“Hidden” Momentum in a Spinning Sphere?

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(August 16, 2012; updated June 30, 2017)

1 Problem

A spinning sphere at rest has zero total momentum. Deduce the mass/energy and the momentum of the spinning sphere as observed in an inertial frame where the sphere has velocity $\mathbf{v}$ perpendicular to its axis of rotation, taking “relativistic” effects into account. Give results accurate to order $1/c^2$ where c is the speed of light in vacuum.

Also deduce the location of the center of mass/energy of the moving, spinning sphere.

Does the moving sphere have “hidden” momentum [1]?

$$\mathbf{P}_{\text{hidden}} \equiv \mathbf{P} - M\mathbf{v}_{\text{cm}} - \oint_{\text{boundary}} (\mathbf{x} - \mathbf{x}_{\text{cm}})\,(\mathbf{p} - \rho\mathbf{v}_b) \cdot d\mathbf{Area} = -\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\text{cm}})\,d\text{Vol}, \qquad (1)$$

where $\mathbf{P}$ is the total momentum of the subsystem, $M = U/c^2$ is its total “mass”, U is its total energy, c is the speed of light in vacuum, $\mathbf{x}_{\text{cm}}$ is its center of mass/energy, $\mathbf{v}_{\text{cm}} = d\mathbf{x}_{\text{cm}}/dt$, $\mathbf{p}$ is its momentum density, $\rho = u/c^2$ is its “mass” density, u is its energy density, $\mathbf{v}_b$ is the velocity (field) of its boundary, and,

$$f^\mu = \frac{\partial T^{\mu\nu}}{\partial x^\nu}\,, \qquad (2)$$

is the 4-force density exerted on the subsystem by the rest of the system, with $T^{\mu\nu}$ being the stress-energy-momentum 4-tensor of the subsystem.[1]

2 Solution

2.1 Mass and Momentum

In its rest frame the sphere has mass M_0 and is not spinning.

In the lab frame of the observer the geometric center (centroid) of the sphere has velocity $\mathbf{v}_0 = v_0\,\mathbf{x}$.

Taking the sphere to be centered on the origin in the frame in which its center is at rest and the sphere is spinning (the * frame[2]), a volume element (with rest mass dM_0) about $\mathbf{r}^\star = (x^\star, y^\star, z^\star)$ has velocity,

$$\mathbf{u}^\star = \boldsymbol{\omega} \times \mathbf{r}^\star = (-\omega y^\star, \omega x^\star, 0), \qquad (3)$$

[1]As discussed in sec. 3 of [1], we do not advocate replacing $\mathbf{v}_{\text{cm}}$ by $\mathbf{v}_{\text{centroid}} = \mathbf{v}_0$ in definition (1), where the centroid is the geometric center of the sphere in the lab frame.

[2]The * frame is obtained from the lab frame by a Lorentz transformation with velocity $-\mathbf{v}_0$. The rest frame of the sphere differs from the * frame by a rotation about the $\boldsymbol{\omega}$-axis.

defining $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}$ to be the angular velocity of the sphere in the * frame, where ωa is small compared to the speed of sound inside the sphere. In the lab frame this element has velocity,

$$\mathbf{u} = \frac{\mathbf{v}_0 + \mathbf{u}_\parallel^\star + \mathbf{u}_\perp^\star/\gamma_0}{1 + \mathbf{v}_0 \cdot \mathbf{u}^\star/c^2} = \frac{(v_0 - \omega y^\star)\,\hat{\mathbf{x}} + \omega x^\star\,\hat{\mathbf{y}}/\gamma_0}{1 - \omega v_0 y^\star/c^2}, \qquad \text{where} \qquad \gamma_0 = \frac{1}{\sqrt{1 - v_0^2/c^2}}. \tag{4}$$

