Forces on Halves of a Uniformly Magnetized Sphere

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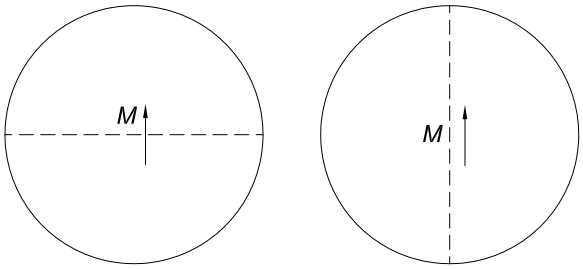
1 Problem

Deduce the force between two halves of a uniformly magnetized sphere, for halves defined by planes perpendicular and parallel to the magnetization density $\mathbf{M}$, assuming the sphere to be in vacuum.



Show that the force between hemispheres in the left figure is the same as that between one such hemisphere and either a perfectly conducting plate, or a plate of infinite magnetic permeability.

This problem was inspired by e-discussions with J. Castro Paredes. The force between two rectangular-prism permanent magnets has been deduced in [1]. For cylindrical magnets, see [2, 3], and for spheres, see [4].

2 Solution for Two Hemispheres

2.1 Fields of a Uniformly Magnetized Sphere

That we should discuss two magnetic fields, $\mathbf{B}$ and $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$ in Gaussian units ($\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ in SI units), in materials that support a (magnetization) density $\mathbf{M}$ of magnetic dipole moments was first noted by W. Thomson in 1871, eq. (r), sec. 517 of [5], and acknowledged in Art. 399 of Maxwell's *Treatise* [6]. That a uniformly magnetized sphere has $\mathbf{H}_{\rm in} = -4\pi\mathbf{M}/3$ throughout its interior was stated by Thomson (1851) in a footnote to sec. 48 of [7],[1] and demonstrated in greater detail in a footnote to sec. 610 of [5]. Hence, $\mathbf{B}_{\rm in} = 8\pi\mathbf{M}/3$.

Outside the sphere of radius a, where $\mathbf{B}_{\rm ext} = \mathbf{H}_{\rm ext}$, these fields are simply those of a magnetic dipole moment $\mathbf{m} = 4\pi a^3\mathbf{M}/3$, the total magnetic moment of the sphere. This result is implicit in Thomson's discussion in secs. 632-633 of [5], though not explicity stated.[2]

[1] The sign of $\mathbf{H}_{\rm in}$ was not clearly stated by Thomson (who did not at that time distinguish between $\mathbf{B}$ and $\mathbf{H}$).

[2] For a textbook discussion, see sec. 5.10 of [8].

In sum, the fields of a sphere of radius a with uniform magnetization density $\mathbf{M}$, centered on the origin, are,

$$\mathbf{B}_{\rm in} = \frac{8\pi\mathbf{M}}{3}\,, \quad \mathbf{H}_{\rm in} = -\frac{\mathbf{B}_{\rm in}}{2} = -\frac{4\pi\mathbf{M}}{3}\,, \quad \mathbf{B}_{\rm ext} = \mathbf{H}_{\rm ext} = \frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3}\,, \quad \mathbf{m} = \frac{4\pi a^3}{3}\mathbf{M}. \tag{1}$$

2.2 Forces via the Maxwell Stress Tensor in Vacuum

Use of the Maxwell stress tensor is generally the most reliable method for computing forces on magnetic materials [9]. However, there is some ambiguity as to the form of the stress tensor inside permanent magnets, so we first consider that the between two hemispheres of a uniformly magnetized sphere are to be computed after a small gap (sometimes called a "virtual air gap" [10, 11]) has been opened between the hemispheres.[3] Then, the force on a hemisphere can be evaluated via a surface entirely in vacuum which encloses the hemisphere. On such a surface the (magnetic part of) the Maxwell stress tensor is simply,

$$\mathsf{T}_{ij} = \frac{B_i B_j}{4\pi} - \delta_{ij}\frac{B^2}{8\pi}\,. \tag{2}$$

For hemispheres with bases perpendicular to $\mathbf{M}$, where $\mathbf{B}$ is normal to the bases and so continuous across their surfaces, $\mathbf{B}_{\rm gap} = \mathbf{H}_{\rm gap} = \mathbf{B}_{\rm in}$, while for hemispheres with bases parallel to $\mathbf{M}$, where $\mathbf{H}$ is parallel to the bases and so continuous across their surfaces, $\mathbf{B}_{\rm gap} = \mathbf{H}_{\rm gap} = \mathbf{H}_{\rm in}$.

2.2.1 Bases of the Hemispheres Perpendicular to M

For hemispheres of a sphere centered on the origin, with bases perpendicular to $\mathbf{M} = M\,\hat{\mathbf{z}}$, we compute the force on the "lower" hemisphere ($z < 0$) using a hemispherical surface whose base is the plane $z = 0$ and whose radius is so large that the fields on the hemispherical surface can be neglected. Then, the force is in the z-direction, and since $B_{z,\rm gap} = B_{z,\rm in} = 8\pi M/3$, while $B_z(r > a, z = 0) = -m/r^3 = 4\pi a^3 M/r^3$, where we also use a spherical coordinate system (r, θ, ϕ) with θ measured with respect to the z-axis,

$$\begin{aligned} F_z &= 2\pi\int_0^\infty \mathsf{T}_{zz}\, r\, dr = \frac{1}{4}\int_0^\infty B_z^2\, r\, dr = \frac{1}{4}\int_0^a \frac{64\pi^2 M^2}{9} r\, dr + \frac{1}{4}\int_a^\infty \frac{16\pi^2 a^6 M^2}{9r^6} r\, dr \\ &= \frac{8\pi^2 a^2 M^2}{9} + \frac{\pi^2 a^2 M^2}{9} = \pi^2 a^2 M^2. \end{aligned} \tag{3}$$

The force (3) is positive, meaning that the lower hemisphere is attracted to the upper.[4]

[3] Once a small gap is opened, physical surfaces are created, at which there might be surface forces that didn't exist prior in the absence of the gap. Hence, it is not clear that a stress tensor for a material without a gap should correctly predict forces once a gap is opened. However, if the magnetic force across the gap is repulsive, we might expect there to be a repulsive magnetic force across an imaginary surface inside the material at the location of the eventual gap, even if the magnitude of the repulsive force is different with and without the gap.

[4] As a check, we can evaluate the force on the lower hemisphere using a surface of integration that closely surrounds the hemisphere, just outside it. The integral over the base of this surface is again $8\pi^2 a^2 M^2\,\hat{\mathbf{z}}/9$,

2.2.2 Bases of the Hemispheres Parallel to M

For hemispheres with bases parallel to $\mathbf{M} = M\,\hat{\mathbf{z}}$ we compute the force on the "left" hemisphere ($y < 0$) using a hemispherical surface whose base is the plane $y = 0$ (the x-z plane) and whose radius is so large that the fields on the hemispherical surface can be neglected. Then, the force is in the y-direction, and since $B_{z,\rm gap} = H_{z,\rm gap} = H_{z,\rm in} = -4\pi M/3$, while $\mathbf{B}(r > a, y = 0) = (3\cos\theta\,\hat{\mathbf{r}} - \hat{\mathbf{z}})m/r^3 = 4\pi a^3 M(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})/3r^3$ and the area element is $r\,dr\,d\theta\,\hat{\mathbf{y}}$, we find,

$$\begin{aligned} F_y &= 2\int_0^\infty r\,dr\int_0^\pi d\theta\,\mathsf{T}_{yy} = -\frac{1}{4\pi}\int_0^\infty r\,dr\int_0^\pi d\theta\,B^2 \\ &= -\frac{\pi a^2/2}{4\pi}\,(4\pi M/3)^2 - \frac{1}{4\pi}\left(4\pi a^3 M/3\right)^2\int_0^\infty r\,dr\int_0^\pi d\theta\,\frac{3\cos^2\theta+1}{r^6} \\ &= -\frac{2\pi^2 a^2 M^2}{9} - \frac{4\pi a^6 M^2}{9}\frac{1}{4a^4}\frac{5\pi}{2} = -\frac{\pi^2 a^2 M^2}{2}\,. \end{aligned} \tag{5}$$

The force (5) is negative, meaning that the force between the two hemispheres is repulsive (with magnitude one half that, eq. (3), for hemispheres with bases perpendicular to $\mathbf{M}$).

