Sommerfeld's Diffraction Problem

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1 Problem

Discuss the scattering (diffraction) of a plane electromagnetic wave of angular frequency ω by a perfectly conducting half plane, considering the latter to be the limit of a parabolic cylinder. Relate the results to the electromagnetic version of Babinet's principle for complementary screens [1, 2, 3].[1]

2 Solution

Diffraction of a scalar light wave by a half plane was considered by Fresnel (1818) in his great *mémoire* [7], where a graph of the intensity was presented on p. 383,

COURBE DES INTENSITÉS DE LUMIÈRE EN DEHORS ET EN DEDANS DE L'OMBRE GÉOMÉTRIQUE D'UN ÉCRAN INDÉFINIMENT ÉTENDU.

Experiments in which polarization effects in scattering of light from a knife edge were first performed by Gouy in 1883 [8]. A partial electromagnetic explanation was given by Poincaré in 1892 [9]. The first "complete" electromagnetic solution was by Sommerfeld in 1895 [10, 11], using a somewhat obscure technique.[2]

Here, we consider a conducting knife edge to be a limiting case of a parabolic cylinder, and take advantage of the fact that Helmholtz wave equation for a scalar wavefunction $\psi\, e^{-i\omega t}$,

$$(\nabla^2 + k^2)\psi = 0, \tag{1}$$

where $k = \omega/c$ and c is the speed of light in vacuum, is separable in parabolic-cylindrical coordinates. Then, using an expansion of a plane wave in parabolic-cylindrical coordinates [14, 15, 16], the problem of scattering of a plane wave by a surface of constant coordinates can be given a series solution. This technique was first employed for scattering by a conducting cylinder [17], using the expansion of a plane wave in cylindrical coordinates found by Jacobi [18]. This technique can be used for surfaces of constant coordinate in the 11 coordinate systems in which the Helmholtz equation is separable [19], and permits formal solutions to a larger set of electromagnetic scattering problems than is commonly acknowledged. The

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[1]A closely related problem is the diffraction of a plane electromagnetic wave by a perfectly absorbing half plane [4, 5, 6].

[2]This technique was characterized by Poincaré [12] as *extrémement ingéniuse.* Another early review of Sommerfeld's method is given in [13].

example of a conducting strip as a limit of an elliptical cylinder is considered in sec. 3 of [20].

The relevance of parabolic-cylindrical coordinates to the knife-edge problem was realized by Lamb [21, 22], who used them to give an alternative derivation of Sommerfeld's solution, which alternative we follow here.[3] Other studies related to that of the present note include [23, 26]-[32]. See also the Appendix below.

The conducting screen lies in the plane $y = 0$ and occupies the region $x > 0$, as shown in the figure below (from [21]).
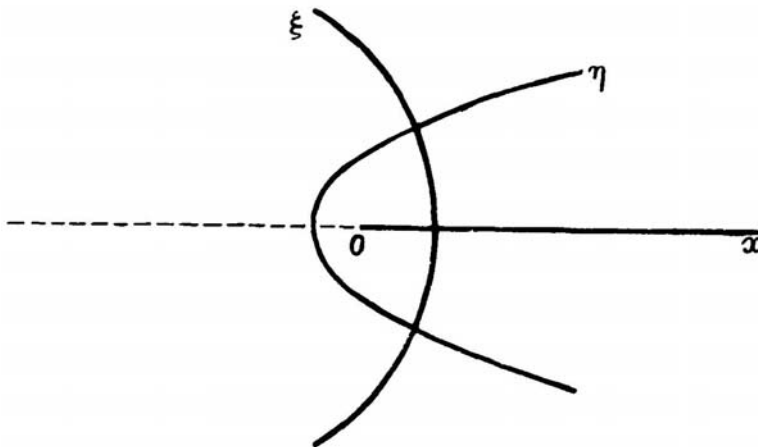
In addition to ordinary cylindrical coordinates (r, ϕ, z), we consider parabolic-cylindrical coordinates (ξ, η, z) which are related (for $0 \le \phi \le 2\pi$) by,[4]

$$
\begin{aligned}
x + iy = \quad r\, e^{i\phi} \quad &= \frac{(\xi + i\eta)^2}{k}, \\
x = \frac{\xi^2 - \eta^2}{k}, \qquad y &= \frac{2\xi\eta}{k}, \\
\xi = \sqrt{kr}\cos\frac{\phi}{2}, \qquad \eta &= \sqrt{kr}\sin\frac{\phi}{2}.
\end{aligned}
\tag{2}
$$

In this convention, the sign of ξ at a point $(\xi, \eta) = (x, y)$ is the same as the sign of y, while η is always non-negative. The half plane $(x > 0, y = 0)$ corresponds to $\eta = 0$, and the complementary half plane $(x < 0, y = 0)$ has coordinate $\xi = 0$.

In this two-dimensional problem the incident wave vector lies in the x-y plane. In case of incident electric field polarized parallel to the edge of the conducting half-plane screen, $i.e.$, in the z-direction, the current on the screen, and the scattered electric field, have only a z-component, so that an analysis can be based on the scalar wavefunction $\psi = E_z$. The boundary condition at the surface of the screen is $\psi = 0$ in this case.

If the incident electric field is polarized perpendicular to the edge of the screen, then the scattered electric field lies in the x-y plane, while both the incident and scattered magnetic field have only z-components. In this case an analysis can be based on the scalar wavefunction

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[3]Lamb's discussion seems little known, and is occasionally rediscovered, as in [24, 25].

[4]Parabolic-cylindrical coordinates are often defined with k replaced by 2 in eqs. (2). However, the convention used here (following Lamb) avoids the appearance of numerous factors of $\sqrt{k/2}$ in later equations.

$\psi = B_z$, for which the boundary condition at the surface of the screen is that the normal derivative $\partial \psi / \partial y = 0$.

It is useful to note that the scattered fields in case of a thin, plane conducting screen obey symmetries perhaps first noted in detail by Meixner [33] (see also [34]),

$$\mathbf{E}_x^s(x, -y, z) = \mathbf{E}_x^s(x, y, z), \qquad \mathbf{B}_x^s(x, -y, z) = -\mathbf{B}_x^s(x, y, z), \qquad (3)$$
$$\mathbf{E}_y^s(x, -y, z) = -\mathbf{E}_y^s(x, y, z), \qquad \mathbf{B}_y^s(x, -y, z) = \mathbf{B}_y^s(x, y, z), \qquad (4)$$
$$\mathbf{E}_z^s(x, -y, z) = \mathbf{E}_z^s(x, y, z), \qquad \mathbf{B}_z^s(x, -y, z) = -\mathbf{B}_z^s(x, y, z). \qquad (5)$$

The symmetries for E_z^s and B_z^s were considered by Lamb [21] to be "evident", but they depend on there being no current flow from one side of the screen to the other, as otherwise the magnetic field energy in finite volumes surrounding the edge of the screen would be infinite.