The relativistic mass dM (= energy/c^2) of the volume element in the lab frame is given by,

$$dM = \gamma_u dM_0 = \frac{dM_0}{\sqrt{1 - u^2/c^2}} \approx dM_0 \left(1 + \frac{u^2}{2c^2}\right) \approx dM_0 \left(1 + \frac{v_0^2 - 2\omega v_0 y^\star + \omega^2(x^{\star 2} + y^{\star 2})}{2c^2}\right), \tag{5}$$

keeps terms only to order $1/c^2$. Hence,

$$M = \int dM \approx M_0 \left(1 + \frac{v_0^2}{2c^2} + \frac{k\omega^2 a^2}{2c^2}\right), \tag{6}$$

recalling that the moment of inertia of the sphere in its rest frame is $I_0 = \int (x^{\star 2} + y^{\star 2})\, dM_0 = kM_0^2 a^2$ with $k = 2/5$. Note that the relativistic mass M in the lab frame does not equal $\gamma_0 M_0 \approx M_0(1 + v_0^2/2c^2)$ as holds for a nonspinning sphere.[3]

The x-component of the momentum of the sphere in the lab frame is,

$$\begin{aligned} P_x &= \int u_x\, dM \approx \int (v_0 - \omega y^\star) \left(1 + \frac{\omega v_0 y^\star}{c^2}\right) dM_0 \left(1 + \frac{v_0^2 - 2\omega v_0 y^\star + \omega^2(x^{\star 2} + y^{\star 2})}{2c^2}\right) \\ &\approx M_0 v_0 \left(1 + \frac{v_0^2}{2c^2} + \frac{k\omega^2 a^2}{2c^2}\right) = M v_0. \end{aligned} \tag{7}$$

That is, the lab-frame momentum is related by,

$$\mathbf{P} = M\mathbf{v}_0 = M\mathbf{v}_{\rm cm}, \tag{8}$$

noting that the center of mass/energy of the sphere has velocity $\mathbf{v}_0$ in the lab frame (as can be confirmed using eq. (11) below). As such, the momentum in the lab frame is not "hidden".

This last statement is consistent with the definition (1) in that the boundary of the (sub) system can be taken outside the sphere, so that the boundary integral vanishes and $\mathbf{P}_{\rm hidden} = 0$.

Also, for a system in isolation, such as the present example, the 4-divergence of the stress tensor is zero, so that $f^0 = 0$ in particular. Thus, $\mathbf{P}_{\rm hidden} = 0$ according to the second form of eq. (1) as well.[4]

---

[3]The mass $M^\star$ of the spinning sphere whose center is at rest in the * frame follows from eq. (6) with $v_0 = 0$.

[4]A basic consequence of the definition (1) is that an isolated system can have no "hidden" momentum, so it should not be a surprise that $\mathbf{P}_{\rm hidden} = 0$ in the present example.

2.2 Location of the Center of Mass/Energy (First Analysis)

The position $\mathbf{x}_{\rm cm}$ of the sphere in the lab frame is given by,

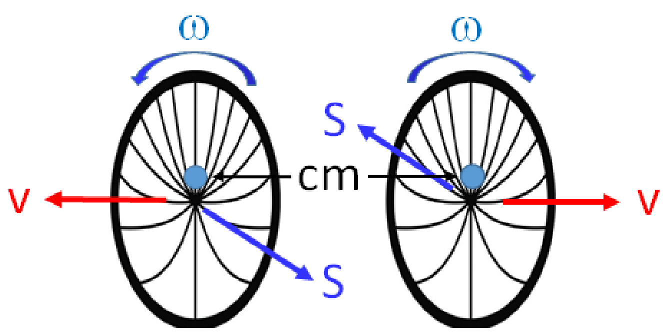
$$
\begin{aligned}
M\mathbf{x}_{\rm cm} &= \int \mathbf{x}\, dM \approx \int (x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})\, dM_0 \left(1 + \frac{v_0^2 - 2\omega v_0 y^\star + \omega^2(x^{\star 2} + y^{\star 2})}{2c^2}\right) \\
&\approx \int \left[(x^\star + v_0 t)\left(1 - \frac{v_0^2}{2c^2}\right)\hat{\mathbf{x}} + y^\star\,\hat{\mathbf{y}} + z^\star\,\hat{\mathbf{z}}\right] dM_0 \left(1 + \frac{v_0^2 - 2\omega v_0 y^\star + \omega^2(x^{\star 2} + y^{\star 2})}{2c^2}\right) \\
&\approx Mv_0 t\,\hat{\mathbf{x}} - \frac{v_0 k M a^2 \omega}{2c^2}\hat{\mathbf{y}} \approx M\mathbf{x}_0 - \frac{\mathbf{S}_0 \times \mathbf{v}_0}{2c^2},
\end{aligned} \tag{9}
$$

