Orbital and Spin Angular Momentum of the Fields of a Turnstile Antenna

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1 Problem

A "turnstile" antenna [1, 2] consists of a pair of linear dipole antennas oriented at 90° to each other, and driven 90° out of phase, as shown in Fig. 1.

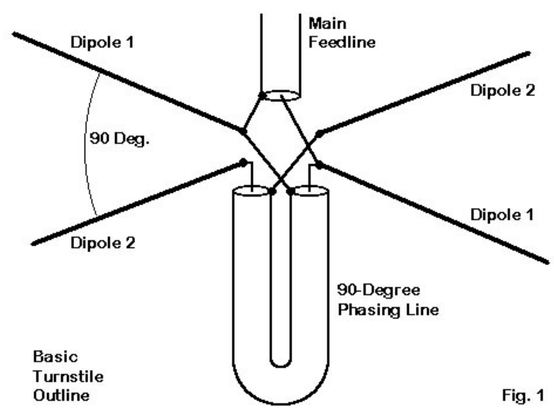

Figure 1: A "turnstile" antenna. From [2].

The linear antennas could be either dipoles as shown in the figure, or simply monopoles. If a pair of loops antennas is used instead, the configuration is called an "eggbeater" antenna.

Consider the case that the length of the linear antennas is small compared to a wavelength, so that it suffices to characterize each antenna by its electric dipole $\mathbf{p}_{1,2}e^{-i\omega t}$, where the magnitudes p_1 and p_2 are equal but their phases differ by 90°, the directions of the two moment differs by 90°, *i.e.*, $\mathbf{p}_1 \cdot \mathbf{p}_2 = 0$, and ω is the angular frequency.

Discuss the angular momentum of the fields of a turnstile antenna.

2 Solution

We consider a basic turnstile antenna whose component antennas lie in the x-y plane at a common point. Then, we can write the total electric dipole moment of the antenna system as,[1]

$$\mathbf{p} = \mathbf{p}_0\, e^{-i\omega t} = p_0\, (\hat{\mathbf{x}} + i\hat{\mathbf{y}})\, e^{-i\omega t}, \tag{1}$$

[1]Note that $\hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}}$ in a spherical coordinate system (r, θ, ϕ).

where p_0 is a real constant.

In Gaussian units (and in vacuum) these fields can be written as (see, for example, sec. 9.1 of [3]),

$$\mathbf{E}(\mathbf{r},t) = \left(\frac{k^2}{r}(\hat{\mathbf{r}}\times\mathbf{p}_0)\times\hat{\mathbf{r}} + \left(-\frac{ik}{r^2}+\frac{1}{r^3}\right)(3(\mathbf{p}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}_0)\right) e^{i(kr-\omega t)}, \tag{2}$$

$$\mathbf{B}(\mathbf{r},t) = \left(\frac{k^2}{r}+\frac{ik}{r^2}\right)\hat{\mathbf{r}}\times\mathbf{p}_0\, e^{i(kr-\omega t)}. \tag{3}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector from the center of the dipole to the observer, c is the speed of light, and $k = \omega/c$. The time-average Poynting vector is, in spherical coordinates (r, θ, ϕ),

$$\langle\mathbf{S}\rangle = \frac{cRe(\mathbf{E}\times\mathbf{B}^\star)}{8\pi} \tag{4}$$

$$= \frac{ck^4(1+\cos^2\theta)}{8\pi r^2}\hat{\mathbf{r}} - \frac{c}{8\pi}Re\left[\left(\frac{2k^2}{r^4}-\frac{ik^3}{r^3}-\frac{ik}{r^5}\right)\left(2(\mathbf{p}_0\cdot\hat{\mathbf{r}})\mathbf{p}_0^\star + (3(\mathbf{p}_0\cdot\hat{\mathbf{r}})(\mathbf{p}_0^\star\cdot\hat{\mathbf{r}})+2p_0^2)\,\hat{\mathbf{r}}\right)\right].$$

The time average density of field momentum is $\langle\mathbf{S}\rangle/c^2$, so the time-average density of field angular momentum is,

$$\langle\mathbf{l}\rangle = \mathbf{r}\times\frac{\langle\mathbf{S}\rangle}{c^2} = -\frac{1}{4\pi c}Re\left[\left(\frac{2k^2}{r^3}-\frac{ik^3}{r^2}-\frac{ik}{r^4}\right)(\mathbf{p}_0\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\times\mathbf{p}_0^\star\right]. \tag{5}$$