2.1   Normal Incidence

We first consider incident waves of the form $e^{i(ky-\omega t)}$, which are normally incident on the screen from $y < 0$. In the following we suppress the time-dependent factor $e^{-i\omega t}$.

2.1.1   Electric Field Parallel to the Edge

We consider the scalar wavefunction $\psi = E_z$, which can be written as,

$$\psi = E_0\, e^{iky} + \psi^s(x, y), \qquad (6)$$

where ψ^s is the scattered wavefunction.

Lamb surmised (sec. 2 of [21]) that the scattered wave has the form,

$$\psi^s(x, y) = u(x, y)\, e^{iky} + v(x, y)\, e^{-iky}, \qquad (7)$$

where u and v are associated with the transmitted and reflected parts of the scattered wave. Since ψ^s obeys Helmholtz' wave equation, whose form in the parabolic-cylindrical coordinates of eq. (2) is,

$$\frac{\partial^2 \psi^s}{\partial \xi^2} + \frac{\partial^2 \psi^s}{\partial \eta^2} + 4(\xi^2 + \eta^2)\psi^s = 0, \qquad (8)$$

the functions u and v obey the differential equations,

$$\frac{\partial^2 u, v}{\partial \xi^2} + \frac{\partial^2 u, v}{\partial \eta^2} \pm 4i\left(\eta \frac{\partial u, v}{\partial \xi} + \xi \frac{\partial u, v}{\partial \eta}\right) = 0, \qquad (9)$$

where the $+$ sign is for u and the $-$ sign is for v. Taking $u, v(\xi, \eta) = f(\xi \pm \eta) \equiv f(\zeta_{u,v})$ eq. (9) becomes,

$$f'' + 2i\zeta_{u,v} f' = 0, \qquad (10)$$

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for both u and v, which integrates to,

$$f(\zeta_{u,v}) = a + b \int_0^{\zeta_{u,v}} e^{-i\zeta^2}\, d\zeta. \tag{11}$$

That is,

$$u = f(\zeta_u) = a_u + b_u \int_0^{\xi+\eta} e^{-i\zeta^2}\, d\zeta, \qquad v = f(\zeta_v) = a_v + b_v \int_0^{\xi-\eta} e^{-i\zeta^2}\, d\zeta, \tag{12}$$

and,

$$\psi^s = a_u\, e^{iky} + a_v\, e^{-iky} + b_u\, e^{iky} \int_0^{\xi+\eta} e^{-i\zeta^2}\, d\zeta + b_v\, e^{-iky} \int_0^{\xi-\eta} e^{-i\zeta^2}\, d\zeta. \tag{13}$$

To determine the four constants a_u, a_v, b_u and b_v we first note that for $x \to -\infty$ and small y, *i.e.*, for small ξ and $\eta \to \infty$, the scattered wave is negligible. In this region we must separately have,

$$a_u + b_u \int_0^{\eta} e^{-i\zeta^2}\, d\zeta \to 0, \qquad a_v + b_v \int_0^{-\eta} e^{-i\zeta^2}\, d\zeta = a_v - b_v \int_0^{\eta} e^{-i\zeta^2}\, d\zeta \to 0. \tag{14}$$

Using the fact (Dwight 858.560 [36]),

$$\int_0^{\infty} e^{-i\zeta^2}\, d\zeta = \frac{\sqrt{\pi}}{2}\, e^{-i\pi/4}, \tag{15}$$

we have that,

$$a_u + \frac{\sqrt{\pi}}{2} b_u\, e^{-i\pi/4} = 0, \qquad a_v - \frac{\sqrt{\pi}}{2} b_v\, e^{-i\pi/4} = 0. \tag{16}$$

Finally, the condition that $E_z = E_0\, e^{iky} + \psi^s$ vanish on the surface of the conducting screen implies that,

$$-E_0 = \psi^s(x > 0, 0) = \psi^s(\xi, 0) = a_u + a_v + (b_u + b_v) \int_0^{\xi} e^{-i\zeta^2}\, d\zeta, \tag{17}$$

and hence,

$$a_u + a_v = -E_0, \qquad b_v = -b_u. \tag{18}$$

Combining this with eq. (16) we find[5]

$$a_u = a_v = -\frac{E_0}{2}, \qquad b_u = -b_v = \frac{E_0}{\sqrt{\pi}}\, e^{i\pi/4}, \tag{20}$$

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[5]In view of eq. (20), and recalling that the sign of ξ is taken to be the same as that of y, we have that,

$$\begin{aligned}
\psi^s(x, -y) &= a_u\, e^{-iky} + a_v\, e^{iky} + b_u\, e^{-iky} \int_0^{-\xi+\eta} e^{-i\zeta^2}\, d\zeta + b_v\, e^{iky} \int_0^{-(\xi+\eta)} e^{-i\zeta^2}\, d\zeta \\
&= a_v\, e^{-iky} + a_u\, e^{iky} + b_v\, e^{-iky} \int_0^{\xi-\eta} e^{-i\zeta^2}\, d\zeta + b_u\, e^{iky} \int_0^{\xi+\eta} e^{-i\zeta^2}\, d\zeta = \psi^s(x, y). \quad (19)
\end{aligned}$$

That is, $\psi^s = E_z^s$ satisfies the symmetry (5), as expected.

and,

$$\psi^s = E_z^s = E_0 \left(\frac{e^{i(ky+\pi/4)}}{\sqrt{\pi}} \int_0^{\xi+\eta} e^{-i\zeta^2} \, d\zeta - \frac{e^{-i(ky-\pi/4)}}{\sqrt{\pi}} \int_0^{\xi-\eta} e^{-i\zeta^2} \, d\zeta - \frac{e^{iky} + e^{-iky}}{2} \right). \quad (21)$$

2.1.2 Electric Field Perpendicular to the Edge

In this case the current on the conducting screen are in the x-direction, so the scattered electric field lies in the x-y plane, while both the incident and scattered magnetic field are parallel to the edge. It is therefore convenient to consider the scalar wavefunction $\psi = B_z = E_0 \, e^{iky} + \psi^s(x, y)$.

