Orbital and Spin Angular Momentum of Electromagnetic Fields

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1 Problem

Poynting [5] identified the flux of energy in the electromagnetic fields $\{\mathbf{E},\mathbf{B}\}$ (in a medium with relative permittivity $\epsilon = 1$ and relative permeability $\mu = 1$) with the vector,

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}, \tag{1}$$

in Gaussian units, where c is the speed of light in vacuum. Thomson [8], Poincaré [15] and Abraham [18] recognized the additional role of the Poynting vector as being proportional to the density of linear momentum stored in the electromagnetic field,

$$\mathbf{p}_{\rm EM} = \frac{\mathbf{S}}{c^2} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}, \tag{2}$$

although these arguments most clearly show that the volume integral,

$$\mathbf{P}_{\rm EM} = \int \mathbf{p}_{\rm EM}\, d\mathrm{Vol}, \tag{3}$$

rather than the integrand (2), has physical significance. This suggests that the density of angular momentum, and the total angular momentum, stored in the electromagnetic field can be written as,

$$\mathbf{l}_{\rm EM} = \mathbf{r}\times\mathbf{p}_{\rm EM} = \mathbf{r}\times\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}, \qquad \text{and} \qquad \mathbf{L}_{\rm EM} = \int \mathbf{r}\times\mathbf{p}_{\rm EM}\, d\mathrm{Vol} = \int \mathbf{r}\times\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,\mathrm{Vol}. \tag{4}$$

Show that the Helmholtz decomposition [2] of a vector field $\mathbf{F}$ into irrotational and rotational parts,

$$\mathbf{F} = \mathbf{F}_{\rm irr} + \mathbf{F}_{\rm rot}, \tag{5}$$

where,

$$\boldsymbol{\nabla}\times\mathbf{F}_{\rm irr} = 0, \qquad \text{and} \qquad \boldsymbol{\nabla}\cdot\mathbf{F}_{\rm rot} = 0, \tag{6}$$

at **all** points in space, leads to alternative forms for the densities of momentum and angular momentum in the electromagnetic fields of a system of charges e_i of rest masses m_i and velocities $\mathbf{v}_i$:

$$\mathbf{P}_{\rm total} = \mathbf{P}_{\rm mech} + \mathbf{P}_{\rm EM} = \sum_i \mathbf{p}_{{\rm canonical},\,i} + \int \mathbf{p}_{\rm EM,orbital}\, d\mathrm{Vol}, \tag{7}$$

and,

$$\mathbf{L}_{\rm total} = \mathbf{L}_{\rm mech} + \mathbf{L}_{\rm EM} = \sum_i \mathbf{l}_{{\rm canonical},\,i} + \int \mathbf{l}_{\rm EM,orbital}\, d\mathrm{Vol} + \int \mathbf{l}_{\rm EM,spin}\, d\mathrm{Vol}, \tag{8}$$

where $\mathbf{P}_{\rm mech} = \sum_i \mathbf{p}_{{\rm mech},i} = \sum_i \gamma_i, m_i v_i, \qquad \gamma_i = 1/\sqrt{1 - v_i^2/c^2},$

$$\mathbf{p}_{{\rm canonical},i} = \gamma_i m \mathbf{v}_i + \mathbf{p}_{{\rm EM,canonical},\,i}\,, \qquad \mathbf{p}_{{\rm EM,canonical},\,i} = \frac{e_i \mathbf{A}_{\rm rot}(\mathbf{r}_i)}{c}, \qquad (9)$$

$$\mathbf{p}_{\rm EM,orbital} = \frac{\sum_{j=1}^3 E_{{\rm rot},\,j} \boldsymbol{\nabla} A_{{\rm rot},\,j}}{4\pi c}\,, \qquad (10)$$

$$\mathbf{L}_{\rm mech} = \sum_i \mathbf{l}_{{\rm mech},i}, \qquad \mathbf{l}_{\rm mech,i} = r_i \times \mathbf{p}_{{\rm mech},i}, \text{ and,}$$

$$\mathbf{l}_{{\rm canonical},\,i} = \mathbf{r} \times \mathbf{p}_{{\rm canonical},\,i}\,, \qquad \mathbf{l}_{\rm EM,orbital} = \mathbf{r} \times \mathbf{p}_{\rm EM,orbital}, \qquad \mathbf{l}_{\rm EM,spin} = \frac{\mathbf{E}_{\rm rot} \times \mathbf{A}_{\rm rot}}{4\pi c}\,, \quad (11)$$

where $\mathbf{A}_{\rm rot}$ is the rotational part of the (gauge-dependent) vector potential $\mathbf{A}$, and $\mathbf{A}_{\rm rot}(\mathbf{r}_i)$ is the rotational part of the vector potential at charge i due to all other sources.[1,2,3]

2 Solution

2.1 Some History (Sept. 22, 2021)

2.1.1 Darboux and Poincaré

In 1878, Darboux [4] noted that there is a constant of the motion in the interaction of an electric charge with a magnetic pole (approximated as the tip of a long, axially magnetized needle). Poincaré [12, 25, 118] later elaborated on this theme.

The first recognition of the invariant of Darboux and Poincaré as relating to angular momentum was by Banderet (1946) [44].

2.1.2 Righi and Sadowsky

In 1894, Righi [10] made a (negative) attempt to detect rotation of a plate that absorbed circularly polarized light.

In 1899, Sadowsky published two articles [13, 14] in an obscure Russian journal, arguing that circularly polarized light carries angular momentum, and reporting a (negative) experimental search for this (also citing Righi). Sadowsky's papers were cited in [34, 36].

2.1.3 J.J. Thomson

In 1891, Thomson noted [7] that a sheet of electric displacement $\mathbf{D}$ (parallel to the surface) which moves perpendicular to its surface with velocity $\mathbf{v}$ must be accompanied by a sheet of magnetic field $\mathbf{H} = \mathbf{v}/c \times \mathbf{D}$ according to the free-space Maxwell equation $\boldsymbol{\nabla} \times \mathbf{H} =$

[1]It is claimed in eq. (B.26), p. 17, of [60] that $\mathbf{A}_{\rm rot}$ (called $\mathbf{A}_\perp$ there) is gauge invariant, but it can be modified by so-called restricted gauge transformations, as discussed in Appendix B.1 below (from [99]).

[2]The rotational electric field, $\mathbf{E}_{\rm rot}$, is nonzero only in time-dependent situations, according to eq. (37), so the quantities $\mathbf{p}_{\rm EM,orbital}$, $\mathbf{l}_{\rm EM,orbital}$ and $\mathbf{l}_{\rm EM,spin}$ are nonzero only in such cases.

[3]The various total momenta and angular momenta in eqs. (7)-(8) are the volume integrals of the densities in eqs. (9)-(11).

$(1/c)\,\partial\mathbf{D}/\partial t$.[4] Then, the motion of the energy density of these sheets implies there is also a momentum density, eqs. (2) and (6) of [7],

$$\mathbf{p}_{\rm EM}^{\rm (Thomson)} = \frac{\mathbf{D}\times\mathbf{H}}{4\pi c}\,. \qquad (12)$$

In 1893, Thomson transcribed much of his 1891 paper into the beginning of *Recent Researches* [8], adding the remark (p. 9) that the momentum density (12) is closely related to the Poynting vector [5, 9],[5,6]

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}. \qquad (13)$$

In vacuum, the field momentum density is simply,

$$\mathbf{p}_{\rm EM}^{\rm (E-B)} = \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,, \qquad (14)$$

which we consider now. Then, the total field momentum of a system is,

$$\mathbf{P}_{\rm EM}^{\rm (E-B)} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d{\rm Vol}. \qquad (15)$$

In 1904, Thomson [116, 20, 21, 22] considered the field momentum, and field angular momentum,

$$\mathbf{L}_{\rm EM}^{\rm (E-B)} = \int \mathbf{r}\times\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\,d{\rm Vol}, \qquad (16)$$

for various examples, including an electric charge together with a magnetic pole, both at rest. He computed that the field momentum in zero in the latter example according to eq. (14), while the field angular momentum is,

$$\mathbf{L}_{\rm EM}^{\rm (E-B)} = -\frac{ep}{c}\,\hat{\mathbf{r}}, \qquad (17)$$

according to eq. (15). That is, Thomson clarified in 1904 that the term $-ep\,\hat{\mathbf{r}}/c$ in Poincaré's sixth equation [12] has the physical significance of angular momentum stored in the electromagnetic field of the system. However, Thomson did not reference Poincaré in his discussion.

Thomson also discussed the case of an electric charge together with either an Ampèrian or Gilbertian magnetic dipole $\mathbf{m}$, finding that,

$$\mathbf{P}_{\rm EM}^{\rm (G)} = 0, \qquad \mathbf{P}_{\rm EM}^{\rm (A)} = \frac{\mathbf{E}\times\mathbf{m}^{\rm (A)}}{c}\,, \qquad (18)$$

[4]Variants of this argument were given by Heaviside in 1891, sec. 45 of [6], and much later in sec. 18-4 of [48], where it was noted that Faraday's law, $\boldsymbol{\nabla}\times\mathbf{E} = -(1/c)\,\partial\mathbf{B}/\partial t$, combined with the Maxwell equation for $\mathbf{H}$ implies that $v = c$ in vacuum, which point seems to have been initially overlooked by Thomson, although noted in sec. 265 of [11].

[5]The idea that an energy flux vector is the product of energy density and energy flow velocity seems to be due to Umov [3], based on Euler's continuity equation [1] for mass flow, $\boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = -\partial\rho/\partial t$.

[6]Thomson argued, in effect, that the field momentum density (14) is related by $\mathbf{p}_{\rm EM} = \mathbf{S}/c^2 = u\mathbf{v}/c^2$ [7, 8]. See also eq. (19), p. 79 of [6], and p. 6 of [30].

Thomson did not compute the electromagnetic field angular momentum for these cases, but the result is,

$$\mathbf{L}_{\rm EM} = \int \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d{\rm Vol} = \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{m}}{c}\,, \tag{19}$$

for both Ampèrian and Gilbertian magnetic dipoles.

Thomson's insights, like those of Darboux and Poincaré on this topic, were ahead of their time, and also went largely unnoticed for many years.

2.1.4 Poynting

In 1909, Poynting [23] suggested that a beam of circularly polarized light carries angular momentum, but he did not relate this to his energy-flow vector (1).

Related discussions were given in [24, 26, 27, 28, 29].

2.1.5 Darwin

In 1932, Darwin [32] discussed the quantum theory of light and identified orbital spin and angular momenta in the Fourier expansion of the quantum wave function of light. He did not invert the Fourier transform to express the angular momentum in terms of (quantum) **E** and **B** fields.[7]

Without reference to Darwin, Stewart (2004) [88] deduced that,

$$\mathbf{L}_{\rm EM} = \int d^3r\, \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c} = \mathbf{L}_{\rm EM,orbital} + \mathbf{L}_{\rm EM,spin}, \tag{20}$$

where,[8]

$$\mathbf{L}_{\rm EM,orbital} = \int \frac{d^3r}{4\pi c} \int \frac{d^3r'}{4\pi c} \frac{\mathbf{r} \times \mathbf{r}'}{|\mathbf{r} - \mathbf{r}|^3}\, \mathbf{B}(\mathbf{r}) \cdot \frac{\partial \mathbf{B}(\mathbf{r}')}{\partial t}\,, \tag{22}$$

$$\mathbf{L}_{\rm EM,spin} = \int \frac{d^3r}{4\pi c} \int \frac{d^3r'}{4\pi c} \frac{\mathbf{B}(\mathbf{r})}{|\mathbf{r} - \mathbf{r}'|} \times \frac{\partial \mathbf{B}(\mathbf{r}')}{\partial t}\,. \tag{23}$$

While the total field angular momentum can be expressed in terms of a local density, eq. (4), neither of the total orbital or spin angular momenta can be so written; they are global, not local entities.