We now restrict our attention to the far zone where the electromagnetic fields are,

$$\mathbf{B} = k^2\frac{e^{i(kr-\omega t)}}{r}\hat{\mathbf{r}}\times\mathbf{p}_0, \qquad \mathbf{E} = \mathbf{B}\times\hat{\mathbf{r}} = \frac{k^2\, e^{i(kr-\omega t}}{r}(\mathbf{p}_0-(\mathbf{p}_0\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}}), \tag{6}$$

whose components in spherical coordinates are,

$$E_r = B_r = \hat{\mathbf{r}}\cdot\mathbf{B} = 0, \tag{7}$$

$$E_\theta = B_\phi = p_0k^2\,\frac{e^{i(kr-\omega t)}}{r}\cos\theta(\cos\phi+i\sin\phi), \tag{8}$$

$$E_\phi = -B_\theta = -p_0k^2\,\frac{e^{i(kr-\omega t)}}{r}(\sin\phi-i\cos\phi), \tag{9}$$

noting that $\hat{\mathbf{r}}\times\hat{\mathbf{x}} = \sin\phi\,\hat{\boldsymbol{\theta}}+\cos\theta\cos\phi\,\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{r}}\times\hat{\mathbf{y}} = -\cos\phi\,\hat{\boldsymbol{\theta}}+\cos\theta\sin\phi\,\hat{\boldsymbol{\phi}}$. In the plane of the antenna, $\theta = 90°$, the electric field has no θ component, and hence no z component; the turnstile radiation in the horizontal plane is horizontally polarized. In the vertical direction, $\theta = 0°$ or $180°$, the radiation is circularly polarized. For intermediate angles θ the radiation is elliptically polarized.

The magnitudes of the fields are,

$$E = B = \frac{p_0k^2}{r}\sqrt{1+\cos^2\theta}, \tag{10}$$

so the time-averaged radiation pattern is,

$$\frac{d\langle P\rangle}{d\Omega} = \frac{cr^2}{8\pi}B^2 = r^2\langle S_{\mathrm{far},r}\rangle = \frac{cp_0^2k^4}{8\pi}(1+\cos^2\theta). \tag{11}$$

The intensity of the radiation varies by a factor of 2 over the sphere. Compared to other simple antennas, this pattern is remarkably isotropic. The radiated power is greatest for $\theta = 0$ or 180° in which directions the polarization is purely circular. The total time-average radiated power is,

$$\langle P \rangle = \frac{2cp_0^2k^4}{3} \,. \tag{12}$$

The time-average density of angular momentum in the far zone is,[2]

$$\langle \mathbf{l}_{\rm far} \rangle = \frac{k^3}{4\pi cr^2} Re\left[i(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} \times \mathbf{p}_0^\star\right] = -\frac{p_0^2k^3}{4\pi cr^2}\sin\theta\,\hat{\boldsymbol{\theta}}. \tag{13}$$

This density flows radially outward at the speed of light, and we can speak of the rate of radiation of angular momentum in the far zone as,

$$\frac{d\langle \mathbf{L} \rangle}{dt d\Omega} = cr^2 \langle \mathbf{l}_{\rm far} \rangle = -\frac{cp_0^2k^3}{4\pi}\sin\theta\,\hat{\boldsymbol{\theta}} = \frac{cp_0^2k^3}{4\pi}(\sin^2\theta\,\hat{\mathbf{z}} - \sin\theta\cos\theta\,\hat{\boldsymbol{\rho}}), \tag{14}$$

where $\hat{\boldsymbol{\rho}}$ is the radial unit vector in cylindrical coordinates (ρ, ϕ, z). Integrating over solid angle, we find that,

$$\frac{d\langle \mathbf{L} \rangle}{dt} = \frac{2cp_0^2k^3}{3}\,\hat{\mathbf{z}} = \frac{\langle P \rangle}{\omega}\,\hat{\mathbf{z}}, \tag{15}$$

which seems consistent with the notion that photons have angular momentum $J = \hbar = U/\omega$.

2.1 Orbital and Spin Angular Momentum

The formalism that $\mathbf{l} = \mathbf{r} \times \mathbf{S}/c^2$ implies that the density (and flow) of angular momentum has no radial component. However, we say that the fields along the z-axis are circularly polarized (and elliptically polarized in general). This suggests that there should be a description in which we can identify an angular momentum with a radial component for the fields along the z-axis.

A decomposition has been given in [4, 5] whereby the total field angular momentum can be written in three terms,

$$\int \mathbf{l}\, d{\rm Vol} = \sum_i \mathbf{l}_{{\rm canonical},\,i} + \int \mathbf{l}_{\rm EM,orbital}\, d{\rm Vol} + \int \mathbf{l}_{\rm EM,spin}\, d{\rm Vol}, \tag{16}$$

$$\mathbf{l}_{{\rm canonical},\,i} = \mathbf{r} \times \mathbf{p}_{{\rm canonical},\,i}\,, \qquad \mathbf{l}_{\rm EM,orbital} = \mathbf{r} \times \mathbf{p}_{\rm EM,orbital}, \qquad \mathbf{l}_{\rm EM,spin} = \frac{\mathbf{E}_{\rm rot} \times \mathbf{A}_{\rm rot}}{4\pi c}\,, \tag{17}$$

and,[3]

$$\mathbf{p}_{\rm EM,orbital} = \frac{\sum_{j=1}^3 E_{{\rm rot},\,j}\boldsymbol{\nabla} A_{{\rm rot},\,j}}{4\pi c}\,, \tag{18}$$

where $\mathbf{A}_{\rm rot}$ is the (gauge-invariant) rotational part of the (gauge-dependent) vector potential $\mathbf{A}$. That is, $\mathbf{A}_{\rm rot}$ is the vector potential in the Coulomb gauge.

