Spherical Capacitor with Anisotropic Permeability

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1 Problem

What is the capacitance of a spherical capacitor with conductors of radii a and b, and permeability that varies with polar angle θ,

$$\epsilon(\cos\theta) = \epsilon_0 \sum_{n=0}^{\infty} a_n P_n(\cos\theta), \tag{1}$$

for $a < r < b$ in a spherical coordinate system (r, θ, ϕ). Here, a_n is a constant and P_n is a Legendre polynomial (with $P_n(1) = 1$).

This problem appears as no. 3.11 in [1], and was posed on a recent entrance exam in Spain for radiologists.

2 Solution

This is a statics problem, so the electric field obeys $\boldsymbol{\nabla} \times \mathbf{E} = 0$, and hence can be deduced from a scalar potential V according to $\mathbf{E} = -\boldsymbol{\nabla} V$. The problem is axially symmetric, so $V = V(r, \theta)$. However, $\nabla^2 V$ does not equal zero, so,

$$V(r, \theta) \neq \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}} \right) P_n(\cos\theta). \tag{2}$$

As there is no "free" charge in the region $a < r < b$, the electric displacement field $\mathbf{D} = \epsilon \mathbf{E}$ obeys,

$$\rho_{\text{free}} = 0 = \boldsymbol{\nabla} \cdot \mathbf{D} = \boldsymbol{\nabla} \cdot (\epsilon \mathbf{E}) = -\boldsymbol{\nabla} \cdot (\epsilon \boldsymbol{\nabla} V) = -\epsilon \nabla^2 V - \boldsymbol{\nabla} V \cdot \boldsymbol{\nabla} \epsilon, \tag{3}$$

and hence,

$$\nabla^2 V = -\frac{1}{\epsilon r^2} \frac{\partial V}{\partial \theta} \frac{d\epsilon}{d\theta} = -\frac{1 - \cos^2\theta}{r^2} \frac{\partial V}{\partial \cos\theta} \frac{\epsilon'}{\epsilon}, \tag{4}$$

where $\epsilon' = d\epsilon/d\cos\theta$. This is a linear, second-order differential equation in the potential V. We seek a separated solution that is the sum of terms of the form $V(r, \theta) = R(r)\Theta(\cos\theta)$, for which eq. (4) implies that,

$$\frac{1}{R} \frac{d(r^2 R')}{dr} = -\frac{1}{\Theta} \frac{d[(1 - \cos^2\theta)\Theta']}{d\cos\theta} - (1 - \cos^2\theta) \frac{\Theta'}{\Theta} \frac{\epsilon'}{\epsilon} \equiv n(n+1), \tag{5}$$

where $R' = dR/dr$, $\Theta' = d\Theta/d\cos\theta$ and n is a separation constant. As usual, the radial function R has the form,

$$R_n = A_n r + \frac{B_n}{r^{n+1}}, \tag{6}$$

where A_n and B_n are constants. The θ equation is,

$$\frac{d[(1-\cos^2\theta)\Theta']}{d\cos\theta} + (1-\cos^2\theta)\frac{\epsilon'}{\epsilon}\Theta' + n(n+1)\Theta = 0, \tag{7}$$

We label the solutions to eq. (7) by $f_n(\cos\theta)$, and note that $f_0 = 1$. The potential V can now be written as,

$$V(r,\theta) = \sum_n \left(A_n r^n + \frac{B_n}{r^{n+1}}\right) f_n(\cos\theta), \tag{8}$$

where the index n might be either continuous or discrete. The coefficient A_0 of the constant term in the potential can be set to zero, as only potential differences are relevant to the electric field (and to capacitance).

The nonzero electric-field components are ($E_\phi = 0$ for azimuthal symmetry),

$$E_r = -\frac{\partial V}{\partial r} = -\sum_n \left(nA_n r^{n-1} - \frac{(n+1)B_n}{r^{n+2}}\right) f_n(\cos\theta)\,, \tag{9}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = -\sum_n \left(A_n r^{n-1} + \frac{B_n}{r^{n+2}}\right) \frac{df_n(\cos\theta)}{d\theta}\,. \tag{10}$$