An implication is that the separation of the angular momentum of (classical) electromagnetic fields into orbital and spin components not crisp in general — as perhaps is to be expected since the concept of intrinsic/spin angular momentum is not "classical".

[7] A version of Darwin's argument is given, without attribution, in Appendix VI of [52].

[8] In 2010, Bialynicki-Birula [106] argued that Darwin's Fourier decomposition for the total angular momentum of classical electromagnetic fields inverts to,

$$\mathbf{L}_{\rm EM,spin} = \int \frac{d^3r}{4\pi c} \mathbf{E}(\mathbf{r}) \times \int \frac{d^3r'}{4\pi} \frac{\boldsymbol{\nabla}' \times \mathbf{B}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\,. \tag{21}$$

2.1.6 Beth

The first direct experimental evidence of the angular momentum of light was obtained in 1934 by Beth [34, 36] (at Princeton University) by observations of small rotations of quartz plates on absorption of circularly polarized light.[9]

2.1.7 Vector Spherical Harmonics

Multipoles of electromagnetic fields can be described via so-called vector spherical harmonics,[10] typically with two indices, l and m. Versions with three indices, J (total angular momentum), L (orbital angular momentum) and M, have been described in sec. 7.3, p. 208 of [55].

Vector spherical harmonics have applications to radiation emitted by "point" sources such as atoms and nuclei. However, they have not found great utility in the description of the angular momentum of "beams" of light, which are generated by spatially extended sources (lasers, *etc.*).

2.1.8 Henriot, Rosenfeld and Humblet

The identification of orbital and spin parts, eqs. (10)-(11), of the electromagnetic angular momentum was anticipated by Henriot (1936) [37], but may be due to Rosenfeld (1940) [39, 41], who examined "classical" field theories of particles of various spin.[11] For the latter, see also [38, 40]. The flux of orbital and spin angular momentum was considered by Humblet (1943) [42]. These considerations are little represented in treatises on classical electrodynamics, but they are well summarized in chap. 1 of [60] (which our secs. 2.3-2.5 largely follow). See also chap. 9 of [95], and [106].

2.1.9 Cohen-Tannoudji

Our decomposition (10)-(11) is discussed in Complement B_1, p. 45 of [60] (1989), which emphasizes reciprocal (Fourier) space.

However, the statement in sec. 1.B.4(ii), p. 17, that the rotational part $\mathbf{A}_{\rm rot}$ (called $\mathbf{A}_\perp$ in [60]) of the vector potential $\mathbf{A}$ is gauge invariant in misleading. There exists an infinite set of restricted gauge transformations that change $\mathbf{A}_{\rm rot}$ while maintaining the Coulomb-gauge condition that $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$ (see Appendix B.1 below). Hence, the view that gauge-invariant quantities are "physical/real" does not apply to $\mathbf{A}_{\rm rot}$, which is better thought of as "unphysical/unreal" although very useful mathematically.[12]

[9] Direct experimental evidence for the momentum/radiation pressure of light was first given by Lebedev (1901) [16], and nearly simultaneously by Nichols and Hull [17, 19].

[10] Vector spherical harmonics were perhaps first introduced in [33] (1934), and independently in [35]. Pedagogic discussions include [58, 61] and secs. 9.7-9.8 of [82].

[11] A version of our eqs. (10)-(11) first appeared as eqs. (93)-(94) of [39].

[12] The great successes of quantum gauge theories based on 4-vector potentials (where a key feature is that the potentials can be varied "arbitrarily" with no affect on observable results) leads many people to consider these potentials to be "physical", or even "real". The present author does not subscribe to this view.

Likewise, the decomposition (10)-(11) gives the impression that $\mathbf{L}_{\rm EM,orbital}$ and $\mathbf{L}_{\rm EM,spin}$ can be expressed in terms of local densities. However, these densities are not gauge invariant, and so are not "physical" in the sense mentioned above.

2.1.10 Allen *et al.*

While "exact" descriptions of the angular momentum "beams" of light with an axis have proved elusive, great progress has been made using the so-called paraxial approximation, starting with the 1992 paper [62]. Here, the so-called Gaussian-Laguerre wavefunctions[13] emphasize the total angular momentum, and its component along the beam axis. However, these functions do not conveniently distinguish between orbital and spin angular momenta.

Subsequent papers which emphasize the Gaussian-Laguerre wavefunctions include [65, 70, 72, 73, 76, 79, 80, 83].[14]

2.1.11 Orbital and Spin Angular Momenta of Optical Beams

Discussions of the orbital and spin angular momentum of light, which are not simply separable, generally fall into two camps, that of Darwin (which uses only $\mathbf{E}$ and $\mathbf{B}$ fields) [63, 66, 67, 68, 69, 74, 75, 77, 81, 84, 85, 88, 89, 90, 92, 93, 96, 101, 102, 104, 108, 106, 107, 119, 120, 123], and of Rosenfeld (which uses $\mathbf{E}$ and $\mathbf{A}_{\rm rot}$) [45, 46, 51, 53, 59, 64, 78, 91, 97, 103, 105, 111, 114, 115, 122].

Of course, neither of these approaches leads to "simple" insights.

2.1.12 Laser Physics *vs.* Particle Physics

The contrasting points of view as to the orbital and spin angular momenta of light of the laser- and particle-physics communities has been reviewed by Leader [112, 121], and the many references therein. See also [113]. Leader favored the decomposition (10)-(11) (which he mistakenly called gauge invariant), apparently because successful computations can be made using it (in a particular gauge).

2.2 Helmholtz Decomposition of the Electromagnetic Fields and the Coulomb Gauge

Helmholtz [2, 99] showed (in a hydrodynamic context) that any vector field, say $\mathbf{F}$, which vanishes suitably quickly at infinity can be decomposed according to eq. (5), where the irrotational and rotational (or solenoidal)[15] components $\mathbf{F}_{\rm irr}$ and $\mathbf{F}_{\rm rot}$ obey eq. (6).

[13] Gaussian-Laguerre wavefunctions were introduced in the Appendix of [47] (1962).

[14] The author's derivation of Gaussian-Laguerre beams as approximations to "exact" solutions of the Helmholtz equation in oblate-spheroidal coordinates is at [86].

[15] The irrotational and rotational/solenoidal components $\mathbf{F}_{\rm irr}$ and $\mathbf{F}_{\rm rot}$ are called the longitudinal and transverse components, $\mathbf{F}_{\parallel}$ and $\mathbf{F}_{\perp}$ respectively, by some people. The latter nomenclature derives from plane waves, $\mathbf{F} = \mathbf{F}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, to which the proof of Helmholtz decomposition does not formally apply, but which is readily written as $\mathbf{F}_{\rm irr} = \mathbf{F}_{\parallel} = (\mathbf{F}\cdot\hat{\mathbf{k}})\,\hat{\mathbf{k}}$ and $\mathbf{F}_{\rm rot} = \mathbf{F}_{\perp} = \mathbf{F} - \mathbf{F}_{\parallel}$ such that $\mathbf{F}_{\perp}\cdot\mathbf{k} = 0$, and the irrotational/longitudinal and rotational/solenoidal/transverse components of $\mathbf{F}$ are parallel and perpendicular, respectively, to the wave vector $\mathbf{k}$. *However,* $\hat{\mathbf{k}}$ *is not defined for* $\mathbf{k} = 0$*, and neglect of this*

Helmholtz also showed that,[16]

$$\mathbf{F}_{\rm irr}(\mathbf{r}) = -\boldsymbol{\nabla} \int \frac{\boldsymbol{\nabla}' \cdot \mathbf{F}(\mathbf{r}')}{4\pi R}\, d{\rm Vol}', \qquad \text{and} \qquad \mathbf{F}_{\rm rot}(\mathbf{r}) = \boldsymbol{\nabla} \times \int \frac{\boldsymbol{\nabla}' \times \mathbf{F}(\mathbf{r}')}{4\pi R}\, d{\rm Vol}', \qquad (25)$$

where $R = |\mathbf{r} - \mathbf{r}'|$. Time does not appear in eq. (25), which indicates that the vector field $\mathbf{F}$ at some point $\mathbf{r}$ (and some time t) can be reconstructed from knowledge of its vector derivatives, $\boldsymbol{\nabla} \cdot \mathbf{F}$ and $\boldsymbol{\nabla} \times \mathbf{F}$, over all space (at the same time t).

An important historical significance of the Helmholtz decomposition (5) and (24) was in showing that Maxwell's equations, which give prescriptions for the derivatives of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$, are mathematically sufficient to determine those fields.

In this note we consider only media with relative permittivity $\epsilon = 1$ and relative permeability $\mu = 1$, so that Maxwell's equations can be written (in Gaussian units) in terms of the macroscopic charge and current densities, ρ and $\mathbf{J}$, as,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho, \qquad (26)$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad (27)$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \qquad (28)$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}, \qquad (29)$$

where c is the speed of light in vacuum.

It follows from eq. (28) that the magnetic field $\mathbf{B}$ is purely rotational in the sense of Helmholtz,

$$\mathbf{B}_{\rm rot} = \mathbf{B}. \qquad (30)$$

In general, the electric field $\mathbf{E}$ has both irrotational and rotational components. In Appendix A it is shown that the irrotational part of $\mathbf{E}$ at time t is the static (Coulomb) field that would exist if the charge density $\rho(\mathbf{r}, t)$ had been unchanged for all earlier times,

$$\mathbf{E}_{\rm irr}(\mathbf{r}, t) = \mathbf{E}^{\rm (C)} = \int \frac{\rho(\mathbf{r}', t)\hat{\mathbf{R}}}{R^2}\, d{\rm Vol}' = \sum_i e_i \frac{\hat{\mathbf{R}}_i}{R_i^2}, \qquad (31)$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, and in the microscopic view, e_i is the electric charge of particle i. Thus, the electric field can be purely rotational only if the macroscopic charge density ρ is everywhere zero; in the microscopic view $\mathbf{E}_{\rm irr} = 0$ only if all particles are electrically neutral.

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can lead to misunderstandings, such as that mentioned in footnote 1.

The author prefers the terms irrotational and rotational to describe the global argument of Helmholtz, because the terms longitudinal and transverse fields commonly describe only local aspects of vector fields.

[16] Using the identity that $(\boldsymbol{\nabla}' \times \mathbf{F}(\mathbf{r}'))/R = \boldsymbol{\nabla}' \times (\mathbf{F}/R) + \mathbf{F} \times \boldsymbol{\nabla}'(1/R) = \boldsymbol{\nabla}' \times (\mathbf{F}/R) + \boldsymbol{\nabla}(1/R) \times \mathbf{F}$, we can also write,

$$\mathbf{F}_{\rm rot}(\mathbf{r}) = \boldsymbol{\nabla} \times \int \boldsymbol{\nabla}' \times \frac{\mathbf{F}(\mathbf{r}')}{4\pi R}\, d{\rm Vol}' + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \int \frac{\mathbf{F}(\mathbf{r}')}{4\pi R}\, d{\rm Vol}' = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \int \frac{\mathbf{F}(\mathbf{r}')}{4\pi R}\, d{\rm Vol}', \qquad (24)$$

for fields $\mathbf{F}$ that vanish quickly enough at infinity.

That the irrotational part of the electric field can be calculated from the instantaneous charge distribution cautions us that the Helmholtz decomposition (5) does not imply independent physical significance for the partial fields $\mathbf{E}_{\rm irr}$ and $\mathbf{E}_{\rm rot}$. In general, only the total electric field $\mathbf{E}$ has the physical significance of propagation at the speed of light.