[2] If the dipole moment $\mathbf{p}_0$ were purely real, as for a small linear antenna, no angular momentum would be radiated.

[3] The canonical momentum $\mathbf{p}_{{\rm canonical},\,i}$ is nonzero only at the positions of charges e_i, and does not contribute to angular momentum in the far zone.

The electric field is related to the Coulomb-gauge potentials by,

$$\mathbf{E} = -\boldsymbol{\nabla} V^{(\mathrm{C})} - \frac{1}{c}\frac{\partial \mathbf{A}^{(\mathrm{C})}}{\partial t} = -\boldsymbol{\nabla} V^{(\mathrm{C})} - \frac{1}{c}\frac{\partial \mathbf{A}_{\mathrm{rot}}}{\partial t} \approx ik\mathbf{A}_{\mathrm{rot}} = \mathbf{E}_{\mathrm{rot}}, \tag{19}$$

where the approximation hold in the far zone, noting that $-\boldsymbol{\nabla} V^{(\mathrm{C})}$ is the instantaneous electric dipole field, which falls off as $1/r^3$. Then, in the far zone we have that the time-average spin-angular-momentum density, according to eq. (17), is,

$$\begin{aligned}
\langle \mathbf{l}_{\mathrm{EM,spin}} \rangle &= \frac{Re(\mathbf{E}_{\mathrm{rot}} \times \mathbf{A}^{\star}_{\mathrm{rot}})}{8\pi c} = \frac{Re(i\mathbf{E} \times \mathbf{E}^{\star})}{8\pi ck} = -\frac{Im(\mathbf{E} \times \mathbf{E}^{\star})}{8\pi ck} \\
&= \frac{-p_0^2 k^3}{8\pi c r^2} Im\left[((\hat{\mathbf{r}} \times \mathbf{p}_0) \times \hat{\mathbf{r}}) \times ((\hat{\mathbf{r}} \times \mathbf{p}_0^{\star}) \times \hat{\mathbf{r}})\right] \\
&= \frac{-p_0^2 k^3}{8\pi c r^2} Im\left[(\mathbf{p}_0 - (\mathbf{p}_0 \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}}) \times (\mathbf{p}_0^{\star} - (\mathbf{p}_0^{\star} \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}})\right] \\
&= \frac{-p_0^2 k^3}{8\pi c r^2} Im\left[\mathbf{p}_0 \times \mathbf{p}_0^{\star} - 2i Im[(\mathbf{p}_0 \cdot \hat{\mathbf{r}})\,\hat{\mathbf{r}} \times \mathbf{p}_0^{\star}]\right] = \frac{p_0^2 k^3}{4\pi c r^2}(\hat{\mathbf{z}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \\
&= \frac{p_0^2 k^3}{4\pi c r^2} \cos\theta\,\hat{\mathbf{r}},
\end{aligned} \tag{20}$$

noting that $\hat{\mathbf{z}} = \cos\theta\,\mathbf{r} - \sin\theta\,\hat{\boldsymbol{\theta}}$. This is an appealing result, in that the spin angular momentum is full strength along the z-axis and vanishing in the x-y plane where the polarization is linear.

The time-average orbital-angular-momentum density in the far zone is, according to eqs. (17)-(18),

$$\langle \mathbf{l}_{\mathrm{EM,orbital}} \rangle = \mathbf{r} \times \langle \mathbf{p}_{\mathrm{EM,orbital}} \rangle = \mathbf{r} \times \frac{Re\left(\sum_{j=1}^{3} E_{\mathrm{rot},\,j} \boldsymbol{\nabla} A^{\star}_{\mathrm{rot},\,j}\right)}{8\pi c}. \tag{21}$$

We expect that this angular momentum density to vary as $1/r^2$, so we must evaluate $\langle \mathbf{p}_{\mathrm{EM,orbital}} \rangle$ to order $1/r^3$, which requires keeping terms in $\mathbf{E}$ and $\mathbf{A}_{\mathrm{rot}}$ to order $1/r^2$.

However, neither $\langle \mathbf{l}_{\mathrm{far}} \rangle$ of eq. (13) nor $\langle \mathbf{l}_{\mathrm{EM,orbital}} \rangle$ has a radial component, so that $\langle \mathbf{l}_{\mathrm{EM,spin}} \rangle + \langle \mathbf{l}_{\mathrm{EM,orbital}} \rangle$ does not equal $\langle \mathbf{l}_{\mathrm{far}} \rangle$.

That is, while the notion of a classical "spin" angular momentum is appealing as a precursor to a quantum analysis, it is not fully consistent in a classical-only view.

References

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