2.3 Forces via Effective Magnetic Poles

The forces on the magnetization of the media can also considered as due to a density of effective magnetic poles. Some care is required to use this approach, since a true magnetic pole density ρ_M would imply $\boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi\rho_M$, and the bulk force density on these poles would be $\mathbf{F} = \rho_M\mathbf{H}$ [12].[5] However, in reality $0 = \boldsymbol{\nabla}\cdot\mathbf{B} = \boldsymbol{\nabla}\cdot(\mathbf{H} + 4\pi\mathbf{M})$, so we write,

$$\boldsymbol{\nabla}\cdot\mathbf{H} = -4\pi\boldsymbol{\nabla}\cdot\mathbf{M} = 4\pi\rho_{M,\rm eff}, \tag{6}$$

and we identify $\rho_{M,\rm eff} = -\boldsymbol{\nabla}\cdot\mathbf{M}$ as the volume density of effective magnetic poles. Inside a uniform magnetization density, as considered here, $\mathbf{B} = \mu\mathbf{H}$ and $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ together imply that $\rho_{M,\rm eff} = 0$. However, a surface density $\sigma_{M,\rm eff}$ of effective poles can exist on the surface of magnetized object, and we see that Gauss' law for the field $\mathbf{H}$ implies that,

$$\sigma_{M,\rm eff} = -\frac{\mathbf{H}\cdot\hat{\mathbf{n}}}{4\pi}\,, \tag{7}$$

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while the z-component of the integral over the hemispherical surface of radius a^+, where the magnetic field is $\mathbf{B} = (3\cos\theta\,\hat{\mathbf{r}} - \hat{\mathbf{z}})m/a^3 = (2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})4\pi M/3$ and the area element is $2\pi a^2\,d\cos\theta\,\hat{\mathbf{r}}$, is,

$$\begin{aligned} F_z &= 2\pi a^2\int_{-1}^0 (\cos\theta\mathsf{T}_{rr} - \sin\theta\mathsf{T}_{\theta r})\,d\cos\theta = \frac{a^2}{2}\int_{-1}^0\left[\cos\theta\left(B_r^2 - \frac{B^2}{2}\right) - \sin\theta B_r B_\theta\right]\,d\cos\theta \\ &= \frac{a^2}{2}\int_{-1}^0\left\{\cos\theta\left[\left(\frac{8\pi M\cos\theta}{3}\right)^2 - \frac{1}{2}\left(\frac{4\pi M}{3}\right)^2(3\cos^2\theta+1)\right] - 2\cos\theta\sin^2\theta\left(\frac{4\pi M}{3}\right)^2\right\}\,d\cos\theta \\ &= \frac{4\pi^2 a^2 M^2}{9}\int_{-1}^0\left\{\cos\theta\left[8\cos^2\theta - (3\cos^2\theta+1)\right] - 4\cos\theta(1-\cos^2\theta)\right\}\,d\cos\theta \\ &= \frac{4\pi^2 a^2 M^2}{9}\int_{-1}^0(-5\cos\theta + 9\cos^3\theta)\,d\cos\theta = \frac{\pi^2 a^2 M^2}{9}, \end{aligned} \tag{4}$$

which is the same as the integral $\int_a^\infty B_z^2(z=0)\,r\,dr/4$ in eq. (3).

[5] See [13] for additional discussion of true and effective magnetic charges.

where unit normal $\hat{\mathbf{n}}$ points outwards from the object. The effective surface pole density can also be written in terms of the magnetization $\mathbf{M} = (\mathbf{B} - \mathbf{H})/4\pi$ as,

$$\sigma_{M,\text{eff}} = \mathbf{M} \cdot \hat{\mathbf{n}}, \tag{8}$$

since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ insures that the normal component of $\mathbf{B}$ is continuous at the interface.

The force on the surface density of effective magnetic poles is,

$$\mathbf{F} = \sigma_{M,\text{eff}} \mathbf{B}_{\text{eff}} = \sigma_{M,\text{eff}} \frac{\mathbf{B}^+ + \mathbf{B}^-}{2}\,, \tag{9}$$

where $\mathbf{B}^+$ and $\mathbf{B}^-$ are the magnetic fields on the two sides of the surface where $\sigma_{M,\text{eff}}$ resides, since the effective poles (which are representations of effects of Ampèrian currents) couple to the macroscopic average of the microscopic magnetic field $\mathbf{B}$.[6,7,8]

A sphere of radius a with uniform magnetization $\mathbf{M} = M\,\hat{\mathbf{z}}$ has effective surface pole density,

$$\sigma_{M,\text{eff}}(r = a^-) = M \cos\theta. \tag{10}$$

2.3.1 Bases of the Hemispheres Perpendicular to M

The lower hemisphere also has an effective surface pole density on its base,

$$\sigma_{M,\text{eff,base}} = M, \tag{11}$$

and the field $\mathbf{B}_{\text{eff,base}}$ which acts on this density is just $\mathbf{B}_{\text{in}} = 8\pi M\,\hat{\mathbf{z}}/3$. The net force on the effective pole densities on the lower hemisphere has only a z-component,

$$\begin{aligned} F_z &= \int_S \sigma_{M,\text{eff}} B_{z,\text{eff}}\, d\text{Area} = \int_{\text{base}} \sigma_{M,\text{eff,base}} B_{z,\text{eff,base}}\, d\text{Area} + \int_{\text{hemi}} \sigma_{M,\text{eff,hemi}} B_{z,\text{eff,hemi}}\, d\text{Area} \\ &= \pi a^2 M \frac{8\pi M}{3} + 2\pi a^2 \int_{-1}^{0} \frac{M\cos\theta}{2} \left[\frac{4\pi M}{3}(3\cos^2\theta - 1) + \frac{8\pi M}{3}\right] d\cos\theta \\ &= \frac{8\pi^2 a^2 M^2}{3} + \frac{4\pi^2 a^2 M^2}{3}\left(-\frac{3}{4} + \frac{1}{2} - 1\right) = \pi^2 a^2 M^2, \end{aligned} \tag{12}$$

as previously found in eq. (3).[9]

[6] Poisson [12] worked exclusively with the magnetic field $\mathbf{H}$, but realized that the effective force on a true (Gilbertian) magnetic pole p is not necessarily $\mathbf{F} = p\mathbf{H}$ if the pole is at rest inside a bulk medium, which results in an altered force on the pole depending on the assumed shape of the surrounding cavity. W. Thomson (Lord Kelvin) noted in 1871, sec. 517 of [5], that for a pole in a disk-shaped cavity with axis parallel to the magnetization $\mathbf{M}$ of the medium, the force would be $\mathbf{F} = p(\mathbf{H} + 4\pi\mathbf{M})$, and therefore he introduced the magnetic field $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ "according to the electromagnetic definition" (in Gaussian units). In sec. 400 of his *Treatise* [6], Maxwell followed Thomson in stating that the effective force on a true magnetic pole is usefully considered to be $\mathbf{F} = p\mathbf{B}$ (Gaussian units). This convention for the effective force on a true (Gilbertian) magnetic pole is the same as the "true" force on an effective (Ampèrian) magnetic pole, which latter is the topic of the sec. 2.2 of this note.