An explicit expression for the rotational part of the electric field can be given in the Darwin approximation (Appendix C), in which electrodynamics is considered only to order $1/c^2$,

$$\mathbf{E}_{\rm rot} = -\sum_i \frac{e_i}{2c^2 R_i}\left[\mathbf{a}_i + (\mathbf{a}_i \cdot \hat{\mathbf{R}}_i)\hat{\mathbf{R}}_i + \frac{3(\mathbf{v}_i \cdot \hat{\mathbf{R}}_{\mathbf{i}})^2 - v_i^2}{R_i}\hat{\mathbf{R}}_i\right], \tag{32}$$

where $\mathbf{a}_i$ and $\mathbf{v}_i$ are the acceleration and velocity of particle i.

The electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ can be related to a scalar potential V and a vector potential $\mathbf{A}$ according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \tag{33}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{34}$$

The vector field $-\boldsymbol{\nabla} V$ is purely longitudinal, but in general the vector potential $\mathbf{A}$ has both longitudinal and transverse components.

The potentials V and $\mathbf{A}$ are not unique, but can be redefined in a systematic way such that the fields $\mathbf{E}$ and $\mathbf{B}$ are invariant under such redefinition. A particular choice of the potentials is called a choice of gauge,[17] and the relations (28)-(29) are said to be gauge invariant. The gauge transformation,

$$\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\chi, \qquad V \to V - \frac{1}{c}\frac{\partial \chi}{\partial t}, \tag{35}$$

leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unchanged.

If we work in the Coulomb gauge (see Appendix B), where $\boldsymbol{\nabla} \cdot \mathbf{A}^{(\rm C)} = 0$, then $\mathbf{A}^{(\rm C)}_{\rm irr} = 0$ and $\mathbf{A}^{(\rm C)}_{\rm rot} = \mathbf{A}^{(\rm C)} = \mathbf{A}^{(\rm C)}_{\rm rot}$, so that,

$$\mathbf{E} = -\boldsymbol{\nabla} V^{(\rm C)} - \frac{\partial \mathbf{A}^{(\rm C)}}{\partial t} = -\boldsymbol{\nabla} V^{(\rm C)} - \frac{\partial \mathbf{A}^{(\rm C)}_{\rm rot}}{\partial t} = \mathbf{E}_{\rm irr} + \mathbf{E}_{\rm rot}, \tag{36}$$

where,

$$\mathbf{E}_{\rm irr} = -\boldsymbol{\nabla} V^{(\rm C)}, \qquad \mathbf{E}_{\rm rot} = -\frac{1}{c}\frac{\partial \mathbf{A}^{(\rm C)}}{\partial t} = -\frac{1}{c}\frac{\partial \mathbf{A}^{(\rm C)}_{\rm rot}}{\partial t}. \tag{37}$$

If we work in some other gauge with potentials $\mathbf{A}$ and V where the vector potential has both irrotational and rotational parts, $\mathbf{A} = \mathbf{A}_{\rm irr} + \mathbf{A}_{\rm rot}$, then the decomposition of the electric field is,[18]

$$\mathbf{E}_{\rm irr} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}_{\rm irr}}{\partial t}, \qquad \mathbf{E}_{\rm rot} = -\frac{1}{c}\frac{\partial \mathbf{A}_{\rm rot}}{\partial t}. \tag{38}$$

[17] More precisely, a particular set of potentials is said to satisfy a gauge condition on $\boldsymbol{\nabla} \cdot \mathbf{A}$, and in general there exists a restricted set of gauge-transformation functions χ that lead to alternative potentials which also satisfy the gauge condition.

[18] Equations (37)-(38) imply that a general $\mathbf{A}_{\rm rot}$ can differ from $\mathbf{A}^{(\rm C)}_{\rm rot}$ by a time-independent vector field.

The decomposition (36)-(38) of the electric field $\mathbf{E}$ into irrotational and rotational fields $\mathbf{E}_{\rm irr}$ and $\mathbf{E}_{\rm rot}$ is gauge invariant, but the simplicity of eq. (37) gives a special importance to the Coulomb gauge. However, one must remain cautious about assigning a direct physical significance to $\mathbf{A}_{\rm rot}$ because it leads to the field $\mathbf{E}_{\rm rot}$ which has components that propagate instantaneously.[19]

2.3 Total Energy of an Electromagnetic System

The electromagnetic energy $U_{\rm EM}$ of a system of charges can be written as,

$$U_{\rm EM} = \int \frac{E^2 + B^2}{8\pi}\, d{\rm Vol}. \tag{39}$$

Using the Parseval-Plancherel identity (96), we can write the electric part of the field energy as,

$$\begin{aligned} U_{\rm E} &= \int \frac{\mathbf{E}\cdot\mathbf{E}}{8\pi}\, d^3\mathbf{r} = \int \frac{\tilde{\mathbf{E}}^\star\cdot\tilde{\mathbf{E}}}{8\pi}\, d^3\mathbf{k} = \int \frac{(\tilde{\mathbf{E}}^\star_{\rm irr} + \tilde{\mathbf{E}}^\star_{\rm rot})\cdot(\tilde{\mathbf{E}}_{\rm irr} + \tilde{\mathbf{E}}_{\rm rot})}{8\pi}\, d^3\mathbf{k} \\ &= \int \frac{\tilde{\mathbf{E}}^\star_{\rm irr}\cdot\tilde{\mathbf{E}}_{\rm irr} + \tilde{\mathbf{E}}^\star_{\rm rot}\cdot\tilde{\mathbf{E}}_{\rm rot}}{8\pi}\, d^3\mathbf{k} = \int \frac{E^2_{\rm irr} + E^2_{\rm rot}}{8\pi}\, d^3\mathbf{r} \equiv U_{\rm E,irr} + U_{\rm E,rot}. \end{aligned} \tag{40}$$

Since $\boldsymbol{\nabla}\cdot\mathbf{E}_{\rm irr} = 4\pi\rho$ and $\mathbf{E}_{\rm irr} = -\boldsymbol{\nabla} V^{(\rm C)}$ (Appendix B), the field energy $U_{\rm E,irr}$ can be transformed in the usual way to the instantaneous Coulomb energy,

$$U_{\rm E,irr} = \int \frac{-\mathbf{E}_{\rm irr}\cdot\boldsymbol{\nabla} V^{(\rm C)}}{8\pi}\, d{\rm Vol} = \int \frac{\rho V^{(\rm C)}}{2}\, d{\rm Vol} = \frac{1}{2}\sum_{i\neq j} e_i V^{(\rm C)}(\mathbf{r}_i) = \frac{1}{2}\sum_{i\neq j}\frac{e_i e_j}{R_{ij}} = U^{(\rm C)}, \tag{41}$$

where $V^{(\rm C)}(\mathbf{r}_i)$ is the instantaneous Coulomb potential at charge i due to other charges. Also, since $\mathbf{B} = \mathbf{B}_{\rm rot}$, the field energy can be written as,

$$U_{\rm EM} = U^{(\rm C)} + U_{\rm E,rot} + U_{\rm B,rot} = U^{(\rm C)} + U_{\rm EM,rot}\,, \tag{42}$$

where $U_{\rm EM,rot} = U_{\rm E,rot} + U_{\rm B,rot}$.

2.3.1 Total Energy in the Darwin Approximation

In the Darwin approximation the total energy of a system of particles of rest masses m_i and electric charges e_i is given by eq. (112),

$$U = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{3 m_i v_i^4}{8c^2} + \frac{1}{2}\sum_{i\neq j}\frac{e_i e_j}{R_{ij}} + \frac{1}{2}\sum_{i\neq j}\frac{e_i e_j}{2c^2 R_{ij}}\left[\mathbf{v}_i\cdot\mathbf{v}_j + (\mathbf{v}_i\cdot\hat{\mathbf{n}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{n}}_{ij})\right]. \tag{43}$$

In this quasistatic approximation the rotational field energies are,

$$U_{\rm E,rot} = 0, \quad U_{\rm M,rot} = \frac{1}{2}\sum_i \frac{e_i \mathbf{v}_i\cdot\mathbf{A}_{\rm rot}(\mathbf{r}_i)}{c} = \frac{1}{2}\sum_{i\neq j}\frac{e_i e_j}{2c^2 R_{ij}}\left[\mathbf{v}_i\cdot\mathbf{v}_j + (\mathbf{v}_i\cdot\hat{\mathbf{n}}_{ij})(\mathbf{v}_j\cdot\hat{\mathbf{n}}_{ij})\right], \tag{44}$$

referring to eqs. (125)-(126), where $\mathbf{A}_{\rm rot}(\mathbf{r}_i)$ is the rotational part of the vector potential at charge i due to other charges.

[19] See, for example, [109].

2.4 Total Momentum of an Electromagnetic System

The total momentum associated with electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ is,

$$\mathbf{P}_{\rm EM} = \int \frac{\mathbf{S}}{c^2}\, d{\rm Vol} = \int \frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\, d{\rm Vol}, \tag{45}$$

where $\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{B}$ is the Poynting vector. We do not consider the Helmholtz decomposition of the Poynting vector, but rather a form based on the Helmholtz decomposition of the electric field,

$$\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2 = \frac{c}{4\pi}\mathbf{E}_{\rm irr}\times\mathbf{B}_{\rm rot} + \frac{c}{4\pi}\mathbf{E}_{\rm rot}\times\mathbf{B}_{\rm rot}. \tag{46}$$

Then, using the Parseval-Plancherel identity (96) and eqs. (88), (91) and (93),

$$\begin{aligned}
\mathbf{P}_{\rm EM,1} &= \int \frac{\mathbf{S}_1}{c^2}\, d{\rm Vol} = \int \frac{\mathbf{E}_{\rm irr}\times\mathbf{B}_{\rm rot}}{4\pi c}\, d^3\mathbf{r} = \int \frac{\tilde{\mathbf{E}}^\star_{\rm irr}\times\tilde{\mathbf{B}}_{\rm rot}}{4\pi c}\, d^3\mathbf{k} \\
&= \int \tilde\rho(\mathbf{k})\frac{4\pi i\hat{\mathbf{k}}}{k}\times\frac{i\mathbf{k}\times\tilde{\mathbf{A}}}{4\pi c}\, d^3\mathbf{k} = \int \frac{\tilde\rho(\mathbf{k})[\tilde{\mathbf{A}} - (\tilde{\mathbf{A}}\cdot\hat{\mathbf{k}})\,\hat{\mathbf{k}}]}{c}\, d^3\mathbf{k} \\
&= \int \frac{\tilde\rho(\mathbf{k})\tilde{\mathbf{A}}_{\rm rot}}{c}\, d^3\mathbf{k} = \int \frac{\rho\mathbf{A}_{\rm rot}}{c}\, d^3\mathbf{r} = \sum_i \frac{e_i\mathbf{A}_{\rm rot}(\mathbf{r}_i)}{c} = \mathbf{P}_{\rm EM,canonical}\,,
\end{aligned} \tag{47}$$

where $\mathbf{A}_{\rm rot}(\mathbf{r}_i)$ is the rotational part of vector potential at particle i due to all other charges.[20,21] Thus, we recognize $\mathbf{P}_{\rm EM,1}$ as the electromagnetic part, $\mathbf{P}_{\rm EM,canonical}$, of the total (gauge-invariant) canonical momentum of the system,

$$\mathbf{P}_{\rm canonical} = \sum_i\left(\mathbf{p}_i + \frac{e_i\mathbf{A}_{\rm rot}(\mathbf{r}_i)}{c}\right) = \mathbf{P}_{\rm mech} + \mathbf{P}_{\rm EM,canonical} = \mathbf{P}_{\rm mech} + \mathbf{P}_{\rm EM,1}, \tag{48}$$

where $\mathbf{p}_i = \gamma_i m_i\mathbf{v}_i$ is the (relativistic) mechanical momentum of particle i. It is often convenient to consider that the electromagnetic part of the canonical momentum of a charge is associated with the charge, although it is more correct to consider this term to be an effect of the interaction between the electromagnetic fields of that charge and the fields of other charges.