where $\mathbf{x}_0 = \mathbf{v}_0 t = v_0 t\,\hat{\mathbf{x}}$ is the geometric center of the moving sphere in the lab frame, $\mathbf{S}_0 = I_0\,\boldsymbol{\omega} = kM_0 a^2\,\boldsymbol{\omega}$ is the (spin) angular momentum of the sphere about its geometric center, and the moment of inertia I_0 is,

$$
\begin{aligned}
I_0 &= \int [(x - v_0 t)^2 + y^2]\, dM \\
&\approx \int \left[x^{\star 2}\left(1 - \frac{v_0^2}{c^2}\right) + y^{\star 2}\right] dM_0 \left(1 + \frac{v_0^2 - 2\omega v_0 y^\star + \omega^2(x^{\star 2} + y^{\star 2})}{2c^2}\right) \\
&\approx \frac{2M_0 a^2}{5}\left(1 + \frac{25\omega^2 a^2}{112c^2}\right) \approx kMa^2.
\end{aligned} \tag{10}
$$

To order $1/c^2$,

$$
\mathbf{x}_{\rm cm} \approx \mathbf{x}_0 - \frac{ka^2\boldsymbol{\omega} \times \mathbf{v}_0}{2c^2} \approx \mathbf{x}_0 - \frac{\mathbf{S}_0 \times \mathbf{v}_0}{2Mc^2}\,. \tag{11}
$$



The center of mass/energy of the moving sphere is not at its geometric center $\mathbf{x}_0$, but is shifted to the side of the sphere that has the higher speed in the lab frame, as the relativistic mass is higher there. The spinning, translating sphere (or better, Lorentz-contracted spheroid) cannot be described as a rigid body, in that the center of mass/energy does not rotate with the matter of the spheroid.

The above analysis did not take into account that the rolling hoop has internal forces/stresses, In the lab frame these stresses are largest near the top of the hoop, where the speed is the greatest. Since there is mass/energy associated with the internal stresses, it seems likely that the position of the center of mass/energy is even higher above the centroid, $\mathbf{x}_0 = \mathbf{v}t$, *than indicated in eq. (11).*

We pursue this additional upward shift in the following section.

2.3 A More General Argument

This section follows [2]. See also [3, 4, 5], sec. 64 of [6][5] and [7, 8, 9]. This topic has an extensive history in considerations of the quantum position operator. For a review, see [10].

The mechanical behavior of a macroscopic subsystem can be described with the aid of its (symmetric) stress-energy-momentum 4-tensor $T^{\mu\nu}$. The quantity,

$$P^\mu = (U/c, P^i) = (U/c, \mathbf{P}) = \int \frac{T^{0\mu}}{c}\, d\text{Vol}. \tag{12}$$

describes the total energy and momentum of the subsystem, although P^μ is not truly a 4-vector unless the subsystem is isolated.[6]

The total mass/energy of the subsystem is,

$$U = \int T^{00}\, d\text{Vol}, \tag{13}$$

and we define the effective mass of the subsystem as,

$$M = \frac{U}{c^2} = \int \frac{T^{00}}{c^2}\, d\text{Vol} = \int \rho\, d\text{Vol}, \tag{14}$$

where we define the effective mass density of the subsystem to be $\rho = T^{00}/c^2$. The center of mass/energy of the subsystem is at position,

$$x^\mu_{\text{cm}} = \frac{1}{U}\int T^{00} x^\mu\ d\text{Vol}, \qquad \mathbf{x}_{\text{cm}} = \frac{1}{M}\int \frac{T^{00}}{c^2}\mathbf{x}\, d\text{Vol}. \tag{15}$$

where $x^\mu = (ct, \mathbf{x})$, as characterizing the coordinates of the center of mass/energy of the subsystem.

