(0,z_0)}{\partial r}\right)^2 > 0\,,$$

where $B = \sqrt{B_r^2 + B_z^2}$ and $B_0 = B_z(0,z_0)$.

Magnetic field due to a uniformly magnetized disk of radius a

$$\Rightarrow \quad \text{Stable for} \quad \frac{a}{2} < z_0 < \frac{a}{\sqrt{2.5}}\,. \qquad \text{(Berry)}$$

Magnetic field due to a current loop of radius a

$$\Rightarrow \quad \text{Stable for} \quad \frac{a}{2} < z_0 < \frac{a}{\sqrt{2}}\,.$$

Euler's disk and its finite-time singularity

Air viscosity makes the rolling speed of a disk go up as its energy goes down.

It is a fact of common experience that if a circular disk (for example, a penny) is spun upon a table, then ultimately it comes to rest quite abruptly, the final stage of motion being characterized by a shudder and a whirring sound of rapidly increasing frequency. As the disk rolls on its rim, the point P of rolling contact describes a circle with angular velocity Ω. In the classical (non-dissipative) theory[1], Ω is constant and the motion persists forever, in stark conflict with observation. Here I show that viscous dissipation in the thin layer of air between the disk and the table is sufficient to account for the observed abruptness of the settling process, during which, paradoxically, Ω increases without limit. I analyse the nature of this 'finite-time singularity', and show how it must be resolved.

Let α be the angle between the plane of the disk and the table. In the classical description, and with the notation defined in Fig. 1, the points P and O are instantaneously at rest in the disk, and the motion is therefore instantaneously one of rotation about line PO with angular velocity ω, say. The angular momentum of the disk is therefore $\mathbf{h} = A\omega\mathbf{e}(t)$, where $A = \frac{1}{4}Ma^2$ is the moment of inertia of the disk of mass M about its diameter; $\mathbf{e}(t)$ is a unit vector in the direction PO; $\mathbf{e}_z$, $\mathbf{e}_d$ are unit vectors in the directions Oz, OC, respectively (see Fig. 1). In a frame of reference rotating with angular velocity $\boldsymbol{\Omega}_d = \Omega\mathbf{e}_z$, the disk rotates about its axis OC with angular velocity $\boldsymbol{\Omega}_d = \Omega_d\mathbf{e}_d$; hence the rolling condition is $\Omega_d = \Omega\cos\alpha$. The absolute angular velocity of the disk is thus $\boldsymbol{\omega} = \Omega(\mathbf{e}_d\cos\alpha - \mathbf{e}_z)$, and so

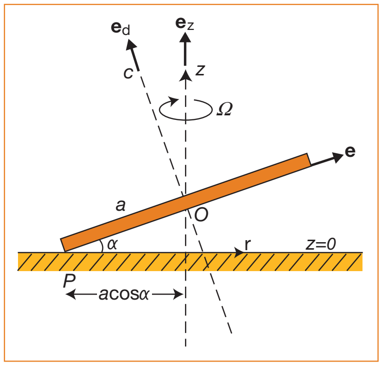


Figure 1 A heavy disk rolls on a horizontal table. The point of rolling contact P moves on a circle with angular velocity Ω. Owing to dissipative effects, the angle α decreases to zero within a finite time and Ω increases in proportion to $\alpha^{-1/2}$.

$\omega = \boldsymbol{\omega}\cdot\mathbf{e} = -\Omega\sin\alpha$.

Euler's equation for the motion of a rigid body is here given by $\mathrm{d}\mathbf{h}/\mathrm{d}t = \boldsymbol{\Omega}\wedge\mathbf{h} = \mathbf{G}$, where $\mathbf{G} = Mga\mathbf{e}_z\wedge\mathbf{e}$ is the gravitational torque relative to P ($\wedge$ indicates the vector product). This immediately gives the result $\Omega^2\sin\alpha = 4g/a$, or, when α is small,

$$\Omega^2\alpha \approx 4g/a \qquad (1)$$

The energy of the motion E is the sum of the kinetic energy $\frac{1}{2}A\omega^2 = \frac{1}{2}Mga\sin\alpha$, and the potential energy $Mga\sin\alpha$, so

$$E = \tfrac{3}{2}Mga\sin\alpha \approx \tfrac{3}{2}Mga\alpha \qquad (2)$$

In the classical theory, α, Ω and E are all constant, and the motion continues indefinitely. As observed above, this is utterly unrealistic.

Let us then consider one of the obvious mechanisms of energy dissipation, namely that associated with the viscosity μ of the surrounding air. When α is small, the dominant contribution to the viscous dissipation comes from the layer of air between the disk and the table, which is subjected to strong shear when Ω is large. We may estimate the rate of dissipation of energy in this layer as follows. Let (r,θ) be polar coordinates with origin at O. For small α, the gap $h(r,\theta,t)$ between the disk and the table is given by $h(r,\theta,t) \approx \alpha(a + r\cos\phi)$, where $\phi = \theta - \Omega t$. We now concede that α is a slowly varying function of time t: we assume that $|\dot\alpha| \ll \Omega$, and make the 'adiabatic' assumption that equation (1) continues to hold. Because the air moves a distance of order a in a time $2\pi/\Omega$, the horizontal velocity $\mathbf{u}_H$ in the layer has order of magnitude $r\Omega\sin\phi$; and as this velocity satisfies the no-slip condition on $z = 0$ and on $z = h$ $(= O(\alpha a))$, the vertical shear $|\partial\mathbf{u}_H/\partial z|$ is of the order $(r\Omega/\alpha a)|\sin\phi|$. The rate of viscous dissipation of energy Φ is given by integrating $\mu(\partial\mathbf{u}_H/\partial z)^2$ over the volume V of the layer of air: this easily gives $\Phi \approx \pi\mu g a^2/\alpha^2$, using equation (1). The fact that $\Phi \to \infty$ as $\alpha \to 0$ should be noted.