The part of the electromagnetic momentum associated with the rotational part of the electric field is,

$$\begin{aligned}
\mathbf{P}_{\rm EM,2} &= \int \frac{\mathbf{E}_{\rm rot}\times\mathbf{B}}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{E}_{\rm rot}\times(\boldsymbol{\nabla}\times\mathbf{A}_{\rm rot})}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\sum_{j=1}^3 E_{\rm rot,j}\boldsymbol{\nabla}A_{\rm rot,j} - (\mathbf{E}_{\rm rot}\cdot\boldsymbol{\nabla})\mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol}
\end{aligned}$$

[20] The rotational vector potential $\mathbf{A}_{\rm rot}$ is not unique as any gauge transformation with a gauge function χ that obeys $\nabla^2\chi = 0$ leads to another rotational vector potential.

[21] While the vector potential $\mathbf{A}_{\rm rot}$ can be nonzero in situations where the electric charge density is everywhere zero, for $\mathbf{P}_{\rm EM,1} = \mathbf{P}_{\rm EM,canonical}$ to be nonzero requires a nonzero charge density, *i.e.*, at least one charge e not balanced by neighboring charges. Then, the electric field is nonzero, the Poynting vector is nonzero, and $\mathbf{P}_{\rm EM}$ according to eq. (45) is nonzero.

$$
\begin{aligned}
&= \int \frac{\sum_{j=1}^{3} E_{\rm rot,j} \boldsymbol{\nabla} A_{\rm rot,j} + (\boldsymbol{\nabla} \cdot \mathbf{E}_{\rm rot}) \mathbf{A}_{\rm rot}}{4\pi c} \, d{\rm Vol} \\
&= \int \sum_{j=1}^{3} \frac{E_{\rm rot,j} \boldsymbol{\nabla} A_{\rm rot,j}}{4\pi c} \, d{\rm Vol} \equiv \int \mathbf{p}_{\rm EM,orbital} \, d{\rm Vol} = \mathbf{P}_{\rm EM,orbital} \, , \qquad (49)
\end{aligned}
$$

where looking ahead to eq (70) we define the momentum density,

$$
\mathbf{p}_{\rm EM,orbital} = \sum_{j=1}^{3} \frac{E_{\rm rot,j} \boldsymbol{\nabla} A_{\rm rot,j}}{4\pi c} \, . \qquad (50)
$$

The total momentum of the system can now be written as,

$$
\begin{aligned}
\mathbf{P}_{\rm total} = \mathbf{P}_{\rm canonical} + \mathbf{P}_{\rm EM,orbital} &= \sum_i \left(\mathbf{p}_i + \frac{e_i \mathbf{A}_{\rm rot}(\mathbf{r}_i)}{c} \right) + \int \sum_{j=1}^{3} \frac{E_{\rm rot,\,j} \boldsymbol{\nabla} A_{\rm rot,\,j}}{4\pi c} \, d{\rm Vol} \\
&= \mathbf{P}_{\rm mech} + \mathbf{P}_{\rm EM} = \mathbf{P}_{\rm mech} + \mathbf{P}_{\rm EM,1} + \mathbf{P}_{\rm EM,2}. \qquad (51)
\end{aligned}
$$

It is often convenient to consider that the electromagnetic part, eq. (47), of the canonical momentum of a charge is associated with the charge, although it is more correct to consider this term to be an effect ("dressing") of the interaction between the electromagnetic fields of that charge and the fields of other charges. In the former view, the momentum $\mathbf{P}_{\rm EM,2}$ associated with the rotational part of the electric field is the only momentum that is "purely" associated with the fields themselves. A pulse of electromagnetic radiation that no longer overlaps with its source charges and currents can be considered as having a purely rotational electric field, such that $\mathbf{P}_{\rm EM,2}$ describes all of the momentum of the pulse.

2.4.1 Momentum of a Circularly Polarized Plane Wave

As an example, consider a circularly polarized electromagnetic plane wave defined by the potentials (which satisfy the conditions for the Coulomb, Gibbs and Lorentz gauges),

$$
\mathbf{A} = \mathbf{A}_{\rm rot} = A_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) e^{i(kz-\omega t)}, \qquad V = 0, \qquad (52)
$$

for which the electromagnetic fields are,

$$
\mathbf{E} = \mathbf{E}_{\rm rot} = -\frac{1}{c} \frac{\partial \mathbf{A}_{\rm rot}}{\partial t} = ik\mathbf{A}_{\rm rot} = ikA_0(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) e^{i(kz-\omega t)}, \qquad (53)
$$

and,

$$
\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}_{\rm rot} = i\mathbf{k} \times \mathbf{A}_{\rm rot} = \hat{\mathbf{k}} \times \mathbf{E}, \qquad (54)
$$

where $\mathbf{k} = k\,\hat{\mathbf{z}}$. The time-average density of electromagnetic momentum associated with the (rotational) electric field is (since no electric charge is associated with the "free" plane wave),

$$
\langle \mathbf{p}_{\rm EM,1} \rangle = 0, \qquad \langle \mathbf{p}_{\rm EM,2} \rangle = \frac{1}{2} \sum_{j=1}^{3} \frac{Re(E^{\star}_{\rm rot,\,j} \boldsymbol{\nabla} A_{\rm rot,\,j})}{4\pi c} = \frac{k^2 A_0^2}{4\pi c} \, \hat{\mathbf{k}}. \qquad (55)
$$

The time-average density of electromagnetic energy is,

$$\langle u \rangle = \frac{1}{2} \frac{|E|^2 + |B|^2}{8\pi} = \frac{k^2 A_0^2}{4\pi}\,, \tag{56}$$

so that,

$$\langle \mathbf{p}_{\mathrm{EM}} \rangle = \frac{c}{8\pi} Re(\mathbf{E}^\star \times \mathbf{B}) = \langle \mathbf{p}_{\mathrm{EM},2} \rangle = \frac{\langle u \rangle}{c}\,\hat{\mathbf{k}}, \tag{57}$$

as expected.

2.4.2 Is There Such a Thing as "Spin Linear Momentum"?

Equations (45), (47) and (49) suggest that densities of momentum stored in an electromagnetic field can be defined as,

$$\mathbf{p}_{\mathrm{EM}} = \mathbf{p}_{\mathrm{EM,canonical}} + \mathbf{p}_{\mathrm{EM,orbital}} = \frac{\rho \mathbf{A}_{\mathrm{rot}}}{c} + \sum_{j=1}^{3} \frac{E_{\mathrm{rot,j}} \boldsymbol{\nabla} A_{\mathrm{rot,j}}}{4\pi c}\,, \tag{58}$$

although only the volume integrals of these densities have clear physical significance.

On comparing eqs. (50) and (69), it is suggestive to identify a density of "spin linear momentum" as,

$$\mathbf{p}_{\mathrm{EM,spin}} = -\frac{(\mathbf{E}_{\mathrm{rot}} \cdot \boldsymbol{\nabla})\mathbf{A}_{\mathrm{rot}}}{4\pi c}\,. \tag{59}$$

However, the significance of this identification is questionable, since the volume integral of $\mathbf{p}_{\mathrm{EM,spin}}$ is zero. Furthermore, $\mathbf{p}_{\mathrm{EM,spin}} = 0$ for a circularly polarized plane wave (52)-(54) whose characterization as carrying spin angular momentum is a primary motivation for the entire present analysis. Hence, we will not consider the notion of "spin linear momentum" further, although this concept has its advocates [101].

2.4.3 Total Momentum in the Darwin Approximation

In the Darwin approximation the total momentum of a system of charges is given by eq. (110),

$$\mathbf{P}_{\mathrm{total}} = \mathbf{P}_{\mathrm{canonical}} = \mathbf{P}_{\mathrm{mech}} + \mathbf{P}_{\mathrm{EM}} = \sum_i m_i \mathbf{v}_i + \sum_i \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \sum_i \frac{e_i \mathbf{A}_{\mathrm{rot}}(\mathbf{r}_i)}{c} \tag{60}$$

In this quasistatic approximation, $\mathbf{P}_{\mathrm{EM},2} = 0$, and all the electromagnetic momentum of the system can be associated with charges via the electromagnetic part of their canonical momenta, which are of order $1/c^2$ since the vector potential is of order $1/c$. Only when electrodynamic effects are considered at higher orders do they include a nonzero contribution to the electromagnetic momentum from the rotational part of the electric field. For example, the (rotational) radiation fields of an oscillating dipole are of order $1/c^2$, so the electromagnetic momentum associated with a pulse of radiation is of order $1/c^5$.

Another result in the Darwin (quasistatic) approximation is based on the simplification of the wave equation (102) for the vector potential to the static equation,

$$\boldsymbol{\nabla}^2 \mathbf{A}_{\mathrm{rot}} \approx -\frac{4\pi}{c} \mathbf{J}_{\mathrm{rot}}. \tag{61}$$

Then [50, 94],

$$\begin{aligned}
\mathbf{P}_{\rm EM,canonical} &= \int \frac{\mathbf{E}_{\rm irr} \times \mathbf{B}_{\rm rot}}{4\pi c}\, d{\rm Vol} = -\int \frac{\boldsymbol{\nabla} V^{(\rm C)} \times \mathbf{B}_{\rm rot}}{4\pi c}\, d{\rm Vol} = \int \frac{V^{(\rm C)} \boldsymbol{\nabla} \times \mathbf{B}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{V^{(\rm C)} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{A}_{\rm rot})}{4\pi c}\, d{\rm Vol} = \int \frac{V^{(\rm C)} [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}_{\rm rot}) - \nabla^2 \mathbf{A}_{\rm rot}]}{4\pi c}\, d{\rm Vol} \\
&\approx \int \frac{V^{(\rm C)} \mathbf{J}_{\rm rot}}{c^2}\, d{\rm Vol}, \qquad (62)
\end{aligned}$$

where $V^{(\rm C)}$ is the instantaneous (Coulomb) potential.

2.4.4 Potential Momentum and "Hidden" Momentum

It is sometimes considered paradoxical that a static electromagnetic system can have nonzero electromagnetic momentum (45). See, for example, [71].

The present analysis offers the perspective that in static configurations the electric field is purely irrotational, so the electromagnetic momentum (45) can be rewritten as,

$$\mathbf{P}_{\rm EM} = \mathbf{P}_{\rm EM,canonical} = \sum_i \frac{e_i \mathbf{A}_{\rm rot}(\mathbf{r}_i)}{c}\,. \qquad (63)$$

This momentum is a kind of electrical potential momentum [56, 98] associated with a charge being at a location with nonzero vector potential (due to other sources). The potential momentum $e\mathbf{A}/c$ of a charge e can be combined with the electrical potential energy eV of that charge, where V is the scalar potential at the location of the charge (due to other sources), into a potential energy-momentum 4-vector,

$$U_{{\rm potential},\,\mu} = \left(eV, \frac{e\mathbf{A}}{c} c\right) = (eV, e\mathbf{A}) = eA_\mu\,. \qquad (64)$$

The implication is that if the vector potential drops to zero, the charge takes on a mechanical momentum (in addition to any initial mechanical momentum) equal to its initial electrical potential momentum.

However, this effect is obscured in many apparently simple examples because of the fact [49] that if the center of energy,

$$\mathbf{r}_U = \frac{\int \mathbf{r}\, u_{\rm total}\, d{\rm Vol}}{\int u_{\rm total}\, d{\rm Vol}}\,, \qquad (65)$$

of a system with total-energy density $u_{\rm total}$ is at rest, then the total momentum of the system must be zero. If a static system is at rest (except for the steady currents that generate the vector potential), its center of energy will also be at rest, and the total momentum of the system must be zero. Such a system must posses a nonzero mechanical momentum equal and opposite to the electrical potential momentum (63). If the vector potential drops to zero in such a way that the center of energy remains at rest, then the mechanical momentum of the system drops to zero as well. In such cases the electrical potential momentum and the mechanical momentum are "hidden" [110].