In general, the lab-frame quantity x^μ_{cm} is not a 4-vector, and it is not the Lorentz transformation x^μ_0 of the quantity,

$$x^{\star\mu}_0 \equiv x^{\star\mu}_{\text{cm}} = \frac{1}{U^\star}\int T^{\star 00} x^{\star\mu}\ d\text{Vol}^\star, \tag{16}$$

where the $\star$ frame is the (instantaneous) frame of the subsystem in which its total 3-momentum is zero (but where the angular velocity is still $\boldsymbol{\omega}$),

$$0 = P^{\star i} = \int \frac{T^{\star 0i}}{c}\, d\text{Vol}^\star. \tag{17}$$

We denote the lab-frame transform of the quantity $x^{\star\mu}_0$ as x^μ_0, which we will call the centroid.[7] Note that $x^0_0 = ct = x^0_{\text{cm}}$. The velocity of the boost from the * frame to the lab frame is

[5]In thermodynamics a closed subsystem can have exchange of energy, but not matter, with other subsystem, whereas an isolated subsystem has no exchange of mass/energy. The term closed system in [2, 6] corresponds to the term isolated system of thermodynamics.

[6]In case of a nonisolated system, P^μ of eq. (12) has been called a "false" 4-vector [11].

[7]The coordinates x^μ_0 are called those of the proper center of mass in [6].

$\mathbf{v}_{\rm cm}$ of eq. (35). Hence, the velocity $\mathbf{v}_0$ of the centroid is the same as the velocity $\mathbf{v}_{\rm cm}$ of the center of mass/energy, even though the position $\mathbf{x}_0$ is not necessarily that same as $\mathbf{x}_{\rm cm}$.

As seen in sec. 2.2 the difference between $\mathbf{x}_{\rm cm}$ and $\mathbf{x}_0$ in the lab frame is related to the presence of angular momentum in the subsystem, so we introduce the quantity,

$$L^{\mu\nu} = \int \frac{x^\mu T^{0\nu} - x^\nu T^{0\mu}}{c}\, d{\rm Vol}, \tag{18}$$

as the (antisymmetric) angular momentum 4-tensor of the subsystem. Further, we introduce the "spin" angular momentum tensors, defined by,

$$S_0^{\mu\nu} = L^{\mu\nu} - (x_0^\mu P^\nu - x_0^\nu P^\mu), \tag{19}$$

and,

$$S_{\rm cm}^{\mu\nu} = L^{\mu\nu} - (x_{\rm cm}^\mu P^\nu - x_{\rm cm}^\nu P^\mu), \tag{20}$$

which subtract away the angular momentum associated with the energy/momentum of the centroid, and of the center of mass/energy, respectively.

For an antisymmetric 4-tensor $A^{\mu\nu}$ we construct two 3-vectors $\mathbf{a}$ and $\tilde{\mathbf{a}}$ and according to,

$$\mathbf{a} = (a^{23}, a^{31}, a^{12}) \qquad \text{and} \qquad \tilde{\mathbf{a}} = (a^{10}, a^{20}, a^{30}). \tag{21}$$

Then, for either of the spin 4-tensors (19)-(20) we can write,

$$\mathbf{S} = \mathbf{L} - \mathbf{x}\times\mathbf{P}, \qquad \tilde{\mathbf{S}} = \tilde{\mathbf{L}} - Mc\,\mathbf{x} + ct\mathbf{P}, \qquad \text{and so} \qquad \mathbf{x} = \frac{1}{Mc}\left(\tilde{\mathbf{L}} - \tilde{\mathbf{S}} + ct\mathbf{P}\right), \tag{22}$$

where from eqs. (18),

$$\mathbf{L} = \int \mathbf{x}\times\mathbf{p}\, d{\rm Vol}, \qquad \text{and} \qquad \tilde{\mathbf{L}} = Mc\,\mathbf{x}_{\rm cm} - ct\mathbf{P}. \tag{23}$$

In particular, from eqs. (19) and (22)-(23) we obtain an expression for the lab-frame 3-position of the centroid,[8]

$$\mathbf{x}_0 = \mathbf{x}_{\rm cm} - \frac{\tilde{\mathbf{S}}_0}{Mc}\,. \tag{24}$$