The analysis for ψ^s is similar to the preceding sec. 2.1.1, except, that the boundary condition at the screen, $E_x(x > 0, 0) = 0$ implies that $\partial B_z(x > 0, 0)/\partial y = 0$, i.e., $\partial B_z^s(x > 0, 0)/\partial y = \partial B_z^s(\xi, 0)/\partial y = -ikE_0$.

To perform the derivative of the integrals in eq. (13) we need the relations,

$$\frac{\partial \xi}{\partial x} = \frac{\xi}{2r}, \qquad \frac{\partial \eta}{\partial x} = -\frac{\eta}{2r}, \qquad (22)$$

$$\frac{\partial \xi}{\partial y} = \frac{\eta}{2r}, \qquad \frac{\partial \eta}{\partial y} = \frac{\xi}{2r}. \qquad (23)$$

Using these, we find,

$$\frac{\partial \psi^s(x > 0, 0)}{\partial y} = -ikE_0 = ik(a_u - a_v) + (b_u - b_v) \left(ik \int_0^\xi e^{-i\zeta^2} \, d\zeta + \frac{k}{2\xi} e^{-i\xi^2} \right), \qquad (24)$$

and hence,

$$a_u - a_v = -E_0, \qquad b_u = b_v. \qquad (25)$$

Combining this with eq. (16) we find[6]

$$a_u = -a_v = -\frac{E_0}{2}, \qquad b_u = b_v = \frac{E_0}{\sqrt{\pi}} e^{i\pi/4}, \qquad (27)$$

and,

$$\psi^s = B_z^s = E_0 \left(\frac{e^{i(ky+\pi/4)}}{\sqrt{\pi}} \int_0^{\xi+\eta} e^{-i\zeta^2} \, d\zeta + \frac{e^{-i(ky-\pi/4)}}{\sqrt{\pi}} \int_0^{\xi-\eta} e^{-i\zeta^2} \, d\zeta - \frac{e^{iky} - e^{-iky}}{2} \right). \quad (28)$$

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[6]In view of eq. (27), and recalling that the sign of ξ is taken to be the same as that of y, we have that,

$$
\begin{aligned}
\psi^s(x, -y) &= a_u \, e^{-iky} + a_v \, e^{iky} + b_u \, e^{-iky} \int_0^{-\xi+\eta)} e^{-i\zeta^2} \, d\zeta + b_v \, e^{iky} \int_0^{-(\xi+\eta)} e^{-i\zeta^2} \, d\zeta \\
&= -a_v \, e^{-iky} - a_u \, e^{iky} - b_v \, e^{-iky} \int_0^{\xi-\eta} e^{-i\zeta^2} \, d\zeta - b_u \, e^{iky} \int_0^{\xi+\eta} e^{-i\zeta^2} \, d\zeta \\
&= -\psi^s(x, y).
\end{aligned}
\qquad (26)
$$

That is, $\psi^s = B_z^s$ satisfies the symmetry (5), as expected.

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2.1.3 Asymptotic Behavior near the Geometric Shadow

To exhibit the well-known behavior of the fields near the edge of the geometric shadow (first deduced by Fresnel in a scalar theory), it is convenient to recast the integrals in eqs. (21) and (28) as running from ζ to ∞, using eq. (15),

$$\int_0^\zeta e^{-i\zeta^2}\, d\zeta = \frac{\sqrt{\pi}}{2} e^{-i\pi/4} - \int_\zeta^\infty e^{-i\zeta^2}\, d\zeta. \tag{29}$$

Then,

$$\frac{\psi^s}{E_0} = -\frac{e^{i(ky+\pi/4)}}{\sqrt{\pi}} \int_{\xi+\eta}^\infty e^{-i\zeta^2}\, d\zeta \pm \frac{e^{-i(ky-\pi/4)}}{\sqrt{\pi}} \int_{\xi-\eta}^\infty e^{-i\zeta^2}\, d\zeta \mp e^{-iky}. \tag{30}$$

We consider these waves for large $y = 2\xi\eta/k \approx 2\xi^2/k \approx 2\eta^2/k$ and small $x = (\xi^2 - \eta^2)/k \approx (\xi - \eta)\sqrt{2y/k}$, where the total wavefunction $\psi = E_0\, e^{iky} + \psi^s$ has the form,

$$\frac{\psi}{E_0} \approx \pm\frac{e^{-i(ky-\pi/4)}}{\sqrt{\pi}} \int_{\sqrt{kx^2/2y}}^\infty e^{-i\zeta^2}\, d\zeta + e^{iky} \mp e^{-iky} \approx \pm\frac{e^{-i(ky-\pi/4)}}{\sqrt{\pi}} \int_{\sqrt{kx^2/2y}}^\infty e^{-i\zeta^2}\, d\zeta, \tag{31}$$

where the second approximation follows noting that the integral varies more slowly with y than the terms $e^{iky} \mp e^{-iky}$, which can be represented by their average, namely zero. The real and imaginary parts of the remaining integral are related to the Fresnel cosine and sine integrals,

$$C(t) = \int_0^t \cos\frac{\pi s^2}{2}\, ds = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{\pi/2}\,t} \cos s^2\, ds, \quad S(t) = \int_0^t \sin\frac{\pi s^2}{2}\, ds = \sqrt{\frac{2}{\pi}} \int_0^{\sqrt{\pi/2}\,t} \sin s^2\, ds, \tag{32}$$

and the integral itself has the interpretation of the length of the chord on the Cornu spiral [37] from $(\mathrm{Sign}(x)C(\sqrt{2kx^2/\pi y}), \mathrm{Sign}(x)S(\sqrt{2kx^2/\pi y}))$ to (0.5,0.5).
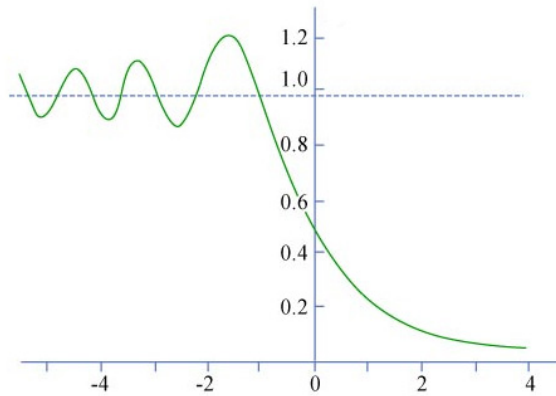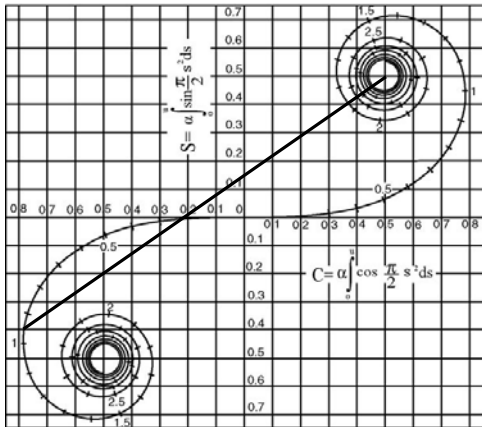
The plot on the right above (from [35]) shows ψ/E_0 as a function of x for $ky = 6\pi$.