[7] Equation (9) is in agreement with prob. 5.12 of [8]. However, the Coulomb Committee in their eq. (1.3-4) [14], and Jefimenko in his eq. (14-9.9a,b) [15], recommended that the field $\mathbf{H}$ be used rather than $\mathbf{B}$ when using the method of effective magnetic poles.

[8] If the object has permeability μ, rather than permanent magnetization, such that its magnetization arises when it is placed in an "initial" field $\mathbf{B}_i$, then this initial field, rather than the effective field $\mathbf{B}_{\text{eff}}$ should be used in eq. (9). See [9] for additional discussion.

[9] Some people claim that the field $\mathbf{H}$ rather than $\mathbf{B}$ should be used when computing the force on effective pole densities. See, for example, sec. IIA of [16]. If we use $\mathbf{H}$ in eq. (12), the predicted force on the lower

2.3.2 Bases of the Hemispheres Parallel to M

Since the magnetization $\mathbf{M}$ is parallel to the surface of the bases of the hemispheres in the plane $y = 0$, there is no effective pole density on these bases (and we don't have to face the delicate question of what is the field $\mathbf{B}_{\rm eff}$ that would act on this pole density).

The force on the left hemisphere ($y < 0$) is then given by an integral over the hemispherical surface,

$$\begin{aligned} F_y &= \int_{\rm hemi} \sigma_{M,{\rm eff},{\rm hemi}} B_{y,{\rm eff},{\rm hemi}}\, d{\rm Area} \\ &= a^2 \int_{-1}^{1} d\cos\theta \int_{\pi}^{2\pi} d\phi\, \frac{M\cos\theta}{2} \left\{ [B_r(r=a^+)\sin\theta + B_\theta(r=a^+)\cos\theta]\sin\phi + 0 \right\} \\ &= \frac{2\pi a^2 M^2}{3} \int_{-1}^{1} d\cos\theta \int_{\pi}^{2\pi} d\phi\, \cos\theta(2\cos\theta\sin\theta + \sin\theta\cos\theta)\sin\phi \qquad (14) \\ &= 2\pi a^2 M^2 \int_{-1}^{1} d\cos\theta\, \cos^2\theta\sqrt{1-\cos^2\theta} \int_{\pi}^{2\pi} d\phi\, \sin\phi = 2\pi a^2 M^2 \frac{\pi}{8}(-2) = -\frac{\pi^2 a^2 M^2}{2}\,, \end{aligned}$$

as previously found in eq. (5), using Dwight 352.01 [17].[10]

2.4 The Biot-Savart Force Law for Bound Current Densities

The present example has no conduction current density, $\mathbf{J}_{\rm cond} = 0$, so we can't use the basic form of the Biot-Savart force law,

$$\mathbf{F} = \frac{1}{c} \int \mathbf{J}_{\rm cond} \times \mathbf{B}_i\, d{\rm Vol}, \qquad (15)$$

where $\mathbf{B}_i$ is the "initial" magnetic field that exists when the conduction currents are zero.[11,12] However, we can consider the forces on the bound current densities associated with the

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hemisphere would be (assuming that a small gap between the hemispheres with $\mathbf{H}_{\rm gap} = \mathbf{B}_{\rm gap} = \mathbf{B}_{\rm in}$),

$$\begin{aligned} F_z &= \int_S \sigma_{M,{\rm eff}} H_{z,{\rm eff}}\, d{\rm Area} = \int_{\rm base} \sigma_{M,{\rm eff},{\rm base}} H_{z,{\rm eff},{\rm base}}\, d{\rm Area} + \int_{\rm hemi} \sigma_{M,{\rm eff},{\rm hemi}} H_{z,{\rm eff},{\rm hemi}}\, d{\rm Area} \\ &= \frac{\pi a^2 M}{2}\left(-\frac{4\pi M}{3} + \frac{8\pi M}{3}\right) + 2\pi a^2 \int_{-1}^{0} \frac{M\cos\theta}{2}\left[\frac{4\pi M}{3}(3\cos^2\theta - 1) - \frac{4\pi M}{3}\right] d\cos\theta \\ &= \frac{2\pi^2 a^2 M^2}{3} + \frac{4\pi^2 a^2 M^2}{3}\left(-\frac{3}{4} + \frac{1}{2} + \frac{1}{2}\right) = \pi^2 a^2 M^2, \qquad (13) \end{aligned}$$

which is the same as when $\mathbf{B}$ is used!

[10] If we use $\mathbf{H}$ rather than $\mathbf{B}$ in eq. (14), the predicted force is the same, since a y-component to the magnetic field exists only outside the sphere, where $\mathbf{B} = \mathbf{H}$. Hence, the present problem does not resolve the issue of whether $\mathbf{B}$ or $\mathbf{H}$ should be used when computing forces via effective magnetic-pole densities.

[11] See, for example, [9]. If one is interested in the force on a subset of the conduction currents, say $\mathbf{J}_{\rm cond,1}$ where $\mathbf{J}_{\rm cond} = \mathbf{J}_{\rm cond,1} + \mathbf{J}_{\rm cond,2}$, then $\mathbf{B}_i$ is the field due to $\mathbf{J}_{\rm cond,2}$ plus that due to any permanent magnetism.

[12] Biot and Savart [18, 19] were concerned with the force on a magnetic pole p due to an electric current (density), and they had no concept of a magnetic field. An expression like eq. (15) was given by Ampère (1825 [21] and by Grassmann (1845) [22], but the first version easily recognizable as the $\mathbf{J}\times\mathbf{B}$ force appeared as eq. (11) of Art. 603 of Maxwell's *Treatise* [6]. Einstein may have been the first to call this the Biot-Savart law, in sec. 2 of [23], and the first such designation in English may be that sec. 7-6 of [24].

magnetization density $\mathbf{M}$ according to,

$$\mathbf{J}_M = c\boldsymbol{\nabla} \times \mathbf{M}, \tag{16}$$

in the bulk, and,

$$\mathbf{K}_M = c\Delta\mathbf{M} \times \hat{\mathbf{n}}, \tag{17}$$

on a surface where $\Delta\mathbf{M}$ is the difference between the magnetization on its two sides, and $\hat{\mathbf{n}}$ is the outward, unit normal vector. The force on such bound current densities is,[13]

$$\mathbf{F} = \frac{1}{c}\int \mathbf{J}_M \times \mathbf{B}\, d\mathrm{Vol} + \frac{1}{c}\int \mathbf{K}_M \times \mathbf{B}_{\text{eff}}\, d\mathrm{Area} \tag{18}$$

where $\mathbf{B}_{\text{eff}}$ is the effective magnetic field at the surface, introduced in eq. (9).