The energy E now satisfies $\mathrm{d}E/\mathrm{d}t = -\Phi$ (neglecting all other dissipation mechanisms). Hence, with E given by equation (2), it follows that

$$\tfrac{3}{2}Mga\,\mathrm{d}\alpha/\mathrm{d}t \approx -\pi\mu g a^2/\alpha^2 \qquad (3)$$

This integrates to give

$$\alpha^3 = 2\pi(t_0 - t)/t_1 \qquad (4)$$

where $t_1 = M/\mu a$, and t_0 is a constant of integration determined by the initial condition: if $\alpha = \alpha_0$ when $t = 0$, then $t_0 = (\alpha_0^3/2\pi)t_1$. What is striking here is that, according to equation (4), α does indeed go to zero at the finite time $t = t_0$. The corresponding behaviour of Ω is $\Omega \approx (t_0 - t)^{-1/6}$, which is certainly singular as $t \to t_0$.

Of course, such a singularity cannot be realized in practice: nature abhors a singularity, and some physical effect must intervene to prevent its occurrence. Here it is not difficult to identify this effect: the vertical acceleration $|\ddot h| = |a\ddot\alpha|$ cannot exceed g in magnitude (as the normal reaction at P must remain positive). From equation (4), this implies that the above theory breaks down at a time τ before t_0, where

$$\tau = t_0 - t \approx (2a/9g)^{3/5}(2\pi/t_1)^{1/5} \qquad (5)$$

A toy, appropriately called Euler's disk[2], is commercially available (Fig. 2; Tangent Toys, Sausalito, California). For this disk, $M = 400$ g, and $a = 3.75$ cm. With these values and with $\mu = 1.78\times10^{-4}$ g cm^{-1} s, $t_1 = M/\mu a \approx 0.8\times10^6$ s, and, if we take $\alpha_0 = 0.1(\approx 6°)$, we find $t_0 \approx 100$ s. This is indeed the order of magnitude (to within $\pm 20\%$) of the observed settling time in many repetitions of the spinning of the disk (with quite variable and ill-controlled initial conditions), that is, there is no doubt that dissipation associated with air friction is sufficient to account for the observed behaviour. The value of τ given by equation (5) is 10^{-2} s for the disk values given above; that is, the behaviour described by equation (4) persists until within 10^{-2} s of the singularity time t_0. At this stage, $\alpha \approx 0.5\times10^{-2}$, $h_0 = a\alpha \approx 0.2$ mm, $\Omega \approx 500$ Hz (and the adiabatic approximation is still well

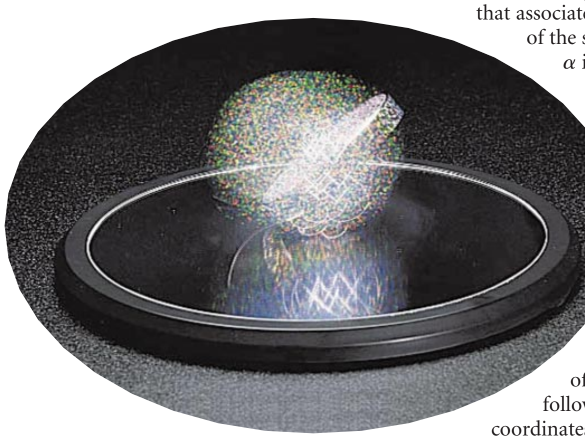
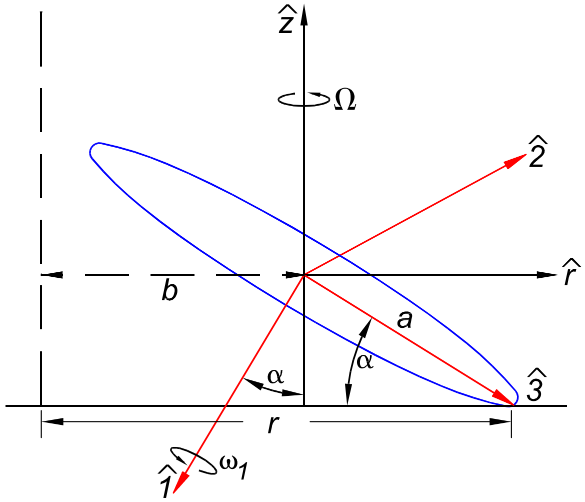


Figure 2 Euler's disk is a chrome-plated steel disk with one edge machined to a smooth radius. If it were not for friction and vibration, the disk would spin for ever. Photo courtesy of Tangent Toys. See http://www.tangenttoy.com/.

H.K. Moffatt

Euler's Disk



Center of disk at rest $\Rightarrow$ $\hat{\mathbf{3}}$ is the instantaneous axis of rotation.