Illustrations of lines of constant phase, of constant intensity, and of the Poynting vector are given in [38, 39].

2.2 Arbitrary Angle of Incidence

Suppose the incident wave has the form $e^{ik(x\sin\alpha+y\cos\alpha)}$, where $-\pi/2 \le \alpha \le \pi/2$ is the angle of incidence with respect to the $-y$ axis. Then, the reflected wave in case of a complete conducting screen at $y = 0$ would have the form $e^{ik(x\sin\alpha-y\cos\alpha)}$. The brilliant "guess" of Lamb (sec. 4 of [21]) is that the forms for the scattered wave found above for normal incidence hold for arbitrary incidence with the modifications,

$$
\begin{aligned}
E_z^{s,\parallel} &= E_0 \frac{e^{i(kx\sin\alpha+ky\cos\alpha+\pi/4)}}{\sqrt{\pi}} \int_0^{\xi_1+\eta_1} e^{-i\zeta^2}\, d\zeta \\
&\quad - E_0 \frac{e^{i(kx\sin\alpha-ky\cos\alpha+\pi/4)}}{\sqrt{\pi}} \int_0^{\xi_2-\eta_2} e^{-i\zeta^2}\, d\zeta \\
&\quad - E_0 \frac{e^{ik(x\sin\alpha+y\cos\alpha)} + e^{ik(x\sin\alpha-y\cos\alpha)}}{2}, \qquad (\mathbf{E}\ \text{parallel to the edge}), \quad (33)
\end{aligned}
$$

$$
\begin{aligned}
B_z^{s,\perp} &= E_0 \frac{e^{i(kx\sin\alpha+ky\cos\alpha+\pi/4)}}{\sqrt{\pi}} \int_0^{\xi_1+\eta_1} e^{-i\zeta^2}\, d\zeta \\
&\quad + E_0 \frac{e^{i(kx\sin\alpha-ky\cos\alpha+\pi/4)}}{\sqrt{\pi}} \int_0^{\xi_2-\eta_2} e^{-i\zeta^2}\, d\zeta \\
&\quad - E_0 \frac{e^{ik(x\sin\alpha+y\cos\alpha)} - e^{ik(x\sin\alpha-y\cos\alpha)}}{2}. \qquad (\mathbf{E}\ \text{perpendicular to the edge}),(34)
\end{aligned}
$$

where,

$$
\xi_1 = \sqrt{kr}\cos\frac{\phi+\alpha}{2}, \qquad \eta_1 = \sqrt{kr}\sin\frac{\phi+\alpha}{2}, \tag{35}
$$

$$
\xi_2 = \sqrt{kr}\cos\frac{\phi-\alpha}{2}, \qquad \eta_2 = \sqrt{kr}\sin\frac{\phi-\alpha}{2}, \tag{36}
$$

With some effort one can verify that these forms satisfy the boundary conditions at the conducting half-plane screen.

A numerical computation of E_z and B_z for $\alpha = 30°$ is shown below.[7] The incident wave moves up and to the right, the first quadrant is the nominal shadow region, and the fourth quadrant contains the standing-wave sum of the incident and reflected waves.
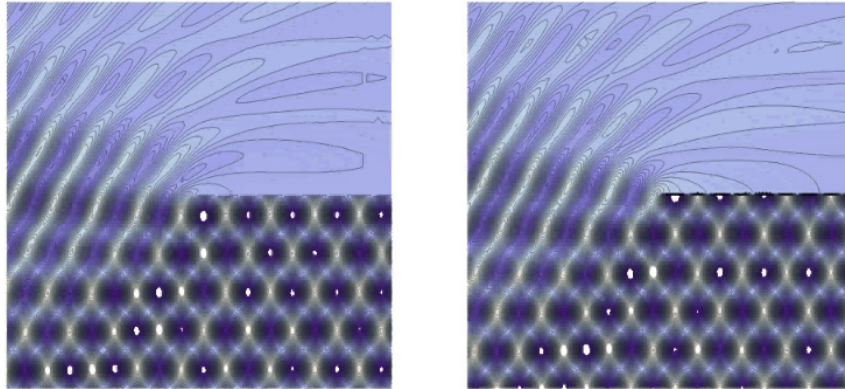
2.3 Complementary Screen and Babinet's Principle

If the (complementary) screen occupies the region $(x < 0, y = 0)$ then it corresponds to $\xi = 0$. That is, the role of coordinates ξ and η is reversed when we go from the original screen to the complementary screen.[8] This reversal has the effect of changing the sign constants b_u and b_v in eq. (14), as well as that of the integrals with upper limits $\xi - \eta$. Hence, the scattered fields in the complementary case are,

$$
\begin{aligned}
E_z^{\prime s, \parallel} &= -E_0 \frac{e^{i(kx\sin\alpha + ky\cos\alpha + \pi/4)}}{\sqrt{\pi}} \int_0^{\xi_1 + \eta_1} e^{-i\zeta^2}\, d\zeta \\
&\quad - E_0 \frac{e^{i(kx\sin\alpha - ky\cos\alpha + \pi/4)}}{\sqrt{\pi}} \int_0^{\xi_2 - \eta_2} e^{-i\zeta^2}\, d\zeta \\
&\quad - E_0 \frac{e^{ik(x\sin\alpha + y\cos\alpha)} + e^{ik(x\sin\alpha - y\cos\alpha)}}{2}, \qquad (\mathbf{E} \text{ parallel to the edge}), \qquad (37)
\end{aligned}
$$

$$
\begin{aligned}
B_z^{\prime s, \perp} &= -E_0 \frac{e^{i(kx\sin\alpha + ky\cos\alpha + \pi/4)}}{\sqrt{\pi}} \int_0^{\xi_1 + \eta_1} e^{-i\zeta^2}\, d\zeta \\
&\quad + E_0 \frac{e^{i(kx\sin\alpha - ky\cos\alpha + \pi/4)}}{\sqrt{\pi}} \int_0^{\xi_2 - \eta_2} e^{-i\zeta^2}\, d\zeta \\
&\quad - E_0 \frac{e^{ik(x\sin\alpha + y\cos\alpha)} - e^{ik(x\sin\alpha - y\cos\alpha)}}{2}. \qquad (\mathbf{E} \text{ perpendicular to the edge}).(38)
\end{aligned}
$$