2.5 Total Angular Momentum of an Electromagnetic System

The angular momentum of the electromagnetic fields of a system of charges can be written in terms of the Poynting vector as,

$$\mathbf{L}_{\rm EM} = \int \mathbf{r} \times \frac{\mathbf{S}}{c^2}\, d{\rm Vol} = \int \frac{\mathbf{r} \times (\mathbf{E} \times \mathbf{B})}{4\pi c}\, d{\rm Vol}, \tag{66}$$

As for the linear momentum of the fields, it is of interest to consider separately the contribution associated with the irrotational and rotational parts of the electric field.

The part of the electromagnetic angular momentum associated with $\mathbf{E}_{\rm irr} = -\boldsymbol{\nabla} V^{(\rm C)}$, for which $\boldsymbol{\nabla} \cdot \mathbf{E}_{\rm rot} = 4\pi\rho$, is,

$$\begin{aligned}
\mathbf{L}_{\rm EM,1} &= \int \frac{\mathbf{r} \times (\mathbf{E}_{\rm irr} \times \mathbf{B})}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{r} \times [\mathbf{E}_{\rm irr} \times (\boldsymbol{\nabla} \times \mathbf{A}_{\rm rot})]}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\sum_{j=1}^3 E_{\rm irr,j} (\mathbf{r} \times \boldsymbol{\nabla}) A_{\rm rot,j} - \mathbf{r} \times (\mathbf{E}_{\rm irr} \cdot \boldsymbol{\nabla}) \mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\sum_{j=1}^3 E_{\rm irr,j} (\mathbf{r} \times \boldsymbol{\nabla}) A_{\rm rot,j} - (\mathbf{E}_{\rm irr} \cdot \boldsymbol{\nabla})(\mathbf{r} \times \mathbf{A}_{\rm rot}) + \mathbf{E}_{\rm irr} \times \mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\sum_{j=1}^3 E_{\rm irr,j} (\mathbf{r} \times \boldsymbol{\nabla}) A_{\rm rot,j} - (\mathbf{E}_{\rm irr} \cdot \boldsymbol{\nabla})(\mathbf{r} \times \mathbf{A}_{\rm rot}) + \mathbf{E}_{\rm irr} \times \mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\sum_{j=1}^3 E_{\rm irr,j} (\mathbf{r} \times \boldsymbol{\nabla}) A_{\rm rot,j} + (\boldsymbol{\nabla} \cdot \mathbf{E}_{\rm irr})(\mathbf{r} \times \mathbf{A}_{\rm rot}) + \mathbf{E}_{\rm irr} \times \mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{-\sum_{j=1}^3 (\nabla_j V^{(\rm C)}) (\mathbf{r} \times \boldsymbol{\nabla}) A_{\rm rot,j} + 4\pi\rho\, (\mathbf{r} \times \mathbf{A}_{\rm rot}) - (\boldsymbol{\nabla} V^{(\rm C)}) \times \mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\sum_{j=1}^3 V^{(\rm C)} \nabla_j (\mathbf{r} \times \boldsymbol{\nabla}) A_{\rm rot,j} + 4\pi\rho\, (\mathbf{r} \times \mathbf{A}_{\rm rot}) + V^{(\rm C)} (\boldsymbol{\nabla} \times \mathbf{A}_{\rm rot})}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{V^{(\rm C)} (\mathbf{r} \times \boldsymbol{\nabla})(\boldsymbol{\nabla} \cdot \mathbf{A}_{\rm rot}) - V^{(\rm C)} (\boldsymbol{\nabla} \times \mathbf{A}_{\rm rot}) + 4\pi\rho\, (\mathbf{r} \times \mathbf{A}_{\rm rot}) + V^{(\rm C)} (\boldsymbol{\nabla} \times \mathbf{A}_{\rm rot})}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\mathbf{r} \times \rho\, \mathbf{A}_{\rm rot}}{c}\, d{\rm Vol} = \sum_i \mathbf{r}_i \times \frac{e_i \mathbf{A}_{\rm rot}(\mathbf{r}_i)}{c} = \sum_i \mathbf{r}_i \times \mathbf{P}_{{\rm EM,canonical},i} \equiv \mathbf{L}_{\rm EM,canonical}\, .
\end{aligned} \tag{67}$$

assuming the various surface integrals that result from integrations by parts vanish for fields the fall off sufficiently quickly at infinity. The sum of $\mathbf{L}_{\rm EM,1} = \mathbf{L}_{\rm EM,canonical}$ and the mechanical angular momentum of the system is,

$$\mathbf{L}_{\rm mech} + \mathbf{L}_{\rm EM,canonical} = \sum_i \mathbf{r}_i \times \left(\mathbf{p}_i + \frac{e_i \mathbf{A}_{\rm rot}(\mathbf{r}_i)}{c} \right) = \mathbf{L}_{\rm canonical}\, , \tag{68}$$

which is the canonical angular momentum of the particles of the system.

Turning to the electromagnetic angular momentum associated with the rotational part of the electric field, for which $\boldsymbol{\nabla} \cdot \mathbf{E}_{\rm rot} = 0$, we have,

$$\mathbf{L}_{\rm EM,2} = \int \frac{\mathbf{r} \times (\mathbf{E}_{\rm rot} \times \mathbf{B})}{4\pi c}\, d{\rm Vol} = \int \frac{\mathbf{r} \times [\mathbf{E}_{\rm rot} \times (\boldsymbol{\nabla} \times \mathbf{A}_{\rm rot})]}{4\pi c}\, d{\rm Vol}$$

$$
\begin{aligned}
&= \int \frac{\sum_{j=1}^{3} E_{\rm rot,j}(\mathbf{r}\times\boldsymbol{\nabla})A_{\rm rot,j} - \mathbf{r}\times(\mathbf{E}_{\rm rot}\cdot\boldsymbol{\nabla})\mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\sum_{j=1}^{3} E_{\rm rot,j}(\mathbf{r}\times\boldsymbol{\nabla})A_{\rm rot,j} - (\mathbf{E}_{\rm rot}\cdot\boldsymbol{\nabla})(\mathbf{r}\times\mathbf{A}_{\rm rot}) + \mathbf{E}_{\rm rot}\times\mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\sum_{j=1}^{3} E_{\rm rot,j}(\mathbf{r}\times\boldsymbol{\nabla})A_{\rm rot,j} - (\mathbf{E}_{\rm rot}\cdot\boldsymbol{\nabla})(\mathbf{r}\times\mathbf{A}_{\rm rot}) + \mathbf{E}_{\rm rot}\times\mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&= \int \frac{\sum_{j=1}^{3} E_{\rm rot,j}(\mathbf{r}\times\boldsymbol{\nabla})A_{\rm rot,j} + (\boldsymbol{\nabla}\cdot\mathbf{E}_{\rm rot})(\mathbf{r}\times\mathbf{A}_{\rm rot}) + \mathbf{E}_{\rm rot}\times\mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&= \int \mathbf{r}\times\frac{\sum_{j=1}^{3} E_{\rm rot,j}\boldsymbol{\nabla}A_{\rm rot,j}}{4\pi c}\, d{\rm Vol} + \int \frac{\mathbf{E}_{\rm rot}\times\mathbf{A}_{\rm rot}}{4\pi c}\, d{\rm Vol} \\
&\equiv \mathbf{L}_{\rm EM,orbital} + \mathbf{L}_{\rm EM,spin}\,, \qquad (69)
\end{aligned}
$$

where the orbital angular momentum,

$$
\mathbf{L}_{\rm EM,orbital} = \int \mathbf{l}_{\rm EM,orbital}\, d{\rm Vol}, \qquad \mathbf{l}_{\rm EM,orbital} = \mathbf{r}\times\frac{\sum_{j=1}^{3} E_{\rm rot,j}\boldsymbol{\nabla}A_{\rm rot,j}}{4\pi c} = \mathbf{r}\times\mathbf{p}_{\rm EM,orbital}\,, \quad (70)
$$

depends on the choice of origin, while,

$$
\mathbf{L}_{\rm EM,spin} = \int \mathbf{l}_{\rm EM,spin}\, d{\rm Vol}, \qquad \mathbf{l}_{\rm EM,spin} = \frac{\mathbf{E}_{\rm rot}\times\mathbf{A}_{\rm rot}}{4\pi c} \quad (71)
$$

is independent of the choice of origin and is therefore an intrinsic property of the fields, which we call the spin angular momentum.

2.5.1 Angular Momentum of a Circularly Polarized Plane Wave

As an example, consider a circularly polarized electromagnetic plane wave, eqs. (52)-(54). The time-average density of spin angular momentum is, according to eq. (11),

$$
\langle \mathbf{l}_{\rm EM,spin}\rangle = \frac{1}{2}\frac{Re(\mathbf{E}^\star_{\rm rot}\times\mathbf{A}_{\rm rot})}{4\pi c} = \frac{Re(-ikA_0\,(\hat{\mathbf{x}}\mp i\hat{\mathbf{y}})\times iA_0\,(\hat{\mathbf{x}}\pm i\hat{\mathbf{y}})}{8\pi c} = \pm\frac{kA_0^2}{4\pi c}\,\hat{\mathbf{k}}. \quad (72)
$$

Thus,

$$
\langle \mathbf{l}_{\rm EM,spin}\rangle = \pm\frac{\langle u\rangle}{\omega}\,\hat{\mathbf{k}}, \quad (73)
$$

in terms of the time-average density $\langle u\rangle = k^2 A_0^2/4\pi$, of electromagnetic energy, which is consistent with the quantum behavior of spin-1 photons.

The time-average density of orbital angular momentum is, according to eqs. (10)-(11),

$$
\langle \mathbf{l}_{\rm EM,orbital}\rangle = \mathbf{r}\times\frac{1}{2}\sum_j \frac{Re(E^\star_{\rm rot,j}\boldsymbol{\nabla}A_{\rm rot,j})}{4\pi c} = \mathbf{r}\times\langle\mathbf{p}_{\rm EM,2}\rangle = \mathbf{r}\times\langle\mathbf{p}_{\rm EM}\rangle = \mathbf{r}\times\left\langle\frac{\mathbf{E}\times\mathbf{B}}{4\pi c}\right\rangle, \quad (74)
$$

recalling eq. (55) for a plane wave. This suggests that the angular momentum integrals obey $\langle\mathbf{L}_{\rm EM,orbital}\rangle = \langle\mathbf{L}_{\rm EM}\rangle$, with the implication that $\langle\mathbf{L}_{\rm EM,spin}\rangle = 0$, in seeming contradiction with eq. (73). Of course, since the fields of an ideal plane wave do not vanish at infinity, the forms (10)-(11) do not necessarily apply to it.

In contrast, the (Darwin) prescription, eqs. (22)-(23), for field angular momentum based only on $\mathbf{E}$ and $\mathbf{B}$ yields $\mathbf{L}_{\rm EM} = \mathbf{L}_{\rm EM,spin}$, with $\mathbf{L}_{\rm EM,orbital} = 0$ for circular polarized plane waves, as discussed in sec. 4 of [88]. See also [89].