$\Rightarrow$ Angular velocity $= \vec{\omega} = \omega\hat{\mathbf{3}}$, and $\mathbf{L} = I_{33}\omega\hat{\mathbf{3}} = kma^2\omega\hat{\mathbf{3}}$.

$$\mathbf{F} = mg\hat{\mathbf{z}}, \qquad \Rightarrow \qquad \mathbf{N} = a\hat{\mathbf{3}} \times mg\hat{\mathbf{z}} = \frac{d\mathbf{L}}{dt},$$

$$\Rightarrow \qquad \frac{d\mathbf{L}}{dt} = \vec{\Omega} \times \mathbf{L}, \qquad \text{where} \qquad \vec{\Omega} = -\frac{g}{ak\omega}\hat{\mathbf{z}}\,.$$

Also, $\vec{\omega} = \Omega\hat{\mathbf{z}} + \omega_{\rm rel}\hat{\mathbf{1}} = (\omega_{\rm rel} - \Omega\cos\alpha)\hat{\mathbf{1}} - \Omega\sin\alpha\hat{\mathbf{3}} = \omega\hat{\mathbf{3}},$

$$\Rightarrow \quad \omega = -\Omega\sin\alpha, \quad \omega_{\rm rel} = \Omega\cos\alpha, \quad \Omega^2 = \frac{g}{ak\sin\alpha}\,.$$

Euler's Disk

As $\alpha \to 0$, the velocity of the point of contact becomes large,

$\Rightarrow$ One hears a high-frequency sound.

But, one sees the rotation of the figure on the face of the disk, whose angular velocity $\Omega - \omega_{\rm rel} = \Omega(1 - \cos\alpha) \to 0$.

The total angular velocity ω also vanishes as $\alpha \to 0$.

Can $v_{\rm contact}$ exceed the speed of sound?

Does air drag become important as $\alpha \to 0$, $\Omega \to \infty$?

Euler's Disk

Energy : $$U = \frac{1}{2} m \dot{h}^2 + \frac{1}{2} I_{33} \omega^2 + mgh \approx \frac{1}{2} m a^2 \dot{\alpha}^2 + \frac{3}{2} m a g \alpha,$$

$\Rightarrow$ Power : $$P = \frac{dU}{dt} \approx m a^2 \dot{\alpha} \ddot{\alpha} + \frac{3}{2} m a g \dot{\alpha} \approx \frac{5}{2} m a g \dot{\alpha}.$$

Rolling friction?

Inelastic collisions with bumps of spacing δ, height $\epsilon\delta$,

$\Rightarrow$ Dissipated power : $$P = -\frac{m g \epsilon \delta}{\delta / a \Omega} = -\epsilon m a g \Omega.$$

General velocity dependent friction : $$P = -\epsilon m a g \Omega^{\beta},$$

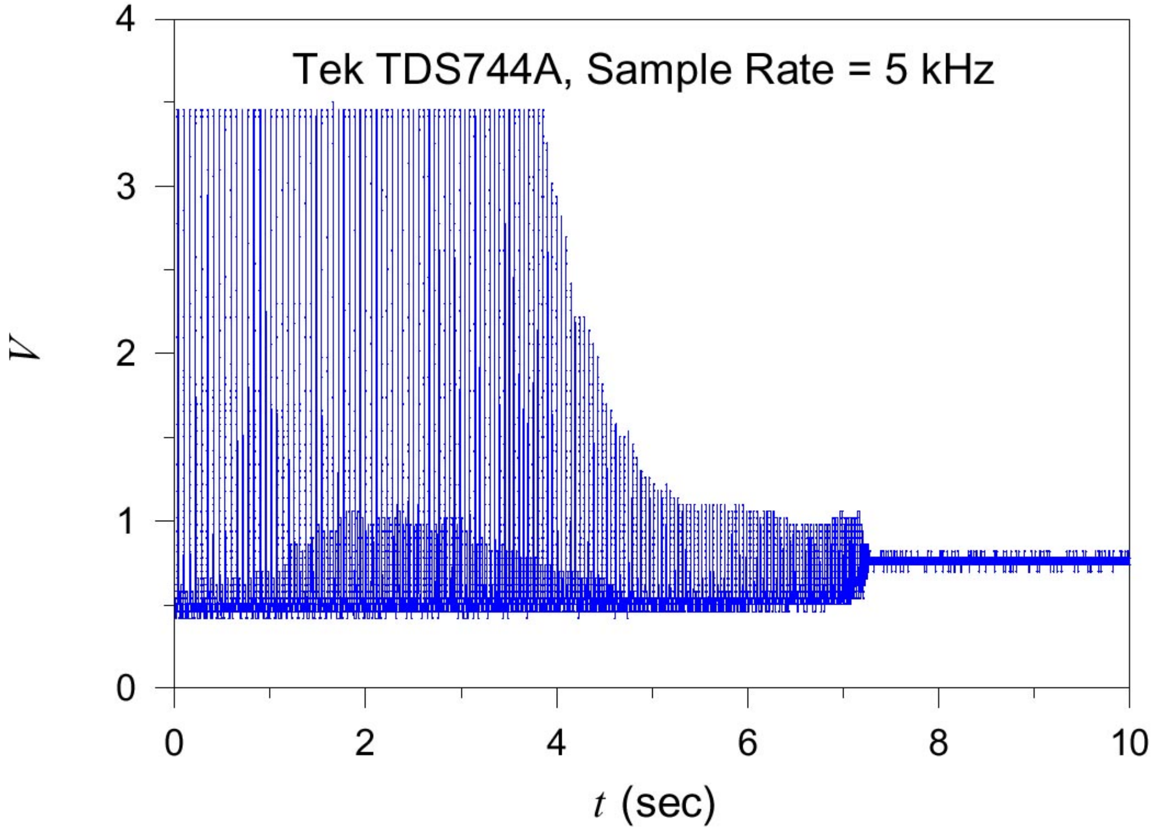
$\beta = 1$-2 for rolling friction (Ruina); $\beta = 4$ for air drag (Moffatt).