In the present example, with uniform magnetization $\mathbf{M}$, the bound current density $\mathbf{J}_M$ is zero, while the bound current density on the surface of the hemispheres is,

$$\mathbf{K}_{M,\text{hemi}} = cM\,\hat{\mathbf{z}} \times \hat{\mathbf{r}} = cM \sin\theta\,\hat{\boldsymbol{\phi}}, \tag{19}$$

and the effective magnetic field at this surface which acts on $\mathbf{K}_{M,\text{hemi}}$ is,

$$\mathbf{B}_{\text{eff,hemi}} = \frac{\mathbf{B}_{\text{in}} + \mathbf{B}_{\text{ext}}(r=a)}{2} = \frac{4\pi M}{3}\hat{\mathbf{z}} + \frac{2\pi M}{3}(3\cos\theta\,\hat{\mathbf{r}} - \hat{\mathbf{z}}) = \frac{8\pi M\cos\theta}{3}\hat{\mathbf{r}} - \frac{2\pi M\sin\theta}{3}\hat{\boldsymbol{\theta}}. \tag{20}$$

The integrand of eq. (18) on a hemispherical surface is then,

$$\begin{aligned}
\frac{\mathbf{K}_{M,\text{hemi}} \times \mathbf{B}_{\text{eff,hemi}}}{c} &= \frac{8\pi M^2 \cos\theta\sin\theta}{3}\hat{\boldsymbol{\theta}} + \frac{2\pi M^2\sin^2\theta}{3}\hat{\mathbf{r}} \\
&= \frac{2\pi M^2}{3}\left[3(\cos^3\theta - \cos\theta)\,\hat{\mathbf{z}} + \sin\theta(3\cos^2\theta + 1)\,\hat{\boldsymbol{\rho}}\right] \qquad (21) \\
&= \frac{2\pi M^2}{3}\left[3(\cos^3\theta - \cos\theta)\,\hat{\mathbf{z}} + \sin\theta(3\cos^2\theta + 1)(\cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}})\right],
\end{aligned}$$

using the transformations $\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{z}} + \cos\theta\,\hat{\boldsymbol{\rho}}$ from unit vectors in spherical coordinates to those in cylindrical coordinates, and then that $\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$.

2.4.1 Bases of the Hemispheres Perpendicular to M

For hemispheres with bases in the plane $z = 0$, the bound surface current density is zero on these bases, $\mathbf{K}_M = c\mathbf{M} \times \pm\hat{\mathbf{z}} = 0$. Hence the only force on the hemispheres (from the perspective of the Biot-Savart law) is that on their hemispherical surfaces.

For the lower hemisphere ($z < 0$), this force has only a net z-component,

$$F_z = 2\pi a^2 \int_{-1}^{0} d\cos\theta\,[\mathbf{K}_M \times \mathbf{B}_{\text{eff}}]_z = 4\pi^2 a^2 M^2 \int_{-1}^{0} d\cos\theta\,(\cos^3\theta - \cos\theta) = \pi^2 a^2 M^2, \tag{22}$$

[13] In case of permeable media, rather than permanent magnets, the magnetic field in the Biot-Savart law is the "initial" field $\mathbf{B}_i$, or equivalently, the total $\mathbf{H}$ field [9]. In view of such caveats, it is generally more straightforward to use the stress tensor to compute the forces.

as previously found in eqs. (3) and (12).

Note that if we supposed that the field which acts on the bound current density were $\mathbf{H}$ rather than $\mathbf{B}$, with $\mathbf{H}_{\rm eff} = [\mathbf{H}_{\rm in} + \mathbf{H}_{\rm ext}(r=a)]/2 = [-\mathbf{B}_{\rm in}/2 + \mathbf{B}_{\rm ext}(r=a)]/2$ as the effective field on the bound current on the spherical surface, then we would not find agreement with the previous calculations of the force on the lower hemisphere.

2.4.2 Bases of the Hemispheres Parallel to M

In this case there bound surface current densities on the bases of the "left" ($y < 0$) and "right" ($y > 0$) hemispheres,

$$\mathbf{K}_{M,\rm base} = cM\,\hat{\mathbf{z}} \times \pm\hat{\mathbf{y}} = \pm cM\,\hat{\mathbf{x}}. \tag{23}$$

However, the effective field $\mathbf{B}_{\rm eff}$ that acts on these surface current densities would appear to be different depending on whether or not a small gap had opened up between the bases of the two hemispheres. If the gap had not opened, we expect that $\mathbf{B}_{\rm eff} = \mathbf{B}_{\rm in}$, while if a small gap exists, then $\mathbf{B}_{\rm eff} = (\mathbf{B}_{\rm in} + \mathbf{B}_{\rm ext})/2 = (\mathbf{B}_{\rm in} + \mathbf{H}_{\rm ext})/2 = (\mathbf{B}_{\rm in} + \mathbf{H}_{\rm in})/2 = \mathbf{B}_{\rm in}/4$, since $\mathbf{H}$ (not $\mathbf{B}$) is parallel to the base, and continuous across its surface if a gap exists.

Using $\mathbf{B}_{\rm eff,base} = \mathbf{B}_{\rm in}/4 = 2\pi M\,\hat{\mathbf{z}}/3$, when the gap exists, the force on the base of the "left" hemisphere according to the Biot-Savart law is,

$$F_{\rm base} = \frac{\pi a^2}{c}\mathbf{K}_{M,\rm base} \times \mathbf{B}_{\rm eff,base} = \pi a^2(-M\,\hat{\mathbf{x}}) \times \frac{2\pi M}{3}\hat{\mathbf{z}} = \frac{2\pi^2 a^2 M^2}{3}\hat{\mathbf{y}}. \tag{24}$$

In addition, there is force on the hemispherical surface, whose y-component is, recalling eq. (21),

$$\begin{aligned} F_{y,\rm hemi} &= \frac{1}{c}\int_{\rm hemi} [\mathbf{K}_{M,\rm hemi} \times \mathbf{B}_{\rm eff,hemi}]_y \, d\text{Area} \\ &= a^2 \int_{-1}^{1} d\cos\theta \int_{\pi}^{2\pi} d\phi\, \frac{2\pi M^2}{3} \sin\theta(3\cos^2\theta + 1)\sin\phi \\ &= -\frac{4\pi a^2 M^2}{3}\int_{-1}^{1} d\cos\theta\, \sqrt{1-\cos^2\theta}\,(3\cos^2\theta + 1) = -\frac{7\pi^2 a^2 M^2}{6}\,, \end{aligned} \tag{25}$$

using Dwight 350.01 and 352.01. Altogether, the force on the left hemisphere is,

$$F_y = F_{y,\rm base} + F_{y,\rm hemi} = \frac{2\pi^2 a^2 M^2}{3} - \frac{7\pi^2 a^2 M^2}{6} = -\frac{\pi^2 a^2 M^2}{2}\,, \tag{26}$$

as previously found in eqs. (5) and (14).

However, if we used $\mathbf{B}_{\rm eff,base}$ before a small gap opened up, we would obtain a different result, that $F_{y,\rm base} = 8\pi^2 a^2 M^2/3$, such that $F_y = 3\pi^2 a^2 M^2/2$. This net positive force implies that a gap would never open up, and that the result (26) would never apply, resulting in disagreement with the expectations from the other methods of calculation. Hence, one becomes suspicious that while the Biot-Savart law can be made to work for permanent magnets, supposing that they support bound current densities, this approach does not correctly identify the location of the force elements within the magnet.