The fields (33)-(34) and (37)-(38) illustrate Babinet's principle for electromagnetic fields [2, 3]),

$$
E_z^{s, \parallel} + B_z^{\prime s, \perp} = E_0\, e^{ik(x\sin\alpha + y\cos\alpha)}, \qquad E_z^{\prime s, \parallel} + B_z^{s, \perp} = E_0\, e^{ik(x\sin\alpha + y\cos\alpha)}. \qquad (39)
$$

That is, the sum of the electric field of one polarization scattered by the one screen and the magnetic field corresponding to the case of electric field of the other polarization scattered by the complementary screen equals the incident field in the absence of the screen.

A Solution via Weber Functions

This Appendix is presently not very satisfactory.

The Helmholtz equation for a scalar wavefunction $\psi(x, y)\, e^{-i\omega t}$ that is independent of z has the form,

$$
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + k^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + k^2\psi = \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} + 4(\xi^2 + \eta^2)\psi = 0, \quad (40)
$$

in parabolic cylinder coordinates (2).

Assuming that $\psi = X(\xi)Y(\eta)$ leads to the differential equations,

$$
\frac{d^2 X}{d\xi^2} + 4\xi^2 X = -CX, \qquad \frac{d^2 Y}{d\eta^2} + 4\eta^2 Y = CY, \qquad (41)
$$

---

[8]Strictly, this statement holds only if we also adopt the convention that ξ is always positive, and the sign of η is the same as the sign of y in the case of the complementary screen.

where C is a separation constant.

Solutions to the differential equations (41) are so-called parabolic-cylinder functions, also known as Weber functions [14, 15, 16, 40, 41, 42, 43],

$$X(\xi) = D_n(2\xi\sqrt{-i}), \quad D_{-n-1}(2\xi\sqrt{i}), \quad \text{and} \quad Y(\eta) = D_n(2\eta\sqrt{-i}), \quad D_{-n-1}(2\eta\sqrt{i}), \quad (42)$$

where n is a non-negative integer (to which the separation constant C is related). Weber functions for negative index are related to those with positive index by,

$$\frac{\sqrt{2\pi}\,i^{n+1}}{\Gamma(-n)}D_{-n-1}(2\xi\sqrt{i}) = D_n(-2\xi\sqrt{-i}) - (-1)^n D_n(2\xi\sqrt{-i}). \tag{43}$$

Setting $n = -m - 1$ we find that,

$$D_m(2\xi\sqrt{i}) = \frac{i^m\,m!}{\sqrt{2\pi}}\left(D_{-m-1}(-2\xi\sqrt{-i}) + (-1)^m D_{-m-1}(2\xi\sqrt{-i})\right), \tag{44}$$

and hence for even $m \geq 0$,

$$\frac{D_m(0)}{D_{-m-1}(0)} = \sqrt{\frac{2}{\pi}}\,i^m m! \qquad \text{(even } m\text{)}. \tag{45}$$

Derivatives of Weber functions (for any n) obey,

$$\frac{dD_n(z)}{dz} = \frac{z}{2}D_n(z) - D_{n+1}(z) = -\frac{z}{2}D_n(z) + D_{n-1}(z), \tag{46}$$

which leads to the relations,

$$D_{n+1}(z) = -e^{z^2/4}\frac{d[e^{-z^2/4}D_n(z)]}{dz}, \qquad D_n(z) = e^{z^2/4}\int_z^\infty e^{-z_0^2/4}D_{n+1}(z_0)\,dz_0. \tag{47}$$

For non-negative integer index n the Weber functions can be written as,

$$D_n(z) = (-1)^n\,e^{z^2/4}\frac{d^n\,e^{-z^2/2}}{dz^n} = 2^{-n/2}\,e^{-z^2/4}H_n(z/\sqrt{2}), \qquad D_0(z) = e^{-z^2/4}, \tag{48}$$

$$D_{-n-1}(z) = e^{z^2/4}\int_z^\infty dz_0\int_{z_0}^\infty dz_1\cdots\int_{z_{n-1}}^\infty e^{-z_n^2/4}\,dz_n, \tag{49}$$

where H_n is a Hermite polynomial. Thus, $D_n(-z) = (-1)^n D_n(z)$ and $D_n(0) = 0$ for odd $n > 0$ and $D_n(0) \neq 0$) for other n. Also,

$$D_{-1}(z\sqrt{2i}) = e^{iz^2/2}\int_{z\sqrt{2i}}^\infty e^{-iz'^2/2}\,dz'. \tag{50}$$

An expansion for a plane wave (with no z-dependence) that propagates at angle ϕ_0 to the positive x-axis is [14, 16],

$$e^{ik(x\cos\phi_0 + y\sin\phi_0)} = \frac{1}{\cos\frac{\phi_0}{2}}\sum_{n=0}^\infty \frac{i^n\tan^n\frac{\phi_0}{2}}{n!}D_n(2\xi\sqrt{-i})D_n(2\eta\sqrt{i}), \tag{51}$$

$$= \frac{1}{\sin\frac{\phi_0}{2}}\sum_{n=0}^\infty \frac{\cot^n\frac{\phi_0}{2}}{i^n n!}D_n(2\xi\sqrt{i})D_n(2\eta\sqrt{-i}). \tag{52}$$

9

For $\phi_0 = 0$ the wave moves in the $+x$ direction and has the simple form,

$$e^{ikx} = e^{ik(\xi^2-\eta^2)/2} = D_0(\xi\sqrt{-2ik})D_0(\eta\sqrt{2ik}), \qquad (53)$$

and similarly for $\phi_0 = \pi$, when the wave moves in the $-x$ direction,

$$e^{-ikx} = e^{-i(\xi^2-\eta^2)} = D_0(2\xi\sqrt{i})D_0(2\eta\sqrt{-i}). \qquad (54)$$

For normal incidence on the screen the angle is $\phi_0 = \pi/2$ and the wave is,

$$e^{iky} = e^{i2\xi\eta} = \sqrt{2}\sum_{n=0}^{\infty}\frac{i^n}{n!}D_n(2\xi\sqrt{-i})D_n(2\eta\sqrt{i}) = \sqrt{2}\sum_{n=0}^{\infty}\frac{1}{i^n n!}D_n(2\xi\sqrt{i})D_n(2\eta\sqrt{-i}). \quad (55)$$

? I believe that the two series in eq. (55) are the complex conjugates of one another. If so, eq. (55) cannot be correct. I have seen one version in which the factor i^n appears in the numerators of both series.