This result was deduced in the lab frame, but it also holds in the * frame, where $\mathbf{x}_0^\star = \mathbf{x}_{\rm cm}^\star$, so it must be that $\tilde{\mathbf{S}}_0^\star = 0$. Then, since $\mathbf{P}^\star = 0$, we have that,

$$S_0^{\star\mu\nu} P_\nu^\star = 0. \tag{25}$$

IF $S_0^{\mu\nu}$ and P^μ are a 4-tensor and a 4-vector, respectively, with respect to Lorentz transformations, then in the lab frame we have that[9]

$$0 = S_0^{\mu\nu} P_\nu, \qquad \tilde{\mathbf{S}}_0\cdot\mathbf{P} = 0 \quad \text{for } \mu = 0, \qquad Mc\,\tilde{\mathbf{S}}_0 = -\mathbf{S}_0\times\mathbf{P}, \quad \text{for } \mu = 1,2,3, \tag{27}$$

[8]Using eq. (20) rather than (19) leads only to $\tilde{\mathbf{S}}_{\rm cm} = 0$, which also follows directly from eq. (20).

[9]In the * frame, $x_0^{\star\mu} = x_{\rm cm}^{\star\mu}$, so we also have that the spin tensors are the same in this frame, $S_0^{\star\mu\nu} = S_{\rm cm}^{\star\mu\nu}$, and in particular $\tilde{\mathbf{S}}_0^\star = \tilde{\mathbf{S}}_{\rm cm}^\star = 0$. Since $\mathbf{P}^\star = 0$, we have that $S_{\rm cm}^{\star\mu\nu} P_\nu^\star = 0$. If $S_{\rm cm}^{\mu\nu}$ were a 4-tensor with respect to Lorentz transformations, then in the lab frame we would have that,

$$0 = S_{\rm cm}^{\mu\nu} P_\nu, \qquad \tilde{\mathbf{S}}_{\rm cm}\cdot\mathbf{P} = 0 \quad \text{for } \mu = 0, \qquad Mc\,\tilde{\mathbf{S}}_{\rm cm} = -\mathbf{S}_{\rm cm}\times\mathbf{P}, \quad \text{for } \mu = 1,2,3. \tag{26}$$

This contradicts the fact that $\tilde{\mathbf{S}}_{\rm cm} = 0$, so we infer that $S_{\rm cm}^{\mu\nu}$ is not a tensor under Lorentz transformations.

and,

$$\mathbf{x}_0 = \mathbf{x}_{\rm cm} - \frac{\mathbf{S}_0 \times \mathbf{P}}{M^2 c^2} = \mathbf{x}_{\rm cm} - \frac{\mathbf{S}_0 \times \mathbf{v}_0}{M c^2}\,. \tag{28}$$

as in eq. (11), but now the term in $\mathbf{S}$ is twice as large.[10,11]

2.3.1 How General is the "General" Argument?

The above argument may not hold for a subsystem that is not isolated from the rest of the Universe, as was tacitly assumed above in eq. (27), and in writing $\mathbf{P} = M\mathbf{v}_0$.

As discussed in sec. 3 of [1], we can deduce a relation (35) between $\mathbf{P}$ and $M\mathbf{v}_0$ for a nonisolated subsystem. The reader may wish to skip ahead to that point.

In detail, from eq. (14) we find,

$$\frac{dM}{dt} = \int \frac{\partial_0 T^{00}}{c}\, d{\rm Vol} + \oint_{\rm boundary} \frac{T^{00}}{c^2}\,(\mathbf{v}_b \cdot d\mathbf{Area}), \tag{29}$$

and,

$$\frac{d}{dt}(M\mathbf{x}_{\rm cm}) = \frac{dM}{dt}\mathbf{x}_{\rm cm} + M\frac{d\mathbf{x}_{\rm cm}}{dt} = \int \frac{\partial_0 T^{00}}{c}\mathbf{x}\, d{\rm Vol} + \oint_{\rm boundary} \frac{T^{00}}{c^2}\,\mathbf{x}\,(\mathbf{v}_b \cdot d\mathbf{Area}), \tag{30}$$

where $x^\mu = (ct, \mathbf{x})$, $\partial_\mu = \partial/\partial x^\mu = (\partial/\partial ct, \boldsymbol{\nabla})$, and $\mathbf{v}_b$ is the velocity (field) of the boundary. Hence,

$$\begin{aligned} M\mathbf{v}_{\rm cm} &= M\frac{d\mathbf{x}_{\rm cm}}{dt} = \int \frac{\partial_0 T^{00}}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\, d{\rm Vol} + \oint_{\rm boundary} \frac{T^{00}}{c^2}\,(\mathbf{x}-\mathbf{x}_{\rm cm})\,(\mathbf{v}_b \cdot d\mathbf{Area}) \\ &= \int \frac{\partial_0 T^{00}}{c}(\mathbf{x}-\mathbf{x}_{\rm cm})\, d{\rm Vol} + \oint_{\rm boundary} (\mathbf{x}-\mathbf{x}_{\rm cm})\,(\rho\mathbf{v}_b \cdot d\mathbf{Area}). \end{aligned} \tag{31}$$