2.5 Is the Stress Tensor Well Defined for Magnetic Materials?

A permanent magnet is a macroscopic quantum system and cannot be described in detail by classical electrodynamics. A particular example is an antiferromagnet, which can have zero macroscopic $\mathbf{M}$, $\mathbf{B}$ and $\mathbf{H}$, yet can have microstructure with negative electromagnetic field energy, as anticipated by Bethe in 1931 [25], and argued by Anderson [26] taking into account the quantum zero-point energy of the electromagnetic field. The quantum character of the energy of permanent magnets suggests that a classical description of force (with its relation to work and energy) will not be generally successful for such materials (nor for electrets[14] = systems with permanent electric dipole moments).

Maxwell's first discussion of a stress tensor in his *Treatise*, Art. 639 of [6], was for a medium for permanent magnetization density $\mathbf{M}$ that is placed in an "initial" magnetic field $\mathbf{H}_i$ (= $\mathbf{B}_i$ inside the volume occupied by magnetization $\mathbf{M}$). By considerations of an interaction energy $U = -\mathbf{M}\cdot\mathbf{H}_i$ (= $-\mathbf{M}\cdot\mathbf{B}_i$), he deduced a (magnetic) stress tensor of the form,

$$\mathsf{T}_{jk} = \frac{B_j H_{i,k}}{4\pi} - \delta_{jk}\frac{H_i^2}{8\pi}\,, \tag{27}$$

where $\mathbf{B} = \mathbf{H}_i + 4\pi M$.[15]

In the present problem $\mathbf{H}_i = \mathbf{B}_i = 0$, so Maxwell's original stress tensor would not apply here.

Note that in the present example consistency of a stress tensor inside the uniformly magnetized sphere with that in a small (vacuum) gap between hemispheres thereof requires that,

$$\mathsf{T}_{zz}(r<a, z=0) = \frac{B_{\rm in}^2}{8\pi} = \frac{8\pi M^2}{9}\,, \qquad \mathsf{T}_{yy}(r<a, y=0) = -\frac{H_{\rm in}^2}{8\pi} = -\frac{2\pi M^2}{9}\,. \tag{28}$$

If we supposed that the fields in eq. (27) were the total fields, rather than the "initial" ones, then we would find $\mathsf{T}_{zz}(r<a, z=0) = -H_{\rm in}^2/4\pi = -4\pi M^2/9$, and $\mathsf{T}_{yy}(r<a, y=0) = -H_{\rm in}^2/8\pi = -2\pi M^2/9$, which agrees with the second of eq. (28), but not with the first.

2.5.1 A Candidate Stress Tensor for Uniform Magnetization

We review the standard derivation [29] of the stress tensor, starting from the Lorentz force density,

$$\mathbf{f} = \rho_{\rm free}\mathbf{E} + \frac{\mathbf{J}_{\rm cond}}{c}\times\mathbf{B} = \frac{d\mathbf{p}_{\rm mech}}{dt}\,, \tag{29}$$

where $\rho_{\rm free}$ is the free-charge density, $\mathbf{J}_{\rm cond}$ is the conduction/free-current density, and $\mathbf{p}_{\rm mech}$ is the density of mechanical momentum in the medium. Using Maxwell's equations, this can

[14] The term electret was coined by Heaviside in 1885, p. 488 of [27].

[15] This $\mathbf{B}$ field is not the total magnetic (induction) field inside the medium, which would be $\mathbf{B}_i + \mathbf{B}_M$, where $\mathbf{B}_m$ is the magnetic field of the medium in the absence of the initial field $\mathbf{H}_i = \mathbf{B}_i$. For a long, thin permanent magnet (needle), it happens that $\mathbf{B}_M \approx 4\pi\mathbf{M}$, but this is a special case.

be written as,

$$\begin{aligned}\frac{d\mathbf{p}_{\rm mech}}{dt} &= \frac{\mathbf{E}}{4\pi}(\boldsymbol{\nabla}\cdot\mathbf{D}) - \frac{\mathbf{B}}{4\pi}\times(\boldsymbol{\nabla}\times\mathbf{H}) + \frac{\mathbf{B}}{4\pi c}\times\frac{\partial\mathbf{D}}{\partial t} \\ &= -\frac{\partial}{\partial t}\frac{\mathbf{D}\times\mathbf{B}}{4\pi c} + \frac{\mathbf{D}}{4\pi c}\times\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{D}) - \frac{\mathbf{B}}{4\pi}\times(\boldsymbol{\nabla}\times\mathbf{H}) \qquad (30)\\ &= -\frac{\partial}{\partial t}\frac{\mathbf{D}\times\mathbf{B}}{4\pi c} - \frac{\mathbf{D}}{4\pi}\times(\boldsymbol{\nabla}\times\mathbf{E}) + \frac{\mathbf{E}}{4\pi}(\boldsymbol{\nabla}\cdot\mathbf{D}) - \frac{\mathbf{B}}{4\pi}\times(\boldsymbol{\nabla}\times\mathbf{H}).\end{aligned}$$

If $\mathbf{D}$ and $\mathbf{E}$ are linear and isotropic, meaning that $\mathbf{D} = \epsilon\mathbf{E}$, where the permittivity ϵ is constant within subvolumes, then,

$$\begin{aligned}\boldsymbol{\nabla}(\mathbf{E}\cdot\mathbf{D}) = \epsilon\boldsymbol{\nabla}(\mathbf{E}\cdot\mathbf{E}) &= 2\epsilon\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) + 2\epsilon(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E} = 2\mathbf{D}\times(\boldsymbol{\nabla}\times\mathbf{E}) + 2(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{D}, \\ -\mathbf{D}\times(\boldsymbol{\nabla}\times\mathbf{E}) &= (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{D} - \boldsymbol{\nabla}\frac{\mathbf{E}\cdot\mathbf{D}}{2}, \qquad (31)\end{aligned}$$

and,

$$\begin{aligned}[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{D}) - \mathbf{D}\times(\boldsymbol{\nabla}\times\mathbf{E})]_i &= \left[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{D}) + (\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{D} - \boldsymbol{\nabla}\frac{\mathbf{E}\cdot\mathbf{D}}{2}\right]_i \\ = E_i\frac{\partial D_j}{\partial x_j} + E_j\frac{\partial D_i}{\partial x_j} - \frac{\partial}{\partial x_i}\frac{\mathbf{E}\cdot\mathbf{D}}{2} &= \frac{\partial}{\partial x_j}\left(E_iD_j - \delta_{ij}\frac{\mathbf{E}\cdot\mathbf{D}}{2}\right). \qquad (32)\end{aligned}$$

A departure from the standard derivation is to consider a medium with uniform magnetization within subvolumes, $\mathbf{B} = \mathbf{H}+4\pi\mathbf{M}$, then $\partial B_i/\partial x_j = \partial H_i/\partial x_j$ within those subvolumes, and (noting that $\boldsymbol{\nabla}\cdot\mathbf{B} = \partial B_j/\partial x_j = 0$),

$$\begin{aligned}[-\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{H})]_i &= [-\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})]_i = \left[(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{H} - \boldsymbol{\nabla}\frac{B^2}{2}\right]_i \\ &= B_j\frac{\partial H_i}{\partial x_j} - \frac{\partial}{\partial x_i}\frac{B^2}{2} = \frac{\partial}{\partial x_j}\left(H_iB_j - \delta_{ij}\frac{B^2}{2}\right). \qquad (33)\end{aligned}$$

Altogether, for a medium with linear electrical fields, but uniform permanent magnetization,

$$\frac{dp_{{\rm mech},i}}{dt} = \frac{1}{4\pi}\frac{\partial}{\partial x_j}\left(E_iD_j + H_iB_j - \delta_{ij}\frac{\mathbf{E}\cdot\mathbf{D} + B^2}{2}\right) = \frac{\partial\mathsf{T}_{ji}}{\partial x_j}, \qquad (34)$$

where the stress tensor is,[16]

$$\mathsf{T}_{ij} = \frac{E_iD_j + B_iH_j}{4\pi} - \delta_{ij}\frac{\mathbf{E}\cdot\mathbf{D} + B^2}{8\pi}. \qquad (35)$$

For the present example this implies,

$$\mathsf{T}_{zz}(r<a, z=0) = -\frac{B_{\rm in}^2}{8\pi} - \frac{B_{\rm in}^2}{8\pi} = -\frac{B_{\rm in}^2}{4\pi} = -\frac{16\pi M^2}{9}, \qquad (36)$$

$$\mathsf{T}_{yy}(r<a, y=0) = -\frac{B_{\rm in}^2}{8\pi} = -\frac{8\pi M^2}{9}, \qquad (37)$$

both of which (perhaps surprisingly) disagree with eq. (28).