According to sec. 16.5 of [42] or 19.3.1 and 19.8.1 of [43], the Weber functions for large $|z| \gg n$ (and $x > 0$) have the leading form[9]

$$D_n(z) \approx z^n e^{-z^2/4}, \qquad (56)$$

such that eq. (51)-(52) can be written for large ξ and η as,

$$e^{ik(x\cos\phi_0+y\sin\phi_0)} \overset{?}{=} \frac{e^{ikx}}{\cos\frac{\phi_0}{2}}\sum_{n=0}^{\infty}\frac{(2iky\tan\frac{\phi_0}{2})^n}{n!} = \frac{e^{ik(x+2y\tan\frac{\phi_0}{2})}}{\cos\frac{\phi_0}{2}} \overset{?}{=} \frac{e^{-ik(x+2y\cot\frac{\phi_0}{2})}}{\sin\frac{\phi_0}{2}}. \qquad (57)$$

Thus, it appears unlikely that the forms (51)-(52) are valid, in which case the results below are not valid.[10]

When such a plane wave is incident on a parabolic cylinder, say of constant η, the scattered wave has the form of an outgoing cylindrical wave at large distances. Consideration of the asymptotic behavior of Weber functions [16] (see also [44]) suggests that the scattered wavefunction ψ^s can be written as[11]

$$\psi^s = \sum_{n=0}^{\infty} a_n D_n(2\xi\sqrt{-i})D_{-n-1}(2\eta\sqrt{-i}) = \sum_{n=0}^{\infty} b_n D_{-n-1}(2\xi\sqrt{-i})D_n(2\eta\sqrt{-i}). \qquad (58)$$

In problem of scattering by a plane conducting screen it is often assumed that the scattered fields obey certain symmetries with respect to change of sign of the distance from the observer to the screen (see, for example, sec. 11.2 of [35]). For the geometry used here, this corresponds to change of sign of y and ξ. Since $D_n(-2\xi\sqrt{-i}) = (-1)^n D_n(2\xi\sqrt{-i})$, these symmetries will not hold in general in the present problem.

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[9]Equation (56) agrees with eq. (48) noting that $H_n(z) \to (2z)^n$ for large z. However, this agreement requires a sign change in eq. (48) compared to eq. (11.2.68) of [16].

[10]In sec. 9 of [14] the limit $n \gg |z|^2$ is considered, but the asymptotic forms used do not seem to me to agree with 19.9.4 of [43] (or p. 403 of [41]), $D_n(z) \approx (-2i)^{n/2}\Gamma(n/2)\,e^{i\sqrt{n+1/2}\,z}/\sqrt{\pi}$. It may be that 19.9.1 was used, which applies for negative n.

[11]I have not seen the second form of eq. (58) elsewhere. For large ξ and η eq. (56) implies that $D_n(2\xi\sqrt{-i})D_{-n-1}(2\eta\sqrt{-i}) \to \cot^n\frac{\phi}{2}\,e^{ikr}/2\sqrt{-ikr}\sin\frac{\phi}{2}$, and that $D_{-n-1}(2\xi\sqrt{-i})D_n(2\eta\sqrt{-i}) \to \tan^n\frac{\phi}{2}\,e^{ikr}/2\sqrt{-ikr}\cos\frac{\phi}{2}$.

A.1 Incident Electric Field Parallel to the Edge of the Screen

When the incident electric field has only a z-component, the resulting currents on the con-ducting screen are in the z-direction, and the scattered electric filed has only a z-component. We therefore take the scalar field component ψ to be E_z, such that the boundary condition is that $\psi = 0$ on the surface of the screen (Dirichlet boundary condition).

For incident angle $0 \leq \phi_0 < \pi/2$ and incident wave amplitude E_0, comparison of eq. (51) and the first form of eq. (58) indicates that for scattering off a conducting parabolic cylinder of coordinate η_0,

$$a_n = -E_0 \frac{i^n \tan^n \frac{\phi_0}{2}}{n! \cos \frac{\phi_0}{2}} \frac{D_n(2\eta_0\sqrt{i})}{D_{-n-1}(2\eta_0\sqrt{-i})}, \tag{59}$$

Going to the limit of a knife edge, we are interested in the case that $\eta_0 = 0$, for which $D_n(0) = 0$ for odd n. Recalling eq. (45), the total electric field (for incident angle $0 \leq \phi_0 < \pi/2$) is the sum of the incident and scattered fields,

$$E_z = E_z^i + E_z^s, \tag{60}$$

$$E_z^i = E_0 \, e^{ik(x\cos\phi_0 + y\sin\phi_0)}, \tag{61}$$

$$E_z^s = -\sqrt{\frac{2}{\pi}} \frac{E_0}{\cos\frac{\phi_0}{2}} \sum_{n \, \text{even}} (-1)^n \tan^n \frac{\phi_0}{2} D_n(2\xi\sqrt{-i}) D_{-n-1}(2\eta\sqrt{-i}). \tag{62}$$

The scattered field E_z^s obeys the symmetry advocated by Born and Wolf [35] that $E_z^s(-y) = E_z^s(y)$, i.e. that $E_z^s(-\xi) = E_z^s(\xi)$, as only terms with even n appear in eq. (62).