While the stress-energy-momentum tensor for an isolated system has zero 4-divergence, this is not necessarily the case for the subsystem under consideration. Rather, the possibly nonzero 4-vector,

$$f^\mu = \partial_\nu T^{\nu\mu} = \partial_\nu T^{\mu\nu}, \tag{32}$$

describes the 4-force density exerted by the subsystem on the rest of the system. If the stress-energy-momentum tensor is nonzero just inside the boundary, the 4-force density can have delta-function contributions on the boundary (as $T^{\mu\nu}$ is defined to be zero outside the boundary), which must be considered carefully in the following. Of course, f^μ is zero outside the boundary of the subsystem.

Using eq. (32) we can write,

$$\partial_0 T^{00} = f^0 - \partial_j T^{0j}, \tag{33}$$

[10] In [5] it is shown that $\mathbf{S}_0 \cdot \mathbf{P} = \mathbf{S}_{\rm cm} \cdot \mathbf{P} = \mathbf{L} \cdot \mathbf{P}$ and that $\mathbf{x}_0 = \mathbf{x}_{\rm cm} + \mathbf{S}_{\rm cm} \times \mathbf{P}/M_0^2 c^2$ (with the + sign miswritten as a $-$).

[11] The result (28) appears in eq. (8) of [12], with $\mathbf{v}_0$ taken to be the velocity of the observer relative to the sphere, *i.e.*, $-\mathbf{v}_0$ of this note. This result was also discussed around eq. (7) of [13], with the claim that $\mathbf{x}_0$ rather than $\mathbf{x}_{\rm cm}$ is the "true" center of mass.

where both f^0 and $\partial_j T^{0j}$ can have delta functions on the boundary. Then, noting that the momentum density $\mathbf{p}$ has components $p_j = T^{0j}/c$, the volume integral in eq. (31) can be written as,

$$\int \frac{\partial_0 T^{00}}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d\mathrm{Vol} = \int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d\mathrm{Vol} - \int \frac{\partial_j T^{0j}}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d\mathrm{Vol}$$
$$= \int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d\mathrm{Vol} - \int \frac{\partial_j [T^{0j}(\mathbf{x} - \mathbf{x}_{\rm cm})]}{c}\, d\mathrm{Vol} + \int \frac{T^{0j}\, \partial_j \mathbf{x}}{c}\, d\mathrm{Vol}$$
$$= \int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d\mathrm{Vol} - \oint_{\rm boundary} \frac{T^{0j}(\mathbf{x} - \mathbf{x}_{\rm cm})}{c}\, d\mathrm{Area}_j + \int \mathbf{p}\, d\mathrm{Vol}$$
$$= \int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d\mathrm{Vol} - \oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} \cdot d\mathbf{Area}) + \mathbf{P}. \qquad (34)$$

Combining eqs. (31) and (34), we have,

$$\mathbf{v}_{\rm cm} = \frac{\mathbf{P}}{M} - \frac{1}{M}\oint_{\rm boundary} (\mathbf{x} - \mathbf{x}_{\rm cm})\,(\mathbf{p} - \rho \mathbf{v}_b) \cdot d\mathbf{Area} + \frac{1}{M}\int \frac{f^0}{c}(\mathbf{x} - \mathbf{x}_{\rm cm})\, d\mathrm{Vol}. \qquad (35)$$

The integrals vanish for an isolated subsystem, in which case $\mathbf{v}_{\rm cm} = \mathbf{P}/M$, as expected.