A survey of four other stress tensors, and field-energy densities, of possible relevance to magnetic materials has been given in sec. 5 of [28], and is reviewed below.

[16]Because $\partial B_i/\partial x_j = \partial H_i/\partial x_j$ the term B_iH_j in eq. (35) could also be written as B_iB_j. Use of the resulting form will be considered in sec. 2.5.3 below.

2.5.2 Linear Materials, with $U = \mathbf{B}\cdot\mathbf{H}/8\pi$

The "classic" stress tensor for linear magnetic media, where $\mathbf{B} = \mu\mathbf{H}$ where μ is the magnetic permeability, was not deduced by Maxwell, but by Lorentz, p. 24 of [29], starting from the "Lorentz" force density, $\mathbf{f} = \rho\mathbf{E} + \mathbf{J}/c \times \mathbf{B}$ (which had been discussed by Maxwell in Art. 599 of [6] but not used to derive a stress tensor). Recalling sec. 2.5.1 above, we see that for a linear magnetic medium, the stress tensor would be,

$$\mathsf{T}_{ij} = \frac{B_i H_j}{4\pi} - \delta_{ij}\frac{\mathbf{B}\cdot\mathbf{H}}{8\pi} \qquad \text{(linear media)}. \tag{38}$$

For the present example this implies,

$$\mathsf{T}_{zz}(r < a, z = 0) = -\frac{B_{\rm in}^2}{8\pi} + \frac{B_{\rm in}^2}{16\pi} = -\frac{4\pi M^2}{9}\,, \tag{39}$$

$$\mathsf{T}_{yy}(r < a, y = 0) = \frac{H_{\rm in}^2}{4\pi} = \frac{4\pi M^2}{9}\,, \tag{40}$$

both of which (not surprisingly) disagree with eq. (28).

2.5.3 Permanent Magnets with $U = B^2/8\pi - \mathbf{B}\cdot\mathbf{M}$ and "M a circulation density like H"

I am unclear as to the meaning of "$\mathbf{M}$ *a circulation density like* $\mathbf{H}$*".*

While the interaction energy density of a magnetization density $\mathbf{M}$ in an "external/initial" field $\mathbf{B}$ can be written as $U_{\rm int} = -\mathbf{B}\cdot\mathbf{M}$, it is doubtful that $U = B^2/8\pi - \mathbf{B}\cdot\mathbf{M}$ represents to total electromagnetic field energy density.[17] But, under this assumption,

$$\mathsf{T}_{ij} = \frac{B_i B_j}{4\pi} - \delta_{ij}\frac{B^2}{8\pi} \qquad \text{(permanent magnet, 1)} \tag{41}$$

For the present example this implies,

$$\mathsf{T}_{zz}(r < a, z = 0) = \frac{B_{\rm in}^2}{8\pi} = \frac{8\pi M^2}{9}\,, \tag{42}$$

$$\mathsf{T}_{yy}(r < a, y = 0) = -\frac{B_{\rm in}^2}{8\pi} = -\frac{H_{\rm in}^2}{2\pi} = -\frac{8\pi M^2}{9}\,. \tag{43}$$

While eq. (42) agrees with eq. (28), eq. (43) does not.

2.5.4 Permanent Magnets with $U = B^2/8\pi - \mathbf{B}\cdot\mathbf{M}$ and "M a flux density like B"

I am unclear as to the meaning of "$\mathbf{M}$ *a flux density like* $\mathbf{B}$*".*

This form has also been advocated in [31].

$$\mathsf{T}_{ij} = \frac{B_i H_j + H_i B_j}{8\pi} - \delta_{ij}\left(\frac{B^2}{8\pi} - \mathbf{B}\cdot\mathbf{M}\right) \qquad \text{(permanent magnet, 2)}. \tag{44}$$

[17] In [28] a factor of the magnetic permeability μ appears in the energy density, $U = B^2/8\pi\mu - \mathbf{B}\cdot\mathbf{M}$, which we omit since the notion of a (linear) permeability seems inconsistent with a permanent magnet.

For the present example this implies,

$$\mathsf{T}_{zz}(r<a,z=0)=-\frac{B_{\rm in}^2}{8\pi}-\frac{B_{\rm in}^2}{8\pi}+MB_{\rm in}=-\frac{16\pi M^2}{9}+\frac{8\pi^2M^3}{3}=\frac{8\pi M^2}{9}\,, \tag{45}$$

$$\mathsf{T}_{yy}(r<a,y=0)=-\frac{B_{\rm in}^2}{8\pi}+MB_{\rm in}=-\frac{8\pi M^2}{9}+\frac{8\pi^2M^3}{3}=\frac{16\pi M^2}{9}\,. \tag{46}$$

While eq. (45) agrees with eq. (28), eq. (43) does not.

2.5.5 Permanent Magnets with $U = H^2/8\pi$ and "M a circulation density like H"

$$\mathsf{T}_{jk}=\frac{B_iB_j}{4\pi}-4\pi M_iM_j-\delta_{ij}\left(\frac{B^2}{8\pi}-2\pi M^2\right)\qquad\text{(permanent magnet, 3)} \tag{47}$$

For the present example this implies,

$$\mathsf{T}_{zz}(r<a,z=0)=\frac{B_{\rm in}^2}{8\pi}-2\pi M^2=\frac{8\pi M^2}{9}-2\pi M^2=-\frac{10\pi M^2}{9}\,, \tag{48}$$

$$\mathsf{T}_{yy}(r<a,y=0)=-\frac{B_{\rm in}^2}{8\pi}+2\pi M^2=\frac{10\pi M^2}{9}\,, \tag{49}$$

neither of which agrees with eq. (28).

2.5.6 Substitution of H for B in the Free-Space Stress Tensor

Although there seems to be little theoretical justification for this, we can consider replacing **B** by **H** in the free-space stress tensor to obtain a possible stress tensor for a permanent magnet,

$$\mathsf{T}_{jk}=\frac{H_iH_j}{4\pi}-\delta_{ij}\frac{H^2}{8\pi}\qquad\text{(permanent magnet, 4)} \tag{50}$$

For the present example this implies,

$$\mathsf{T}_{zz}(r<a,z=0)=\frac{H_{\rm in}^2}{8\pi}=\frac{2\pi M^2}{9}\,, \tag{51}$$

$$\mathsf{T}_{yy}(r<a,y=0)=-\frac{H_{\rm in}^2}{8\pi}=-\frac{2\pi M^2}{9}\,. \tag{52}$$

Then, eq. (52) agrees with eq. (28), but eq. (51) does not.