The magnetic field is related to the electric field by Faraday's law,

$$\mathbf{B} = -\frac{i}{k}\boldsymbol{\nabla} \times \mathbf{E}, \tag{63}$$

where the curl of a vector $\mathbf{V}$ is given in parabolic-cylindrical coordinates by

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{V} &= \frac{k}{2\sqrt{\xi^2 + \eta^2}} \left(\frac{\partial V_z}{\partial \eta} - \frac{2}{k} \frac{\partial \left(\sqrt{\xi^2 + \eta^2} \, V_\eta \right)}{\partial z} \right) \hat{\boldsymbol{\xi}} \\
&+ \frac{k}{2\sqrt{\xi^2 + \eta^2}} \left(\frac{2}{k} \frac{\partial \left(\sqrt{\xi^2 + \eta^2} \, V_\xi \right)}{\partial z} - \frac{\partial V_z}{\partial \xi} \right) \hat{\boldsymbol{\eta}} \\
&+ \frac{1}{\xi^2 + \eta^2} \left(\frac{\partial \left[(\xi^2 + \eta^2) \, V_\eta \right]}{\partial \xi} - \frac{\partial \left[(\xi^2 + \eta^2) \, V_\xi \right]}{\partial \eta} \right) \hat{\mathbf{z}},
\end{aligned} \tag{64}$$

noting that the scale factors are $h_\xi = h_\eta = (2/k)\sqrt{\xi^2 + \eta^2}$, $h_z = 1$. Thus,

$$\mathbf{B} = -\frac{i}{2\sqrt{\xi^2 + \eta^2}} \left(\frac{\partial E_z}{\partial \eta} \hat{\boldsymbol{\xi}} - \frac{\partial E_z}{\partial \xi} \hat{\boldsymbol{\eta}} \right). \tag{65}$$

A.1.1 Normal Incidence

For normal incidence, $\phi_0 = \pi/2$, the forms (60)-(62) simplify slightly to,

$$E_z^i = E_0\, e^{iky}, \tag{66}$$

$$E_z^s = -\frac{E_0}{\sqrt{\pi}} \sum_{n\,\text{even}} (-1)^n D_n(2\xi\sqrt{-i}) D_{-n-1}(2\eta\sqrt{-i}), \tag{67}$$

the latter equation of which is still rather intricate. *So far I have not found reference to any asymptotic expansions for complex argument of large absolute value, so the solutions presented here are more of formal than practical interest. However, these formal solutions provide counterexamples to certain lore about scattering from plane screens, as remarked elsewhere in this note.*

A somewhat different approach to this case was taken by Morse and Feshbach [16], who considered an incident wave in the $+x$ direction, eq. (53), and a conducting screen along the negative y-axis, where $\xi = -\eta$. In this case they found the scattered electric field to be,

$$E_z^s = -\frac{1}{\sqrt{2\pi}} \left[D_0(2\xi\sqrt{-i}) D_{-1}(2\eta\sqrt{-i}) + D_{-1}(2\xi\sqrt{-i}) D_0(2\eta\sqrt{-i}) \right]. \tag{68}$$

Yet another approach was taken by Lamb [21], who is his sec. 3 made the educated guess that for the electric field polarized parallel to the edge of the screen, the scatter field $\psi^s = E_z^s$ obeys,

$$\frac{\partial \psi^s}{\partial x} = C \frac{e^{ikr}}{\sqrt{kr}} \sin\frac{\phi}{2}, \tag{69}$$

which he deftly integrated to find,

$$\frac{\psi^s}{E_0} = \frac{1-i}{\sqrt{2\pi}} \left(e^{-iky} \int_0^{\xi+\eta} e^{-i\zeta^2}\, d\zeta - e^{iky} \int_0^{\xi-\eta} e^{-i\zeta^2}\, d\zeta \right) - \frac{e^{iky} + e^{-iky}}{2}. \tag{70}$$

A.1.2 Complementary Screen

If the conducting screen occupied the half plane $(x < 0, 0, z)$ it would have coordinate $\xi = 0$.

Comparison of eq. (52) and the second form of eq. (58) indicates that for scattering off a conducting parabolic cylinder of coordinate ξ_0,

$$b_n = -\frac{\cot^n \frac{\phi_0}{2}}{i^n n! \sin\frac{\phi_0}{2}} \frac{D_n(2\xi_0\sqrt{i})}{D_{-n-1}(2\xi_0\sqrt{-i})}. \tag{71}$$

Going to the limit of a knife edge, we are interested in the case that $\xi_0 = 0$, for which $D_n(0) = 0$ for odd n. The total electric field (for incident angle $0 \le \phi_0 < \pi/2$) is,

$$E_z' = E_z^i + E_z'^s, \tag{72}$$

$$E_z'^s = -\sqrt{\frac{2}{\pi}} \frac{E_0}{\sin\frac{\phi_0}{2}} \sum_{n\,\text{even}} \cot^n \frac{\phi_0}{2} D_n(2\xi\sqrt{i}) D_{-n-1}(2\eta\sqrt{i}). \tag{73}$$

12

A naïve application of Babinet's principle of complementary screens [1] to the present problem would suggest that $E_z + E_z' = E_z^i$ for $y > 0$, which does not appear to be consistent with eqs. (61)-(63) and (73). To explore whether the electromagnetic version of Babinet's principle [2, 3] holds here, we need to consider the case of incident electric field polarized perpendicular to the z-axis.

A.2 Incident Electric Field Perpendicular to the Edge of the Screen

In this case the incident magnetic field (as well as the scattered magnetic field) is parallel to the edge of the screen, *i.e.*, the z-axis, and we take the scalar field ψ to be B_z. The tangential component of the electric field must vanish at the surface of the perfectly conducting screen, and the fourth Maxwell equation tells us that at the screen $E_x = 0 \propto \partial B_z/\partial y$. For conducting screens that are surfaces of constant η_0 (or of constant ξ_0), the (Neumann) boundary conditions are,

$$\frac{\partial B_z(\eta_0)}{\partial \xi} = 0 \qquad \left(\text{or} \qquad \frac{\partial B_z(\xi_0)}{\partial \eta} = 0 \right). \tag{74}$$

For incident angle $0 \leq \phi_0 < \pi/2$ we again take the incident field to have the form (51) and the scattered field to have the first form of eq. (58). For a screen of constant η_0 the normal derivatives of the incident and scattered fields are,

$$\frac{\partial B_z^i}{\partial \xi} = \frac{2E_0\sqrt{-i}}{\cos\frac{\phi_0}{2}} \sum_{n=0}^{\infty} \frac{i^n \tan^n \frac{\phi_0}{2}}{n!} D_n'(2\xi\sqrt{-i}) D_n(2\eta_0\sqrt{i}), \tag{75}$$

$$\frac{\partial B_z^s}{\partial \xi} = 2\sqrt{-i} \sum_{n=0}^{\infty} a_n D_n'(2\xi\sqrt{-i}) D_{-n-1}(2\eta_0\sqrt{-i}), \tag{76}$$

where $D_n'(z) = dD_n(z)/dz$. The boundary condition (74) again leads to Fourier coefficients a_n as given by eq. (59), so the magnetic field for the case of a conducting screen with $\eta_0 = 0$ is given by,

$$B_z = B_z^i + B_z^s, \tag{77}$$

$$B_z^i = E_0 \, e^{ik(x\cos\phi_0 + y\sin\phi_0)}, \tag{78}$$

$$B_z^s = -\frac{E_0}{\cos\frac{\phi_0}{2}} \sum_{n=0}^{\infty} \frac{i^n \tan^n \frac{\phi_0}{2}}{n!} \frac{D_n(0)}{D_{-n-1}(0)} D_n(2\xi\sqrt{-i}) D_{-n-1}(2\eta\sqrt{-i})$$

$$= -\sqrt{\frac{2}{\pi}} \frac{E_0}{\cos\frac{\phi_0}{2}} \sum_{n \, \text{even}} (-1)^n \tan^n \frac{\phi_0}{2} D_n(2\xi\sqrt{-i}) D_{-n-1}(2\eta\sqrt{-i}). \tag{79}$$