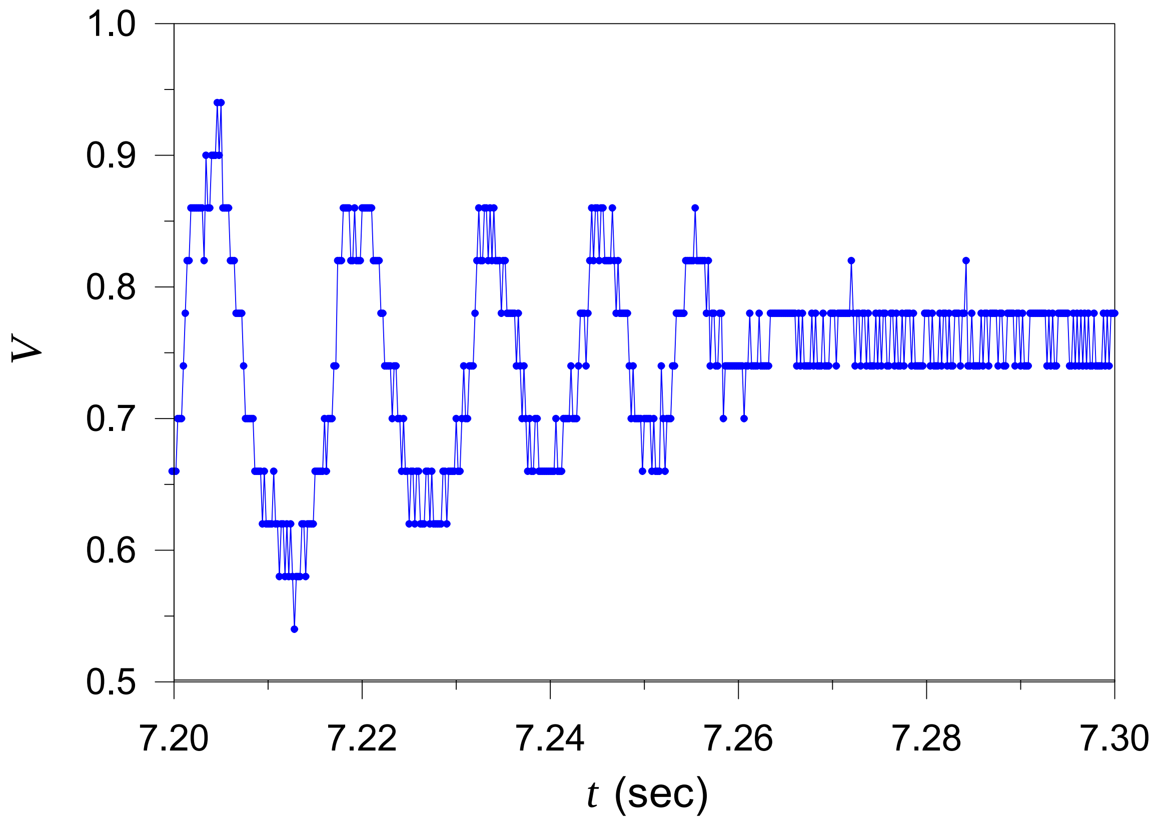
$$\Rightarrow \qquad \Omega(t) = \left(\frac{5g/\epsilon(\beta+2)ak}{t_0 - t} \right)^{1/(\beta+2)} \equiv \left(\frac{C}{t_0 - t} \right)^{1/(\beta+2)}.$$

$\Omega(t)$ appears to have a singularity at a finite time t_0.

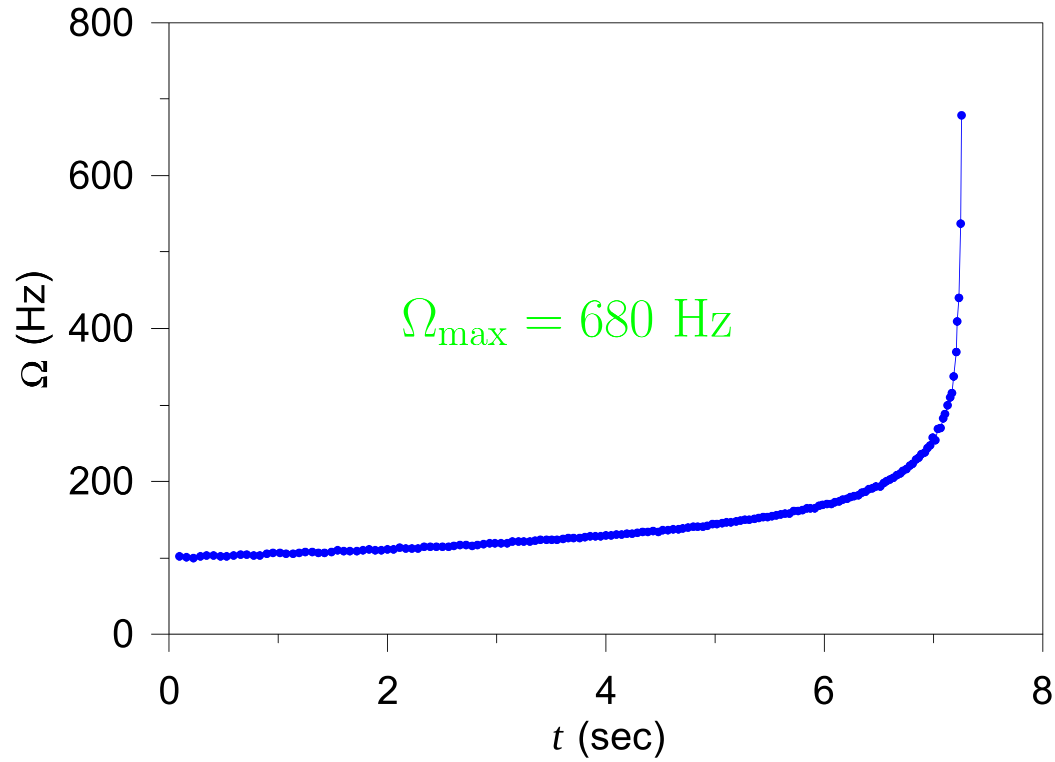
Can we determine β by experiment?