2.5.2 Is There "Really" Such a Thing as Classical Spin Angular Momentum?

Equation (66) suggests the we could define the density of angular momentum in the electromagnetic field as,

$$\mathbf{l}_{\rm EM} = \mathbf{r} \times \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\,. \tag{75}$$

Then, eq. (58) suggests that we can replace $\mathbf{E} \times \mathbf{B}/4\pi c$ by $\mathbf{p}_{\rm EM,canonical} + \mathbf{p}_{\rm EM,orbital}$ to write,

$$\mathbf{l}_{\rm EM} \stackrel{?}{=} \mathbf{r} \times (\mathbf{p}_{\rm EM,canonical} + \mathbf{p}_{\rm EM,orbital}). \tag{76}$$

In contrast, eqs. (67)-(71) suggest that we can also write,

$$\begin{aligned} \mathbf{l}_{\rm EM} &= \mathbf{l}_{\rm EM,canonical} + \mathbf{l}_{\rm EM,orbital} + \mathbf{l}_{\rm EM,spin} = \mathbf{r} \times (\mathbf{p}_{\rm EM,canonical} + \mathbf{p}_{\rm EM,orbital}) + \mathbf{l}_{\rm EM,spin} \\ &= \mathbf{r} \times \frac{\rho \mathbf{A}_{\rm rot}}{c} + \mathbf{r} \times \frac{\sum_{j=1}^{3} E_{\rm rot,j} \boldsymbol{\nabla} A_{\rm rot,j}}{4\pi c} + \frac{\mathbf{E}_{\rm rot} \times \mathbf{A}_{\rm rot}}{4\pi c}\,. \end{aligned} \tag{77}$$

The analysis that has led to the apparent contradiction between eqs. (76) and (77) assumed that the surface integrals that arise during the various integrations by parts can be neglected. This assumption is not valid for plane waves, or for monochromatic waves whose time dependence $e^{-i\omega t}$ implies these waves exist at arbitrarily early and late times. Physical waves have existed only for a finite time, and hence are bounded in space such that the surface integrals are indeed negligible. That is, neglect of the integrals on distant surfaces is a good approximation for physical fields.

Thus, the transformations (47), (49), (67) and (69) do not justify equating the integrand $\mathbf{E} \times \mathbf{B}/4\pi c$ with $\mathbf{p}_{\rm EM,canonical} + \mathbf{p}_{\rm EM,orbital}$, or equating the integrand $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})/4\pi c$ to the form $\mathbf{r} \times (\mathbf{p}_{\rm EM,canonical} + \mathbf{p}_{\rm EM,orbital}) + \mathbf{l}_{\rm EM,spin}$. In particular, the argument that led to eq. (76) does not imply that the volume integral of $\mathbf{r} \times (\mathbf{p}_{\rm EM,canonical} + \mathbf{p}_{\rm EM,orbital})$ equals the total electromagnetic angular momentum (66) of the system. While care must be taken when using the densities of momentum and angular momentum introduced here (and elsewhere), there remains a valid domain of applicability of these concepts, including the "spin" angular momentum density (71).

A related issue is how we should regard the two forms of angular momentum density (75) and (77), both of whose volume integrals yield that same total electromagnetic angular momentum for a bounded system. The form (75) suggests that all electromagnetic angular momentum is "orbital", while the form (77) includes the "intrinsic spin" angular momentum (66).

The situation here is similar to that concerning magnetostatics, where a classical model of, say, iron atoms is that each has a magnetic moment related to the microscopic current *density* $\mathbf{J}_{\rm atom}$ *within the atom,*

$$\mathbf{M}_{\rm atom} = \frac{1}{2c}\int_{\rm atom}(\mathbf{r}-\mathbf{r}_{\rm atom})\times\mathbf{J}_{\rm atom}\,d{\rm Vol} = \int_{\rm atom}\frac{\mathbf{r}\times\mathbf{J}_{\rm atom}}{2c}\,d{\rm Vol} + \frac{\mathbf{r}_{\rm atom}}{2c}\times\int\mathbf{J}_{\rm atom}\,d{\rm Vol}$$
$$= \int_{\rm atom}\frac{\mathbf{r}\times\mathbf{J}_{\rm atom}}{2c}\,d{\rm Vol}, \qquad (78)$$

which is independent of the choice of origin for steady current distributions $\mathbf{J}_{\rm atom}$.[22] *A ferromagnetic magnetic moment is considered to be an intrinsic property of the atom (and related to the "spin" angular momentum of the atom). We can calculate the total magnetic moment of a block of iron as the sum of all atomic moments, which can be transformed into an integral over the macroscopic current density* $\mathbf{J}$,

$$\mathbf{M}_{\rm total} = \sum_{\rm atoms}\mathbf{M}_{\rm atom} = \frac{1}{2c}\int\mathbf{r}\times\sum_{\rm atoms}\mathbf{J}_{\rm atom}\,d{\rm Vol} = \int\frac{\mathbf{r}\times\mathbf{J}}{2c}\,d{\rm Vol}, \qquad (79)$$

where $\mathbf{J}$ *is obtained by averaging the atoms currents* $\mathbf{J}_{\rm atom}$ *over volumes large compared to an atom but small compared to macroscopic scales. We now can define magnetization densities in two ways, microscopic and macroscopic:*

$$\mathbf{m}_{\rm micro} = \frac{\mathbf{M}_{\rm atom}}{{\rm Vol}_{\rm atom}}, \qquad \textit{and} \qquad \mathbf{m}_{\rm macro} = \frac{\mathbf{r}\times\mathbf{J}}{2c}, \qquad (80)$$

such that,

$$\mathbf{M}_{\rm total} = \int\mathbf{m}_{\rm micro}\,d{\rm Vol} = \int\mathbf{m}_{\rm macro}\,d{\rm Vol}. \qquad (81)$$

However, the microscopic and macroscopic magnetization densities are very different; a uniform microscopic density is associated with a macroscopic density that is nonzero only on the surface of the iron block.

Returning to the case of electromagnetic angular momentum, we can certainly consider the form (75) to represent the macroscopic density of electromagnetic angular momentum.[23] It is appealing to argue that the form (77) corresponds to a more microscopic description, in which the intrinsic angular momentum of "particles" of the electromagnetic field is described by the density (71) of "spin" angular momentum. Such an interpretation is not entirely justified by the usual premises of classical electrodynamics, but it is more acceptable from a quantum perspective.[24,25]

[22]Noting that $\nabla\cdot(x_i\mathbf{J}) = \mathbf{J}\cdot\nabla x_i = J_i$, we have that $\int J_i\,d{\rm Vol} = \int\nabla\cdot(x_i\mathbf{J})\,d{\rm Vol} = \oint(x_i\mathbf{J})\cdot d\mathbf{Area} = 0$ for any current distribution that is bounded in space.

[23]Comparison with the case of a uniformly magnetized block of iron suggests that the macroscopic angular momentum of an electromagnetic field with uniform "spin" angular momentum resides on the surface of the field. In the case of a circularly polarized plane wave, the "surface" is at infinity, such that the macroscopic description omits the angular momentum by the neglect of the surface integrals.

[24]It is noteworthy that the formalism of "spin" electromagnetic field angular momentum arose in the context of classical field theories [38, 39] of particles with "spin".

[25]For recent comments on this theme, see [115].

2.5.3 Comments (Oct. 1, 2021)

Despite the computational appeal of the decomposition (11) of the angular momentum of the electromagnetic field into "orbital" and "spin" components expressible as local densities in a particular gauge, this decomposition should not be regarded as "physical". Rather, the gauge-invariant decomposition (22)-(23), following Darwin [32], can be regarded as the "physical" representation of the two types of electromagnetic angular momentum. It remains that Darwin's decomposition does not provide a vision of these angular-momentum components as having local spatial densities (unlike the total electromagnetic angular momentum described by eq. (4)); rather, the "orbital" and "spin" angular momenta of the electromagnetic field are global concepts.

A Appendix: Fourier Transforms

The Fourier transform of a vector field $\mathbf{F}(\mathbf{r})$ in ordinary 3-space is the vector field $\tilde{\mathbf{F}}(\mathbf{k})$ in k-space defined by,

$$\tilde{\mathbf{F}}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int \mathbf{F}(\mathbf{r}) e^{-i\,\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{r}, \tag{82}$$

and the corresponding Fourier integral representation of $\mathbf{F}$ is,

$$\mathbf{F}(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int \tilde{\mathbf{F}}(\mathbf{k}) e^{i,\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{k}. \tag{83}$$

We symbolize the relations (82)-(83) by,

$$\mathbf{F}(\mathbf{r}) \leftrightarrow \mathcal{F}(\mathbf{k}). \tag{84}$$

For example,

$$\frac{1}{r} \leftrightarrow \frac{1}{(2\pi)^{3/2}} \frac{4\pi}{k^2}\,, \qquad \text{and} \qquad \frac{\hat{\mathbf{r}}}{r^2} \leftrightarrow \frac{1}{(2\pi)^{3/2}} \frac{-4\pi i\,\hat{\mathbf{k}}}{k}\,, \tag{85}$$

where $\hat{\mathbf{a}}$ is the unit vector $\mathbf{a}/a$.

The curl and divergence of the field $\mathbf{F}$ have Fourier transforms,

$$\boldsymbol{\nabla} \odot \mathbf{F} = \frac{1}{(2\pi)^{3/2}} \int \boldsymbol{\nabla} \odot (\tilde{\mathbf{F}}(\mathbf{k}) e^{i\,\mathbf{k}\cdot\mathbf{r}})\, d^3\mathbf{k} = \frac{1}{(2\pi)^{3/2}} \int i\,\mathbf{k} \odot \tilde{\mathbf{F}}\, d^3\mathbf{k}, \tag{86}$$

where $\odot$ represents either operation $\cdot$ or $\times$, which implies the relations,

$$\boldsymbol{\nabla} \times \mathbf{F} \leftrightarrow i\,\mathbf{k} \times \tilde{\mathbf{F}}, \qquad \boldsymbol{\nabla} \cdot \mathbf{F} \leftrightarrow i\,\mathbf{k} \cdot \tilde{\mathbf{F}}, \tag{87}$$

For example,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \leftrightarrow \tilde{\mathbf{B}} = i\,\mathbf{k} \times \tilde{\mathbf{A}}. \tag{88}$$

Then, from eq. (6) the Fourier transforms $\tilde{\mathbf{F}}_{\rm irr}$ and $\tilde{\mathbf{F}}_{\rm rot}$ of the irrotational and rotational parts, $\mathbf{F}_{\rm irr}$ and $\mathbf{F}_{\rm rot}$ of a vector field $\mathbf{F}$ obey,

$$\mathbf{k} \times \tilde{\mathbf{F}}_{\rm irr} = 0, \qquad \mathbf{k} \cdot \tilde{\mathbf{F}}_{\rm rot} = 0, \qquad \tilde{\mathbf{F}}_{\rm irr} \cdot \tilde{\mathbf{F}}_{\rm rot} = 0, \tag{89}$$

which together with the relation,

$$\mathbf{F} = \mathbf{F}_{\text{irr}} + \mathbf{F}_{\text{rot}} \leftrightarrow \tilde{\mathbf{F}} = \tilde{\mathbf{F}}_{\text{irr}} + \tilde{\mathbf{F}}_{\text{rot}} \tag{90}$$

imply that,

$$\tilde{\mathbf{F}}_{\text{irr}} = (\tilde{\mathbf{F}} \cdot \hat{\mathbf{k}})\,\hat{\mathbf{k}} = \tilde{\mathbf{F}}_{\parallel}, \qquad \tilde{\mathbf{F}}_{\text{rot}} = \tilde{\mathbf{F}} - \tilde{\mathbf{F}}_{\text{irr}} = \tilde{\mathbf{F}}_{\perp}. \tag{91}$$

As an example, the Maxwell equation (26) has the Fourier transform,

$$i\,\mathbf{k} \cdot \tilde{\mathbf{E}} = 4\pi\tilde{\rho}(\mathbf{k}), \tag{92}$$

where $\tilde{\rho}(\mathbf{k})$ is the transform of $\rho(\mathbf{r})$, so the irrotational part of $\tilde{\mathbf{E}}$ is,

$$\tilde{\mathbf{E}}_{\text{irr}} = \tilde{\rho}(\mathbf{k})\frac{-4\pi i\,\hat{\mathbf{k}}}{k}, \tag{93}$$

which is the product of two Fourier transforms, $\tilde{F} = \tilde{\rho}(\mathbf{k})$ and $\tilde{\mathbf{G}} = -4\pi i\,\hat{\mathbf{k}}/k$. In general, the product $\tilde{F}(\mathbf{k})\tilde{G}(\mathbf{k})$ of the Fourier transforms of scalar fields $F(\mathbf{r})$ and $G(\mathbf{r})$ has the inverse transform,

$$\frac{1}{(2\pi)^{3/2}}\int F(\mathbf{r}')G(\mathbf{r} - \mathbf{r}')\,d^3\mathbf{r}', \tag{94}$$

which is not $F(\mathbf{r})G(\mathbf{r})$ but their spatial convolution. Using eqs. (93)-(94) together with eq. (85), we find the irrotational part of the electric field to be,

$$\mathbf{E}_{\text{irr}} = \int \frac{\rho(\mathbf{r}')\hat{\mathbf{R}}}{R^2}\,d^3\mathbf{r}' = \mathbf{E}^{(\text{C})}, \tag{95}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Thus, the irrotational part of the electric field $\mathbf{E}$ at time t is the instantaneous Coulomb field $\mathbf{E}^{(\text{C})}$ of the electric charge density $\rho(\mathbf{r}, t)$, *i.e.*, its "static" part, as would hold if the present charge density had never been different in the past.