The scattered field B_z^s does not obey the symmetry advocated by Born and Wolf [35] that $B_z^s(-y) = -B_z^s(y)$, *i.e.* that $B_z^s(-\xi) = -B_z^s(\xi)$, as only terms with even n appear in eq. (79). That the symmetry would not hold when the electric field is perpendicular to the edge of the screen was anticipated in [34]. Another counterexample is given in sec. 3.2 of [20].

The form (79) is symmetric in ξ (and hence in y), which is somewhat surprising. Is this a hint of a defect in the present analysis?

13

A.2.1 Complementary Screen

If the conducting screen occupied the half plane $(x < 0, 0, z)$ it would have coordinate $\xi = 0$.

For incident angle $0 \leq \phi_0 < \pi/2$ we now take the incident field to have the form (52) and the scattered field to have the scattered form of eq. (58). For a screen of constant ξ_0 the normal derivatives of the incident and scattered fields are,

$$\frac{\partial B_z'^i}{\partial \eta} = \frac{2E_0\sqrt{-i}}{\sin\frac{\phi_0}{2}} \sum_{n=0}^{\infty} \frac{\cot^n \frac{\phi_0}{2}}{i^n n!} D_n(2\xi_0\sqrt{i}) D_n'(2\eta\sqrt{-i}), \tag{80}$$

$$\frac{\partial B_z'^s}{\partial \eta} = 2\sqrt{-i} \sum_{n=0}^{\infty} b_n D_{-n-1}(2\xi_0\sqrt{-i}) D_n'(2\eta\sqrt{-i}). \tag{81}$$

Comparison of eq. (52) and the second form of eq. (58) indicates that for scattering off a conducting parabolic cylinder of coordinate ξ_0,

$$b_n = -\frac{\cot^n \frac{\phi_0}{2}}{i^n n! \sin\frac{\phi_0}{2}} \frac{D_n(2\xi_0\sqrt{-i})}{D_{-n-1}(2\xi_0\sqrt{-i})} \,. \tag{82}$$

Going to the limit of a knife edge, we are interested in the case that $\xi_0 = 0$, for which $D_n(0) = 0$ for odd n. The total magnetic field (for incident angle $0 \leq \phi_0 < \pi/2$) is,

$$B_z' = B_z^i + B_z'^s, \tag{83}$$

$$B_z'^s = -\sqrt{\frac{2}{\pi}} \frac{E_0}{\sin\frac{\phi_0}{2}} \sum_{n\,\text{even}} \cot^n \frac{\phi_0}{2} D_{-n-1}(2\xi\sqrt{-i}) D_n(2\eta\sqrt{-i}). \tag{84}$$

A.3 Electromagnetic Version of Babinet's Principle

The electromagnetic version of Babinet's principle [2, 3] is that,

$$\mathbf{E}(y > 0) + \mathbf{B}'(y > 0) = \mathbf{E}^i(y > 0), \qquad \mathbf{B}(y > 0) - \mathbf{E}'(y > 0) = \mathbf{B}^i(y > 0), \tag{85}$$

where the plane conducting screen (with $y = 0$) associated with the fields $\mathbf{E}'$ and $\mathbf{B}'$ is the complement of that for the fields $\mathbf{E}$ and $\mathbf{B}$, and the incident fields in the complementary case are the duals,[12]

$$\mathbf{E}'^i = -\mathbf{B}^i, \qquad \mathbf{B}'^i = \mathbf{E}^i. \tag{86}$$

In the present case this implies that $\mathbf{B}'^i = \mathbf{E}^i = E_0 \, e^{ik(x\cos\phi_0 + y\sin\phi_0)} \, \hat{\mathbf{z}}$, as considered in sec. 2.2.1.

It appears to me implausible that Babinet's principle (85) is satisfied by eqs. (60)-(62) and (83)-(84).

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[12]See, for example, sec. 2.1 of [3].

A.3.1 Normal Incidence

The argument given in sec. 5.3.3 of [45] is fairly convincing that Booker's electromagnetic version of Babinet's principle should hold for plane waves at normal incidence on screens with all edges parallel to one rectangular coordinate axis, as in Sommerfeld's problem. For normal incidence with the incident electric field polarized parallel to the z-axis, we have from eq. (62) that,

$$E_z^s = -\frac{2E_0}{\sqrt{\pi}} \sum_{n \text{ even}} (-1)^n D_n(2\xi\sqrt{-i}) D_{-n-1}(2\eta\sqrt{-i}), \tag{87}$$

and from eq. (84) that,

$$B_z'^s = -\frac{2E_0}{\sqrt{\pi}} \sum_{n \text{ even}} D_{-n-1}(2\xi\sqrt{-i}) D_n(2\eta\sqrt{-i}). \tag{88}$$

The electromagnetic version of Babinet's principle would be satisfied if,

$$e^{iky} = \frac{2}{\sqrt{\pi}} \sum_{n \text{ even}} \left((-1)^n D_n(2\xi\sqrt{-i}) D_{-n-1}(2\eta\sqrt{-i}) + D_{-n-1}(2\xi\sqrt{-i}) D_n(\eta\sqrt{-i}) \right). \tag{89}$$

If eq. (55) is correct we can say that,

$$e^{iky} = \frac{\sqrt{2}}{2} \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(D_n(2\xi\sqrt{-i}) D_n(2\eta\sqrt{i}) + (-1)^n D_n(2\xi\sqrt{i}) D_n(2\eta\sqrt{-i}) \right). \tag{90}$$

It is not clear to me that eqs. (89) and (90) are equal. If eq. (89) is to be correct, it seems likely that there exist alternatives to the plane-wave expansions of eqs. (51-(52).

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