Euler's Disk

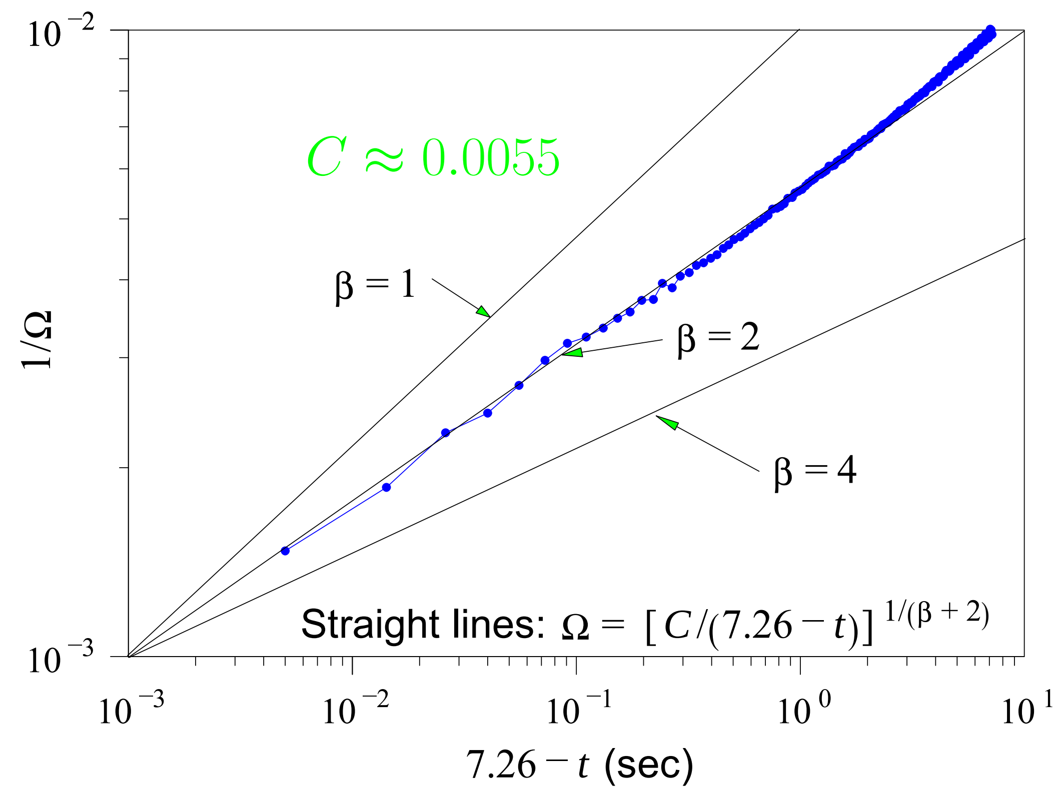

Tek TDS744A, Sample Rate = 5 kHz
V
t (sec)
0
1
2
3
4
0
2
4
6
8
10


V
t (sec)
0.5
0.6
0.7
0.8
0.9
1.0
7.20
7.22
7.24
7.26
7.28
7.30

Euler's Disk

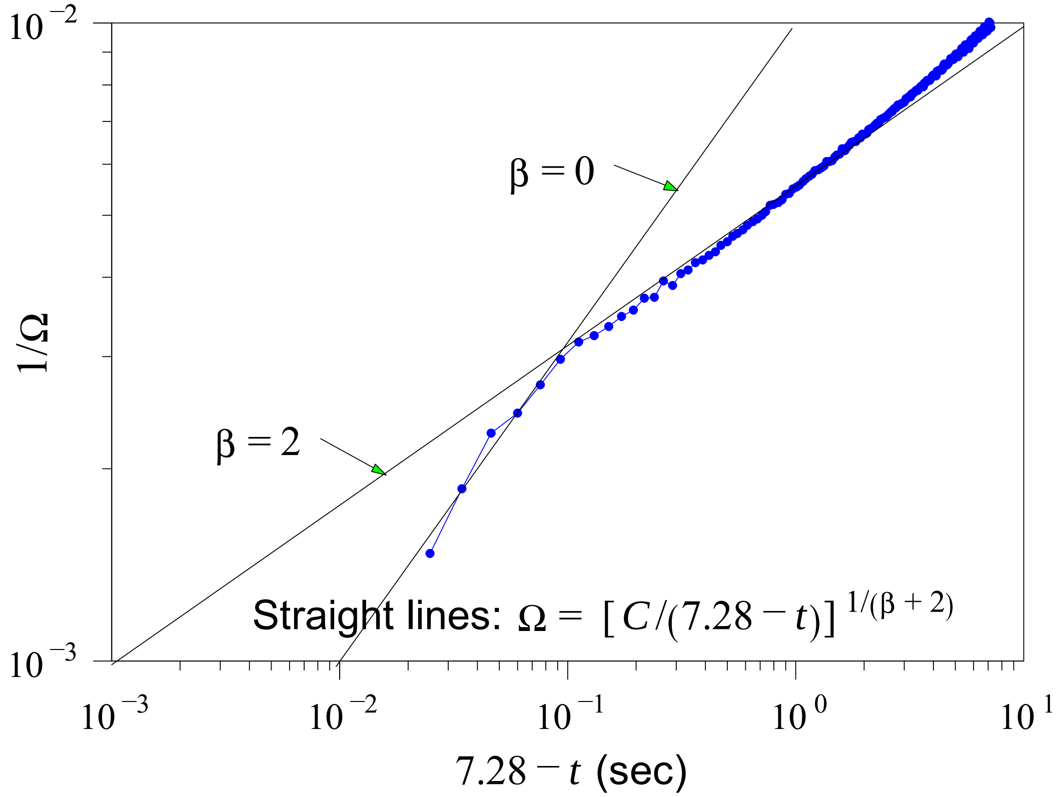


800
600
400
200
0
0
2
4
6
8
Ω (Hz)
t (sec)