The tangential electric field must vanish on the conductors,

$$E_\theta(r=a) = 0 = E_\theta(r=b), \tag{11}$$

which suggests that $A_n = 0 = B_n$ for all nonzero n (for which $df_n(\cos\theta)/d\theta$ is nonzero and varies with θ).[1] That is, the electric field is purely radial, and isotropic despite the form (1) of the permeability,[2]

$$E_r = \frac{B_0}{r^2}\,, \qquad E_\theta = 0, \qquad E_\phi = 0 \qquad (a < r < b). \tag{12}$$

Hence, the components of the electric displacement field $\mathbf{D} = \epsilon\mathbf{E}$ are,

$$D_r = \frac{\epsilon_0 B_0}{r^2}\sum_{n=0}^{\infty} a_n P_n(\cos\theta), \qquad D_\theta = 0, \qquad D_\phi = 0 \qquad (a < r < b) \tag{13}$$

(such that $\boldsymbol{\nabla}\cdot\mathbf{D} = \rho_{\text{free}} = 0$ for $a < r < b$).[3]

[1]This is the one possibly doubtful step in the solution. If index n takes on only a countable set of values the claim surely holds (as $E_\theta = 0$ for all values of θ, which is a continuous parameter), but if n has continuous values the claim is less obvious.

[2]The solution given in [1] assumes this, and that $E_r \propto 1/r^2$, to be obvious, given the conditions (11). However, The electric field outside a conducting sphere does not, in general, have this form, as, for example, in case of an external point charge + conducting sphere. The key to the present problem is the presence of two concentric, conducting spherical surfaces.

[3]This verifies that all equations related to $\mathbf{D} = \epsilon\mathbf{E}$ are satisfied, and we can be confident that the solution is unique, despite the one step above that was not strictly "proven".

In case of a spherical capacitor, we consider that "free" charge Q is placed on the surface $r = a$, and "free" charge $-Q$ on the surface $r = b$. The "free" charge Q on the surface $r = a$ is then related by,

$$Q = \int_{r=a} D_r(r = a)\, d\text{Area} = 2\pi a^2 \int_{-1}^{1} d\cos\theta\, \frac{\epsilon_0 B_0}{a^2} \sum_{n=0}^{\infty} a_n P_n(\cos\theta) = 4\pi\epsilon_0 a_0 B_0. \tag{14}$$

Integration of D_r over the surface $r = b$ leads to the same relation. Hence,

$$B_0 = \frac{Q}{4\pi\epsilon_0 a_0}\,. \tag{15}$$

The potential difference between the two conductors is,

$$\Delta V = -\int_a^b E_r\, dr = -B_0 \int_a^b \frac{dr}{r^2} = B_0 \left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q}{4\pi\epsilon_0 a_0} \frac{b-a}{ab}\,, \tag{16}$$

and the capacitance C is,

$$C = \frac{Q}{\Delta V} = 4\pi\epsilon_0 a_0 \frac{ab}{b-a}\,. \tag{17}$$

The capacitance (and the electric field) is the same as if the permeability were uniform with value $\epsilon = \epsilon_0 a_0$. That is, for permeability which depends only on polar angle θ, only its uniform component (in an expansion in Legendre polynomials as in eq. (1)) influences the capacitance of a spherical capacitor.[4]

The solution can be extended to the case that the permeability varies according to,

$$\epsilon(\cos\theta, \phi) = \sqrt{4\pi}\epsilon_0 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\cos\theta, \phi), \tag{18}$$

with the capacitance then being,

$$C = \frac{Q}{\Delta V} = 4\pi\epsilon_0 a_{00} \frac{ab}{b-a}\,. \tag{19}$$

That is, the potential can again be written in a separated form,

$$V(r, \theta) = \sum_{l,m} \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}}\right) f_l^m(\cos\theta) g_m(\phi), \tag{20}$$

where A_{00} can be set to 0 and $f_0^0 = 1 = g_0$, so we can again argue that the vanishing of the tangential electric field at $r = a$ and b implies that all coefficients except B_{00} vanish.

References

[1] J.A. Cronin, D.F. Greenberg and V.I. Telegdi, *University of Chicago Graduate Problems in Physics* (U. Chicago Press, 1967), http://kirkmcd.princeton.edu/examples/EM/cronin_67.pdf

[4]For example, permeability of the form $\epsilon = \epsilon_0(1 + A\cos^2\theta) = \epsilon_0[(1 + A/3)P_0(\cos\theta) + (2A/3)P_2(\cos\theta)]$ has $a_0 = 1 + A/3$ (and not $a_0 = 1$).