We also note the Parseval-Plancherel identity for two scalar fields $F(\mathbf{r})$ and $G(\mathbf{r})$ with Fourier transforms $\tilde{F}(\mathbf{k})$ and $\tilde{G}(\mathbf{k})$:

$$\int F^\star(\mathbf{r})G(\mathbf{r})\,d^3\mathbf{r} = \int \tilde{F}^\star(\mathbf{k})\tilde{G}(\mathbf{k})\,d^3\mathbf{k}. \tag{96}$$

B Appendix: Coulomb Gauge

The vector potential in the Coulomb gauge is chosen to be purely rotational/transverse,

$$\boldsymbol{\nabla} \cdot \mathbf{A}^{(\text{C})} = 0, \qquad \text{so that,} \qquad \mathbf{A}^{(\text{C})} = \mathbf{A}_{\text{rot}} \qquad \text{(Coulomb).} \tag{97}$$

We restrict our discussion to media for which the relative permittivity is $\epsilon = 1$ and the relative permeability is $\mu = 1$. Then, using eq. (28) in the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi$, the scalar potential V in any gauge obeys,

$$\boldsymbol{\nabla}^2 V + \frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \mathbf{A} = -4\pi\rho, \tag{98}$$

and the Maxwell equation $\boldsymbol{\nabla} \times \mathbf{B} = (4\pi/c)\mathbf{J} + \partial\mathbf{E}/\partial ct$ leads to,

$$\boldsymbol{\nabla}^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{J} + \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c}\frac{\partial V}{\partial t}\right) \tag{99}$$

for the vector potential in any gauge.

Thus, in the Coulomb gauge, eq. (98) becomes Poisson's equation,

$$\boldsymbol{\nabla}^2 V^{(\mathrm{C})} = -4\pi\rho, \tag{100}$$

which has the formal solution,

$$V^{(\mathrm{C})}(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t)}{R}\, d\mathrm{Vol}' \qquad (\text{Coulomb}), \tag{101}$$

where $R = |\mathbf{r} - \mathbf{r}'|$, in which changes in the charge distribution ρ instantaneously affect the potential $V^{(\mathrm{C})}$ at any distance.

In the Coulomb gauge, eq. (99) becomes, using the continuity equation, $\boldsymbol{\nabla}\cdot\mathbf{J} + \partial\rho/\partial t = 0$,

$$\begin{aligned}\boldsymbol{\nabla}^2 \mathbf{A}^{(\mathrm{C})} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}^{(\mathrm{C})}}{\partial t^2} &= -\frac{4\pi}{c}\mathbf{J} + \frac{\boldsymbol{\nabla}}{c}\frac{\partial V^{(\mathrm{C})}}{\partial t} = -\frac{4\pi}{c}\mathbf{J} - \frac{4\pi}{c}\boldsymbol{\nabla}\int \frac{\boldsymbol{\nabla}'\cdot\mathbf{J}(\mathbf{r}', t)}{4\pi R}\, d\mathrm{Vol}' \\ &= -\frac{4\pi}{c}(\mathbf{J} - \mathbf{J}_{\mathrm{irr}}) = -\frac{4\pi}{c}\mathbf{J}_{\mathrm{rot}},\end{aligned} \tag{102}$$

using eqs. (25), (101) and the continuity equation, $\boldsymbol{\nabla}\cdot\mathbf{J} = -\partial\rho/\partial t$. Thus, a formal solution for the (retarded, rotational) vector potential in the Coulomb gauge is,[26]

$$\mathbf{A}^{(\mathrm{C})}_{\mathrm{rot}}(\mathbf{r}, t) = \mathbf{A}^{(\mathrm{C})}(\mathbf{r}, t) = \frac{1}{c}\int \frac{\mathbf{J}_{\mathrm{rot}}(\mathbf{r}', t' = t - R/c)}{R}\, d\mathrm{Vol}' \qquad (\text{Coulomb}), \tag{104}$$

where the rotational part of the current density is given by,

$$\mathbf{J}_{\mathrm{rot}}(\mathbf{r}, t) = \frac{1}{4\pi}\boldsymbol{\nabla}\times\int \frac{\boldsymbol{\nabla}'\times\mathbf{J}(\mathbf{r}', t)}{R}\, d\mathrm{Vol}' = \frac{1}{4\pi}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int \frac{\mathbf{J}(\mathbf{r}', t)}{R}\, d\mathrm{Vol}', \tag{105}$$

which goes to zero at large distance if the current density lies within a bounded volume.

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[26]In case of steady currents, $\boldsymbol{\nabla}\cdot\mathbf{J} = 0$, $\mathbf{J}_{\mathrm{rot}} = \mathbf{J}$, and a formal solution is,

$$\mathbf{A}^{(\mathrm{C})}(\mathbf{r}) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}')}{R}\, d\mathrm{Vol}' \qquad (\text{Coulomb, static}). \tag{103}$$

Other forms of the Coulomb-gauge vector potential are possible, via the restricted gauge transformation $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi$, where the gauge-transformation function obeys Laplace's equation,$\boldsymbol{\nabla}\chi = 0$. However, the uniqueness theorem for solutions to Laplace's equation with Neumann boundary conditions (see sec. 1.9 of [82]), the only possible χ for which $\boldsymbol{\nabla}\chi = 0$ at large distances is the trivial case $\chi = 0$. Hence, any alternative Coulomb-gauge vector potential does not go to zero everywhere at infinity, and the static form (103) plays a somewhat special role.

B.1 Alternative Forms of the Coulomb-Gauge Potentials

While the potentials (101) and (104) can be considered to be the standard forms for the Coulomb gauge, they are not the only possible ones.

If the gauge-transformation function χ obeys Laplace's equation, $\nabla^2\chi = 0$, then the vector potential of the gauge transformation $\mathbf{A}^{(C)} \to \mathbf{A}^{(C)} + \boldsymbol{\nabla}\chi$, $V^{(C)} \to V^{(C)} - \partial\chi/\partial t$, also satisfies the Coulomb-gauge condition (97).[27]

In quasistatic examples of charge and current densities within a bounded volume, and where radiation can be ignored, the standard potentials (101) and (104) go to zero at large distances. Then, all of the alternative Coulomb-gauge potentials, generated by a gauge function χ that obeys Laplace's equation do not go to zero at infinity in all directions. This follows from the uniqueness theorem for solutions to Laplace's equation with either Dirichlet or Neumann boundary conditions (see, for example sec. 1.9 of [82]), since the trivial case $\chi = 0$ has both χ and its derivatives equal to zero (at large distances). So, when one adds the constraint that the Coulomb-gauge potentials must vanish at infinity, then the standard forms (101) and (104) are the only such solutions (if indeed they vanish at infinity).[28]

Of course, only for electro- and/or magnetostatic examples (or for those inside a bounded perfectly conducting surface) can there be potentials that vanish at infinity, so for all examples involving "radiation to infinity" ("real photons"), we must accept the nonuniqueness of Coulomb-gauge potentials.

C Appendix: Darwin's Approximation

The Lagrangian for a charge e of mass m that moves with velocity $\mathbf{v}$ in an external electromagnetic field that is described by (Coulomb-gauge) potentials $V^{(C)}$ and $\mathbf{A}^{(C)}$ can be written (see, for example, sec. 16 of [54]),

$$\mathcal{L} = -mc^2\sqrt{1 - v^2/c^2} - eV^{(C)} + e\frac{\mathbf{v}}{c} \cdot \mathbf{A}^{(C)}. \tag{106}$$

Darwin [31] worked in the Coulomb gauge, and kept term only to order v^2/c^2. Then, the scalar and vector potentials due to a charge e that has velocity $\mathbf{v}$ can be taken as (see sec. 65 of [54] or sec. 12.6 of [82]),

$$V^{(C)} = \frac{e}{R}, \qquad \mathbf{A}^{(C)} = \frac{e[\mathbf{v} + (\mathbf{v}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]}{2cR}, \tag{107}$$

[27]For example, the gauge functions $\chi = \pm Bxy/2$ lead from the axially symmetric vector potential, $\mathbf{A}^{(C)} = BR^2\,\hat{\boldsymbol{\phi}}/2\rho$, of a uniform magnetic field $B\,\hat{\mathbf{z}}$ inside an axially symmetric current distribution on a cylinder of radius R about the z-axis, to the so-called "Landau" potentials, which are nonzero at $\rho = \sqrt{x^2 + y^2} = \infty$. See also sec. 2.1 of [117].

[28]It is claimed in eq. (B.26), p. 17, of [60] that $\mathbf{A}_{\rm rot}$ (called $\mathbf{A}_\perp$ there) is gauge invariant, since the Fourier transform of the gauge transformation $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\chi$ is $\mathbf{A}'_{\mathbf{k}} = \mathbf{A}_{\mathbf{k}} + i\mathbf{k}\,\chi_{\mathbf{k}}$. This makes it appear that the term $i\mathbf{k}\,\chi_{\mathbf{k}}$ contributes only to the irrotational part of $\mathbf{A}'$, since the Fourier transform of $\mathbf{A}_{\rm irr}$ is $\mathbf{A}_{\mathbf{k},{\rm irr}} = (\hat{\mathbf{k}}\cdot\mathbf{A}_{\mathbf{k}})\,\hat{\mathbf{k}}$ (as in eq. (B.14a) of [60]), such that $\mathbf{A}'_{\rm rot} = \mathbf{A}_{\rm rot}$. However, $\hat{\mathbf{k}}$ is undefined for $\mathbf{k} = 0$, such that the Fourier component $\mathbf{A}_{\mathbf{k}=\mathbf{0}}$ is entirely rotational. Then, if $\nabla^2\chi = 0$, its Fourier transform is $0 = -k^2\chi_{\mathbf{k}} = i\mathbf{k}\cdot(i\mathbf{k}\,\chi_{\mathbf{k}})$, such that $i\mathbf{k}\,\chi_{\mathbf{k}}$ can be nonzero for $\mathbf{k} = \mathbf{0}$, in which case $\boldsymbol{\nabla}\chi$ contributes to $\mathbf{A}'_{\rm rot}$ and this differs from $\mathbf{A}_{\rm rot}$.

where $\hat{\mathbf{n}}$ is directed from the charge to the observer, whose (present) distance is R.

Combining equations (106) and (107) for a collections of charged particles, and keeping terms only to order v^2/c^2, we arrive at the Darwin Lagrangian,

$$\mathcal{L} = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{m_i v_i^4}{8c^2} - \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} \left[\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})\right], \qquad (108)$$

where we ignore the constant sum of the rest energies of the particles.