Ωmax = 680 Hz



10 −2
10 −3
1/Ω
C ≈ 0.0055
β = 1
β = 2
β = 4
Straight lines: Ω = [C/(7.26 − t)] 1/(β + 2)
10 −3
10 −2
10 −1
10 0
10 1
7.26 − t (sec)

Euler's Disk



Ambiguity in determining t_0.

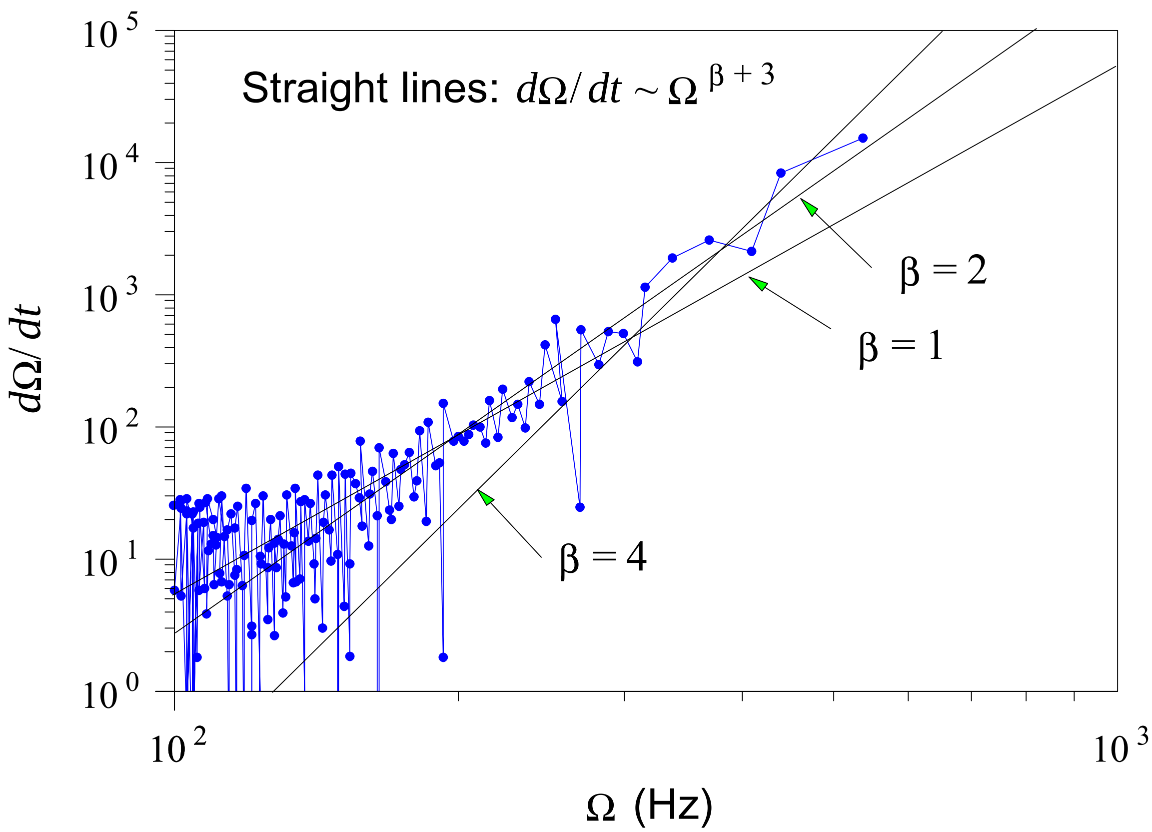
When exactly one cycle is left, $t_0 - t = 2\pi/\Omega(t)$.