Of the six candidate stress tensors surveyed here, eqs. (41) and (50) are the only ones that have the signs of both $\mathsf{T}_{zz}(r<a,z=0)$ and $\mathsf{T}_{yy}(r<a,y=0)$ the same as needed for a successful computation via a stress tensor inside the magnets of the force between its hemispheres.

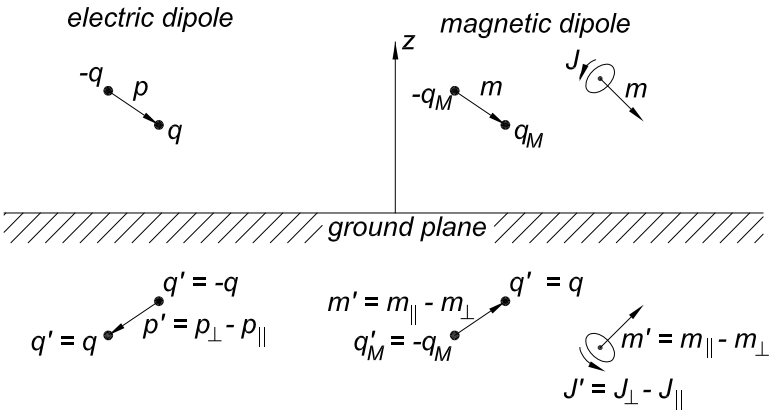
Thus, the survey tends to confirm the impression that a generally applicable stress tensor for permanent magnets cannot be given. It remains that the use of the "virtual-air-gap method", together with the stress tensor for air/vacuum seems reliable (despite its "theoretical" lack of elegance).

3 Solution for Magnetized Hemisphere + Plate

3.1 Perfectly Conducting Plate

Here, we take the view that both the electric and magnetic field vanishes inside a perfect conductor, and hence any exterior electric/magnetic field must be normal/tangential to its surface.[18]

Then, we can devise an image method for a single magnetic dipole above a perfectly conducting plane, in analogy to the image method for electric charge. Recall that for the latter, the image of electric charge q at height z above the conducting plane is a charge $-q$ at distance z below the plane, such that the sum of the electric fields of these two charges is normal to the surface $z = 0$ of the perfect conductor. Then, an electric dipole $\mathbf{p} = \mathbf{p}_\perp + \mathbf{p}_\parallel$ at height z above the surface has image $\mathbf{p}' = \mathbf{p}_\perp - \mathbf{p}_\parallel$ as distance z below the surface, as shown on the left below.



If a magnetic dipole $\mathbf{m}$ is regarded as a (Gilbertian) pair of magnetic charges/poles $\pm q_M$,[19] then the image of each magnetic charge is a magnetic charge of the same sign, such that the total magnetic field is parallel to the surface of the perfect conductor, and hence the image of the dipole is $\mathbf{m}' = \mathbf{m}_\parallel - \mathbf{m}_\perp$, as shown above. Alternatively, if the magnetic dipole is regarded as due to an Ampèrian electric current density $\mathbf{J}$, then the image of this electric current is an electric current $\mathbf{J}' = \mathbf{J}_\perp - \mathbf{J}_\parallel$, such that the image dipole is again $\mathbf{m}' = \mathbf{m}_\parallel - \mathbf{m}_\perp$.[20]

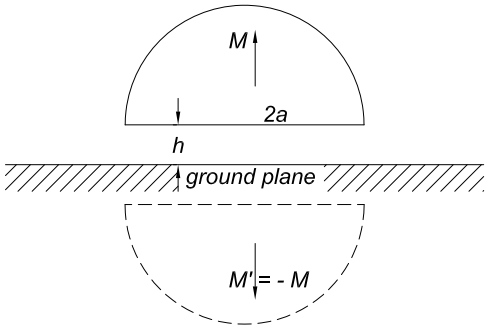
Turning to the case of a hemisphere of radius a and uniform magnetization $\mathbf{M} = M\,\hat{\mathbf{z}}$ with base parallel to a perfectly conducting plane at $z = 0$ and height h above it, the image is an inverted hemisphere with uniform magnetization $\mathbf{M}' = -\mathbf{M}$, as shown in the figure on the next page.

The total magnetic force on the hemisphere can be computed by integrating over the force on its magnetic-dipole elements $\mathbf{M}\,d\mathrm{Vol}$ due to the elements $\mathbf{M}'\,d\mathrm{Vol}'$ in the image hemisphere.

[18] Plasmas are often considered to be perfect conductors, and can be created with a "frozen-in" magnetic field which is time independent (after the plasma is created). For more discussion, see [32].

[19] For a review of how we know that permanent magnets do not actually consist of Gilbertian magnetic dipoles, see [33].

[20] For applications of these prescriptions to electric and magnetic dipole antennas, see [34, 35].



Recall that the force on magnetic dipole $\mathbf{m}$ due to dipole $\mathbf{m}'$ is,

$$\begin{aligned}\mathbf{F} &= \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_{\mathbf{m}'}) = \boldsymbol{\nabla}\left(\frac{3(\mathbf{m}\cdot\mathbf{r})(\mathbf{m}'\cdot\mathbf{r})}{R^5} - \frac{\mathbf{m}\cdot\mathbf{m}'}{R^3}\right) \\ &= 3\frac{(\mathbf{m}\cdot\mathbf{r})\mathbf{m}' + (\mathbf{m}'\cdot\mathbf{r})\mathbf{m} + (\mathbf{m}\cdot\mathbf{m}')\mathbf{r}}{R^5} - 15\frac{(\mathbf{m}\cdot\mathbf{R})(\mathbf{m}'\cdot\mathbf{R})\mathbf{R}}{R^7}\,, \qquad (53)\end{aligned}$$

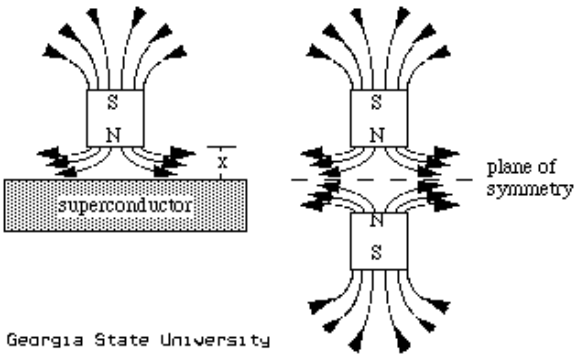
where $\mathbf{R} = \mathbf{x}_{\rm m} - \mathbf{x}_{\rm m'}$ is the position vector from $\mathbf{m}'$ to $\mathbf{m}$. To compute the force on the (upper) hemisphere, we use two spherical coordinates systems, (r,θ,ϕ) centered on the base of the upper hemisphere, and $(r',\theta'\phi')$ centered on the base of the image hemisphere. Then, the (x,y,z) components of $\mathbf{R}$ are,

$$R_x = r\sin\theta\cos\phi - r'\sin\theta'\cos\phi', \qquad (54)$$
$$R_y = r\sin\theta\sin\phi - r'\sin\theta'\sin\phi', \qquad (55)$$
$$R_z = 2h + r\cos\theta - r'\cos\theta'. \qquad (56)$$