The Lagrangian (108) does not depend explicitly on time, so the corresponding Hamiltonian,

$$\begin{aligned} \mathcal{H} &= \sum_i \mathbf{p}_i \cdot \mathbf{v}_i - \mathcal{L} \\ &= \sum_i \frac{p_i^2}{2m_i} - \sum_i \frac{p_i^4}{8m_i^3 c^2} + \sum_{i>j} \frac{e_i e_j}{R_{ij}} - \sum_{i>j} \frac{e_i e_j}{2m_i m_j c^2 R_{ij}} \left[\mathbf{p}_i \cdot \mathbf{p}_j + (\mathbf{p}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{p}_j \cdot \hat{\mathbf{n}}_{ij})\right], \quad (109) \end{aligned}$$

is the conserved energy of the system, where,

$$\begin{aligned} \mathbf{p}_i &= \frac{\partial \mathcal{L}}{\partial \mathbf{v}_i} = m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \sum_{j \neq i} \frac{e_i e_j}{2c^2 R_{ij}} \left[\mathbf{v}_j + \hat{\mathbf{n}}_{ij}(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})\right] \\ &= = m_i \mathbf{v}_i + \frac{m_i v_i^2}{2c^2} \mathbf{v}_i + \frac{e_i \mathbf{A}^{(\mathrm{C})}(\mathbf{r}_i)}{c} \qquad (110) \end{aligned}$$

is the canonical momentum of particle i, and,

$$\mathbf{A}^{(\mathrm{C})}(\mathbf{r}_i) = \sum_{j \neq i} \frac{e_i e_j}{2c^2 R_{ij}} \left[\mathbf{v}_j + \hat{\mathbf{n}}_{ij}(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})\right] \qquad (111)$$

is the vector potential at charge i due to the other charges. Hence, the energy/Hamiltonian is,

$$U = \sum_i \frac{m_i v_i^2}{2} + \sum_i \frac{3m_i v_i^4}{8c^2} + \sum_{i>j} \frac{e_i e_j}{R_{ij}} + \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} \left[\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})\right], \qquad (112)$$

as first derived by Darwin [31].

The part of this Hamiltonian/energy associated with electromagnetic interactions is,

$$\begin{aligned} U_{\mathbf{EM}} &= \frac{1}{2} \sum_{i \neq j} \frac{e_i e_j}{R_{ij}} + \frac{1}{2} \sum_{i \neq j} \frac{e_i e_j}{2c^2 R_{ij}} \left[\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})\right] \\ &= \frac{1}{2} \sum_i e_i \left(V^{(\mathrm{C})}(\mathbf{r}_i) + \frac{\mathbf{v}_i \cdot \mathbf{A}^{(\mathrm{C})}(\mathbf{r}_i)}{c} \right), \qquad (113) \end{aligned}$$

where,

$$V^{(\mathrm{C})}(\mathbf{r}_i) = \sum_{j \neq i} \frac{e_j}{R_{ij}} \qquad (114)$$

is the electric scalar potential at charge i due to other charges.[29]

[29]The integral form of eq. (113),

$$U_{\mathbf{EM}} = \frac{1}{2} \int \left(\rho V^{(\mathrm{C})} + \frac{\mathbf{J} \cdot \mathbf{A}^{(\mathrm{C})}}{c} \right) d\mathrm{Vol}, \qquad (115)$$

C.1 Direct Calculation of the Interaction Electromagnetic Energy in the Darwin Approximation

The interaction electromagnetic energy associated with a set $\{i\}$ of charges e_i can be written,

$$U_{\rm EM} = \sum_{i>j} \int \frac{\mathbf{E}_i \cdot \mathbf{E}_j + \mathbf{B}_i \cdot \mathbf{B}_j}{4\pi}\, d{\rm Vol}. \tag{116}$$

The electric and magnetic fields of a charge e at distance R from an observer follow in the Darwin approximation from the potentials (110),

$$\begin{aligned} \mathbf{E} &= -\boldsymbol{\nabla} V^{(\rm C)} - \frac{\partial \mathbf{A}^{(\rm C)}}{\partial ct} = \frac{e}{R^2}\,\hat{\mathbf{n}} - \frac{e}{2c^2 R}\left[\mathbf{a} + (\mathbf{a}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \frac{3(\mathbf{v}\cdot\hat{\mathbf{n}})^2 - v^2}{R}\hat{\mathbf{n}}\right] \\ &\equiv \mathbf{E}^{(\rm C)} + \mathbf{E}_{\rm rot}, \end{aligned} \tag{117}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}^{(\rm C)} = \frac{e\mathbf{v}\times\hat{\mathbf{n}}}{cR^2}\,, \tag{118}$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the (present) acceleration of the charge,[30] and,

$$\mathbf{E}^{(\rm C)} = \frac{e}{R^2}\,\hat{\mathbf{n}}\,, \qquad \mathbf{E}_{\rm rot} = -\frac{e}{2c^2 R}\left[\mathbf{a} + (\mathbf{a}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + \frac{3(\mathbf{v}\cdot\hat{\mathbf{n}})^2 - v^2}{R}\hat{\mathbf{n}}\right]. \tag{119}$$

See [43] for applications of these relations to considerations of electromagnetic momentum rather than energy.

The potentials (107) are in the Coulomb gauge, so that $\boldsymbol{\nabla}\cdot\mathbf{A}^{(\rm C)} = 0$, and hence,

$$\boldsymbol{\nabla}\cdot\mathbf{E}_{\rm rot} = 0. \tag{120}$$

The electric part of the energy (112) can be written,

$$U_{\rm E} = \sum_{i>j} e_i e_j \int \frac{\hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_j}{4\pi R_i^2 R_j^2}\, d{\rm Vol} + \sum_{i>j}\int\left(\frac{e_i\hat{\mathbf{n}}_i\cdot\mathbf{E}'_j}{4\pi R_i^2} + \frac{e_j\hat{\mathbf{n}}_j\cdot\mathbf{E}'_i}{4\pi R_j^2}\right) d{\rm Vol} + \mathcal{O}\left(\frac{1}{c^4}\right). \tag{121}$$

It is well known (see, for example, the Appendix of [87]), that,

$$\int \frac{\hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_j}{4\pi R_i^2 R_j^2}\, d{\rm Vol} = \frac{1}{R_{ij}}\,. \tag{122}$$

For the second integral in eq. (117), we integrate by parts to find,[31]

$$\int \frac{\hat{\mathbf{n}}_i\cdot\mathbf{E}_{{\rm rot},\,j}}{R_i^2}\, d{\rm Vol} = -\int \mathbf{E}_{{\rm rot},\,j}\cdot\boldsymbol{\nabla}\left(\frac{1}{R_i}\right) d{\rm Vol} = \int \frac{1}{R_i}\boldsymbol{\nabla}\cdot\mathbf{E}_{{\rm rot},\,j}\, d{\rm Vol} = 0. \tag{124}$$

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shows the possibly surprising result that the electromagnetic energy in the Darwin approximation has the form of that for a system of quasistatic charge and current densities ρ and $\mathbf{J}$ (which implies use of the Coulomb gauge; see, for example, sec. 5.16 of [82] or secs. 31 and 33 of [57]).

[30] Sec. 65 of [54] shows that in the Darwin approximation the Liénard-Wiechert potentials (Lorenz gauge) reduce to $V^{(\rm L)} = e/R + (e/2c^2)\partial^2 R/\partial t^2$ and $\mathbf{A}^{(\rm L)} = e\mathbf{v}/cR$, from which eqs. (113)-(115) also follow.

[31] The surface integral resulting from the integration by parts in eq. (124) vanishes as follows:

$$\int \frac{\mathbf{E}_{{\rm rot},\,j}}{R_i}\cdot d\mathbf{Area} = -\int \frac{[\mathbf{a}_j + (\mathbf{a}_j\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]}{2c^2 R_i R_j}\cdot d\mathbf{Area} + \int \frac{(\cdots)}{R_i R_j^2}\cdot d\mathbf{Area} \to -\int \frac{\mathbf{a}_j\cdot\hat{\mathbf{n}}}{c^2}\, d\Omega = 0. \tag{123}$$

Thus, the electric part of the interaction energy is,

$$U_E = \sum_{i>j} \frac{e_i e_i}{R_{ij}}\,, \tag{125}$$

which holds for charges of any velocity when we work in the Coulomb gauge.

The magnetic part of the energy (112) is,

$$\begin{aligned} U_{\rm M} &= \sum_{i>j} \int \frac{\mathbf{B}_i \cdot \mathbf{B}_j}{4\pi}\, d{\rm Vol} = \sum_{i>j} \int \frac{\mathbf{B}_i \cdot \boldsymbol{\nabla} \times \mathbf{A}_j^{\rm (C)}}{4\pi}\, d{\rm Vol} = \sum_{i>j} \int \frac{\mathbf{A}_j^{\rm (C)} \cdot \boldsymbol{\nabla} \times \mathbf{B}_i}{4\pi}\, d{\rm Vol} \\ &= \sum_{i>j} \frac{e_i \mathbf{v}_i \cdot \mathbf{A}_j^{\rm (C)}(\mathbf{r}_i)}{c} = \sum_{i>j} \frac{e_i e_j}{2c^2 R_{ij}} \left[\mathbf{v}_i \cdot \mathbf{v}_j + (\mathbf{v}_i \cdot \hat{\mathbf{n}}_{ij})(\mathbf{v}_j \cdot \hat{\mathbf{n}}_{ij})\right], \end{aligned} \tag{126}$$

where we note that $\mathbf{B} \cdot \boldsymbol{\nabla} \times \mathbf{A} = \epsilon_{lmn} B_l \partial A_n / \partial x_m$, so that integration by parts leads to $-\epsilon_{lmn} A_n B_m \partial B_l / \partial x_m = \epsilon_{nml} A_n \partial B_l / \partial x_m = \mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{B}$ (and not to $-\mathbf{A} \cdot \boldsymbol{\nabla} \times \mathbf{B}$), and that,[32]

$$\boldsymbol{\nabla} \times \mathbf{B}_i = \frac{4\pi}{c} \mathbf{J}_i + \frac{\partial \mathbf{E}_i}{\partial ct} = \frac{4\pi e_i \mathbf{v}_i}{c} \boldsymbol{\delta}(\mathbf{r} - \mathbf{r}_i) - \boldsymbol{\nabla} \frac{\partial V_i^{\rm (C)}}{\partial ct} - \frac{\partial^2 \mathbf{A}_i^{\rm (C)}}{\partial (ct)^2}\,. \tag{128}$$

Thus, we again find the interaction electromagnetic energy $U_{\rm EM} = U_{\rm E} + U_{\rm M}$ to be given by eq. (110).

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[32] In greater detail, the integrand $\mathbf{A}_j \cdot \boldsymbol{\nabla} \times \mathbf{B}_i$ includes the term $\mathbf{A}_j \cdot \partial^2 \mathbf{A}_i / \partial (ct)^2$ which is of order $1/c^4$, while the integral of the term $\mathbf{A}_j \cdot \boldsymbol{\nabla} \partial \phi_i / \partial ct$ vanishes according to,

$$-\int \mathbf{A}_j \cdot \boldsymbol{\nabla} \frac{\partial \phi_i}{\partial ct}\, d{\rm Vol} = -\int \frac{\partial \phi_i}{\partial ct} \mathbf{A}_j \cdot d\mathbf{Area} + \int \frac{\partial \phi_i}{\partial ct} \boldsymbol{\nabla} \cdot \mathbf{A}_j\, d{\rm Vol} = \int \frac{\mathbf{v}_i \cdot \hat{\mathbf{n}}_i}{c R_i^2} \mathbf{A}_j \cdot d\mathbf{Area} \to 0. \tag{127}$$

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