$\beta = 2 \quad \Rightarrow \quad \Omega = (C/2\pi)^{1/3} \approx 580$ Hz, $\qquad t_0 - t \approx 0.011$ s.

Euler's Disk

Avoid use of t_0 via the relation $\dfrac{d\Omega}{dt} \propto \Omega^{\beta+3}$. (Chatterjee)

$d\Omega/dt$ calculated via second differences, $\Rightarrow$ greater error.



Results are not definitive, but it appears that $P_{\rm dissipated} \approx \Omega^2$ as for rolling friction.

It is not excluded that during the last few cycles $P_{\rm dissipated} \approx \Omega^4$, but such an effect is not very prominent.

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Analytical dynamics

Numismatic gyrations

The familiar shuddering motions of spinning coins as they come to rest are not at all intuitive. Moffatt's analysis (*Nature* **404,** 833–834; 2000) identifies air viscosity as the causative factor in coin jitter, so we tested this hypothesis by studying coins spinning in a vacuum. We discovered that the presence of air has little effect on the final motions of the coins, indicating that slippage and friction between the coin's edge and the supporting surface might cause the vibrations that accompany the end of the spin.

Casual observation of various objects spun on a tabletop indicates that compression of trapped air does not qualitatively affect the complex motions of spinning disks. We noted that a ring-shaped bell-jar lid, a short cylinder or the lid of a shoe-polish can — tested with either the rim or the flat side down — show a comparable behaviour: they spin on edge, topple over, then wobble to a shuddering halt. The universality of this motion is surprising in light of the air-viscosity mechanism proposed by Moffatt. As rings do not trap air the way solid disks do, these objects should generate shear forces of different magnitudes. The similar kinetic behaviour of these objects appears to contradict a decisive role for air viscosity.

The Dutch 2.5-guilder coin has magnetic properties that allow it to be spun with a precise frequency on a magnetic stirrer. We placed the coin in a glass desiccator that had a slightly concave bottom, brought it to a spin of approximately 10 Hz, and observed the motions of the slowing coin after the desiccator was lifted carefully from the stirring platform. The desiccator could be evacuated to less than 1 mtorr of air pressure.

Coins *in vacuo* spun on average for 12.5 s; coins in air spun on average for 10.5 s (average of 10 observations each). This difference in time can be attributed to a difference in the time the coin was spinning upright on its edge. The time from the onset of tumbling to standstill did not differ markedly and was about 4 s under both conditions. With or without air, the coin displayed the same characteristic final motions. We conclude that the presence or absence of air may have some effect on the upright duration of the spin, but has little effect on the final whirling motions that bring coins to rest. In contrast, Moffatt's analysis would predict a very long wobbling time for a coin in a vacuum.

We propose an alternative explanation for the jerking motions with which coins lose their spin. A coin toppling from rotation on edge preserves its rotational energy so that the axis of rotation changes from the plane of the coin to one perpendicular to the coin. The coin now must wobble on its edge. As Moffatt indicates, the friction is minimal when the point of contact between the supporting surface and the wobbling coin describes a circle with radius $R\cos(\alpha)$ (see his Fig. 1). But the coin is not free to choose any rotation speed. The gravitational force supplies a moment that interacts with the spin moment and the wobble moment. As a result, the coin is subject to precessing forces that rub the coin's edge in a jerking motion against the tabletop. We believe that this sliding friction temporarily lifts the coin, moving the point of contact between edge and supporting surface in a rapid staccato. It is this friction that brings the coin to a final rest.

The role of surface friction can be readily confirmed with the toy that inspired Moffatt's analysis. When placed on a table rather than on its slippery platform, Euler's disk rapidly comes to rest, illustrating the influence of the roughness of the supporting surface on the spinning time. Air viscosity may play a role in stopping 'theoretical' coins. Real-world coins, thrown on a table, do not need a finite-time singularity to control their spin. Edges rubbing against the tabletop explain the rapid dissipation of monetary momentum.

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Moffatt replies — It is true that there are a number of possible dissipative mechanisms for the rolling disk in addition to viscous dissipation in the surrounding air: vibration of the supporting surface, rolling friction due to plastic deformation at the point of rolling contact, and, as suggested by van den Engh *et al.*, dissipation due to slipping rather than rolling. The 'adiabatic' equation that I used, relating the precessional angular velocity Ω to the angle α, is valid only under the rolling condition, and experiments indicate that this condition is indeed satisfied for the 'toy' Euler's disk rolling on a flat, smooth horizontal glass plate placed on a firm table (V. A. Vladimirov, personal communication). I believe therefore that slipping does not occur in this case.

The problem really is to identify the dominant dissipative mechanism, for a given disk and a given surface, and then to evaluate the associated rate of dissipation of energy as a function of the angle α (which is proportional to the energy). If this rate of dissipation of energy turns out to be proportional to a power of α, where the exponent of this power, λ say, is less than one, then, under the adiabatic approximation, a finite-time singularity (for which Ω becomes infinite) will occur.

The air-viscosity mechanism I described yields $\lambda = -2$ (note that air viscosity is relatively insensitive to pressure, so that partial evacuation of the vessel in which the disk experiment is conducted should have only a small effect). An improved theory that takes account of oscillatory Stokes layers on the disk and supporting surface (L. Bildsten, personal communication) yields $\lambda = -5/4$. If 'rolling' friction is assumed to dissipate energy at a rate proportional to Ω, then $\lambda = -1/2$. Careful experiments under a variety of conditions should distinguish between these various possibilities.