The net force has only a z-component,

$$\begin{aligned}F_x &= \boldsymbol{\nabla}(\mathbf{m}\cdot\mathbf{B}_{\mathbf{m}'}) = \boldsymbol{\nabla}\left(\frac{3(\mathbf{m}\cdot\mathbf{r})(\mathbf{m}'\cdot\mathbf{R})}{r^5} - \frac{\mathbf{m}\cdot\mathbf{m}'}{R^3}\right) \\ &= M^2\int_0^a r^2\,dr\int_0^1 d\cos\theta\int_0^{2\pi}d\phi\int_0^a r'^2\,dr'\int_{-1}^0 d\cos\theta'\int_0^{2\pi}d\phi'\left(\frac{15R_z^3}{R^7} - \frac{9R_z}{R^5}\right) \qquad (57)\end{aligned}$$

This integral is positive although not readily evaluated analytically. For $a \ll h$ it scales as M^2a^6/h^4, while for $h \ll a$ it goes as M^2a^2. A (hemispherical) permanent magnet could be levitated above a perfectly conducting (superconducting) plane.[21]



[21] For a spherical magnet of radius a and vertical magnetization M centered at height h above a horizontal perfectly conducting plane, the magnetic-levitation force from is $8\pi^2M^2a^6/3h^4$. See Appendix B2 of [4],

3.2 High-Permeability Half Space (Refrigerator Magnet)

In this section we consider permanent magnets in vacuum that are outside a permeable half space $z < 0$ in which the (relative) permeability is μ. There are no free/conduction currents in such examples so $\boldsymbol{\nabla} \times \mathbf{H} = 0$ everywhere, and it is convenient to write $\mathbf{H} = -\boldsymbol{\nabla}\Phi_M$ where Φ_M is a scalar potential.

We begin with the case of a (Gilbertian) point magnetic charge q_M (even though these don't seem to exist in Nature) at $(x, y, z) = (0, 0, a)$. As in sec. 2.1.1 of [36] for an electric charge plus dielectric half space, a magnetic image method is to suppose that the magnetic scalar potential Φ_M in the region $z > 0$ is that due to the original point charge q_M at $(0, 0, a)$ plus an image charge q'_M at $(0, 0, -b)$,

$$\Phi_M(x, 0, z > 0) = \frac{q_M}{[x^2 + (z - a)^2]^{1/2}} + \frac{q'_M}{[x^2 + (z + b)^2]^{1/2}}\,, \tag{58}$$

and that the potential in the region $z < 0$ (inside the permeable half space) is that due to the original point charge plus a point charge q''_M at $(0, 0, c)$,

$$\Phi_M(x, 0, z < 0) = \frac{q_M}{[x^2 + (z - a)^2]^{1/2}} + \frac{q''_M}{[x^2 + (z - c)^2]^{1/2}}\,, \tag{59}$$

Continuity of the potential Φ_M across the plane $z = 0$ requires that,

$$b = c, \qquad \text{and} \qquad q''_M = q'_M. \tag{60}$$

Continuity of B_z across the plane $z = 0$ requires that $B_z(x, 0, 0^+) = H_z(x, 0, 0^+) = B_z(x, 0, 0^-) = \mu H_z(x, 0, 0^-)$, *i.e.*,

$$\frac{\partial \Phi_M(x, 0, 0^+)}{\partial z} = \mu \frac{\partial \Phi_M(x, 0, 0^-)}{\partial z}\,, \tag{61}$$

$$\frac{q_M a}{[x^2 + a^2]^{3/2}} - \frac{q'_M b}{[x^2 + b^2]^{3/2}} = \mu \left(\frac{q_M a}{[x^2 + a^2]^{3/2}} + \frac{q'_M b}{[x^2 + b^2]^{3/2}} \right), \tag{62}$$

which implies that,[22]

$$a = b = c, \qquad \text{and} \qquad q''_M = q'_M = -q \frac{\mu - 1}{\mu + 1}\,. \tag{63}$$

The potential and magnetic field $\mathbf{H} = \mathbf{B}/\mu$ in the permeable region $z < 0$ are as if the media were vacuum and the original magnetic charge q_M were replaced by charge $q_M + q''_M = 2q/(\mu + 1)$. In the region $z > 0$ the potential and $\mathbf{B} = \mathbf{H}$ fields are as for the original charge q_M plus an image charge $q' = q'' = -q(\mu - 1)/(\mu + 1)$, both in vacuum.

In the limit that $a = 0$ (such that charge q_M lies on the surface of the permeable medium the $\mathbf{H}$ field is as if the media were vacuum but the charge were $2q_M/(\mu + 1)$. Note that the $\mathbf{H}$ field is radial, and the same in both media despite their differing permeabilities, as

[22] For a metamaterial with relative permeability $\mu = -1$, eqs. (63) and (64) diverge (as would their electric equivalents if the relative permittivity were $\epsilon = -1$). However, metamaterials can have negative permeability (and/or permittivity) only for nonzero frequencies [37], so technically this divergence cannot occur. It remains possible that metamaterials could lead to large "image" forces at low frequencies [38].

is consistent with the requirement that the tangential component of $\mathbf{H}$ be continuous at the interface $z = 0$.

In the limit that $\mu \to \infty$, the potential above the plane is that due to the original charge plus an image charge $-q$ at $(0, 0, -a)$, and the potential below the plane is zero. However, this is not the image prescription for a point magnetic charge above a perfectly conducting plane, where the $\mathbf{B}$ field must be tangential to the surface of the conduction plane such that the image charge is $+q_M$ rather than $-q_M$. That is, a magnetic charge is attracted to a half space of infinite permeability, but repelled from a perfectly conducting (or superconducting) plane.

A magnetic dipole $\mathbf{m}$ can be regarded (insofar as we are only considered with its effect outside the dipole) as due to a pair of equal and opposite magnetic charges $\pm q_M$ separated by distance $\mathbf{d} = \mathbf{m}/q_M$. If this dipole is in vacuum outside a permeable half space, then the image method of point magnetic charges tells us that the potential and fields of the dipole in vacuum are as if the permeably medium were also vacuum but with an image magnetic dipole,

$$\mathbf{m}' = (\mathbf{m}_\perp - \mathbf{m}_\parallel)\frac{\mu - 1}{\mu + 1}, \qquad \text{where} \qquad \mathbf{m} = \mathbf{m}_\perp + \mathbf{m}_\parallel. \tag{64}$$

And, the $\mathbf{H}$ field inside the permeable medium is as if the medium were vacuum but the original magnetic dipole had strength $2\mathbf{m}/(\mu + 1)$.[23]

For a high-permeability medium ($\mu \gg 1$) the image dipole is $\mathbf{m}' = \mathbf{m}_\perp - \mathbf{m}_\parallel$. In particular, for a permanent magnet with magnetization perpendicular to the surface of the permeable medium, the image magnet has the same magnetization $\mathbf{m} = \mathbf{m}_\perp$ as the actual magnet. The magnet is attracted to the medium with a force $\int B^2\, d\text{Area}/8\pi$, where $\mathbf{B}$ is the magnetic field at the symmetry plane of the original magnet plus its image.[24] If the original magnet is a hemisphere of radius a and uniform magnetization $\mathbf{M}$ perpendicular to its base, the combination of original plus image magnet is spherical, and the attractive force is $\pi^2 a^2 M^2$ as found in eq. (3).

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[23] These results are discussed in [39, 40].

[24] As noted above, the field inside the permeable medium is zero in the limit infinite permeability. So, if the force on the hemispherical magnet were computed using the stress tensor just below the surface of the permeable medium, we would find that force to be zero. Instead, it is more appropriate to evaluate the stress tensor in a (virtual) gap between the magnet and the permeable medium (and not to use the average of the stress tensor on the two sides of the surface of the permeable medium).

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