I chose to focus on viscous dissipation because that is the only mechanism for which a fundamental (rather than empirical) description is available, namely that based on the Navier–Stokes equations of fluid dynamics. The fact that the air-viscosity mechanism exhibits the strongest singularity as α tends to zero suggests that this mechanism will always dominate when α is sufficiently small. For larger α and smaller disks (such as the 2.5-guilder coin), rolling friction is an equally plausible candidate (A. Ruina, personal communication), but determination of the associated rate of dissipation of energy (in terms of the physical properties of the disk and the surface) involves solution of the equations of (possibly plastic) deformation in both solids at the moving point of rolling contact, a difficult problem, which, so far as I am aware, still awaits definitive analysis.

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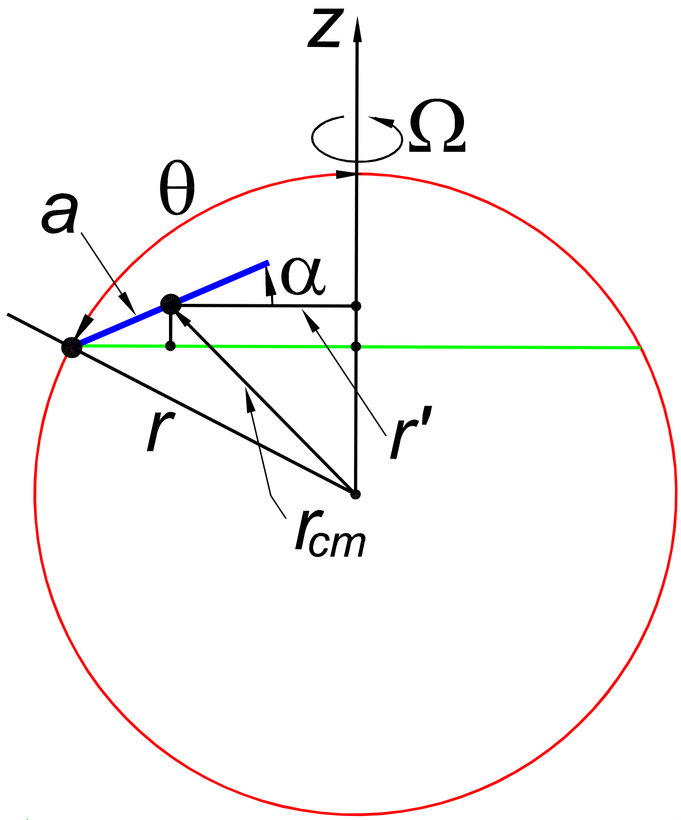
The Globe of Death

There exist stable, horizontal orbits entirely above the equator on the inside of the Globe of Death.

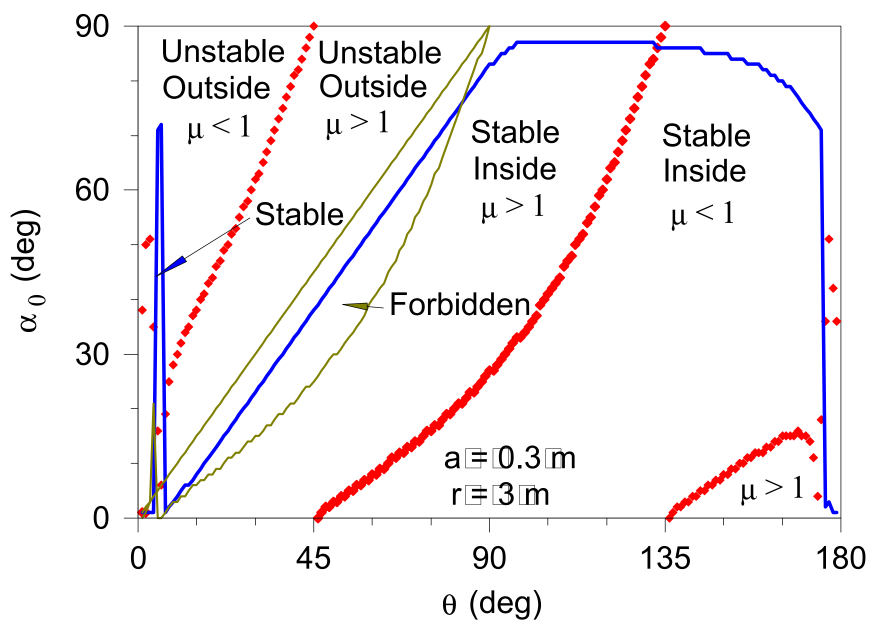
In some motorcycle acts, the globe splits apart at the equator with one or more bikes moving inside the upper hemisphere.

The Globe of Death



$$\Omega^2 = \frac{g\cot\alpha}{(2k+1)r' + ka\cos\alpha} = \frac{g\cot\alpha}{(2k+1)r\sin\theta - (k+1)a\cos\alpha},$$

where $k = 1/4$ for a uniform disk, $= 1/2$ for a hoop, *etc.*.



Coda

"It is not that the phenomena, though familiar and often interesting, are held to be specially important, but it was regarded rather as a point of honour to shew how the mathematical formulation could be effected, even if the solution should prove to be impracticable, or difficult of interpretation."
– Horace Lamb, *Higher Mechanics* (1920),
Sec. 66, "Rolling of a Solid on a Fixed Surface".