The integrals in eq. (35) can be zero even for subsystems that are not isolated. For example, if the subsystems occupy disjoint spatial volumes the integrals are zero. This holds for any reasonable choice of subsystems of an all-mechanical system, as considered in [14]. The volume integral can be nonzero only if the subsystems are both fields, or matter and fields, which occupy that same spatial volume.

We now restrict our discussion to examples in which the integrals are zero, so $\mathbf{v}_{\rm cm} = \mathbf{P}/M$, and in addition the total mass and energy, M and U, of the subsystem are constant in time. In this case, the magnitude of the momentum $\mathbf{P}$, and $P^\mu P_\mu = Mc^2$, are is also constant in time. Furthermore we suppose that the angular momentum tensor $L^{\mu\nu}$ is constant in time, as holds for each of two subsystems that interact via a central force.

We still face the issue of whether relation (27) holds in the lab frame of a nonisolated subsystem.

References

[1] K.T. McDonald, *On the Definition of "Hidden" Momentum* (July 9, 2012),
http://kirkmcd.princeton.edu/examples/hiddendef.pdf

[2] C. Møller, *On the Definition of the Centre of Gravity of an Arbitrary Closed System in the Theory of Relativity*, Comm. Dublin Inst. Adv. Study A **5**, 3 (1949),
http://kirkmcd.princeton.edu/examples/GR/moller_cdias_a5_3_49.pdf

[3] A.D. Fokker, *Relativiteitstheorie* (Noordhoof, Groningen, 1929),
http://kirkmcd.princeton.edu/examples/GR/fokker_relativiteitstheorie_29.pdf
Time and Space, Weight and Inertia (Pergamon, 1965),
http://kirkmcd.princeton.edu/examples/GR/fokker_time_space_65.pdf

[4] A. Papapetrou, *Drehimpuls- und Schwerpunktsatz in der relativistischen Mechanik*, Praktika Acad. Athénes **14**, 540 (1939),
http://kirkmcd.princeton.edu/examples/GR/papapetrou_paa_14_540_39.pdf
Drehimpuls- und Schwerpunktsatz in der Diracschen Theorie, Praktika Acad. Athénes **15**, 504 (1940), http://kirkmcd.princeton.edu/examples/GR/papapetrou_paa_15_504_40.pdf

[5] M.H.L. Pryce, *The mass-centre in the restricted theory of relativity and its connexion with the quantum theory of elementary particles*, Proc. Roy. Soc. London A **195**, 62 (1948), http://kirkmcd.princeton.edu/examples/GR/pryce_prsla_195_62_48.pdf

[6] C. Møller, *The Theory of Relativity*, 1st ed. (Clarendon Press, 1952),
http://kirkmcd.princeton.edu/examples/GR/moller_relativity_52.pdf

[7] H.C. Corben, *Classical and Quantum Theories of Spinning Particles* (Holden-Day, 1968), Chap. II, http://kirkmcd.princeton.edu/examples/QM/corben_68_chap2.pdf

[8] D. Alba, L. Lusanna and M. Pauri, *Centers of mass and rotational kinematics for the relativistic N-body problem in the rest-frame instant form*, J. Math. Phys. **43**, 1677 (2002), http://kirkmcd.princeton.edu/examples/GR/alba_jmp_43_1677_02.pdf

[9] D. Alba, H.W. Crater and L. Lusanna, *Hamiltonian relativistic two-body problem: center of mass and orbit reconstruction*, J. Phys. A **40**, 9585 (2007),
http://kirkmcd.princeton.edu/examples/GR/alba_jpa_40_9585_07.pdf

[10] M. Lorente and P. Roman, *General expressions for the position and spin operators of relativistic systems*, J. Math. Phys. **15**, 70 (1974),
http://kirkmcd.princeton.edu/examples/GR/lorente_jmp_15_70_74.pdf

[11] E. Comay, *Lorentz transformation of a system carrying "Hidden Momentum,"* Am. J. Phys. **68**, 1007 (2000), http://kirkmcd.princeton.edu/examples/EM/comay_ajp_68_1007_00.pdf

[12] L. Filipe Costa *et al.*, *Mathisson's helical motions for a spinning particle: Are they unphysical?* Phys. Rev. D **85**, 024001 (2012),
http://kirkmcd.princeton.edu/examples/GR/filipe_prd_85_024001_12.pdf
See also sec. 3 of [2].

[13] S.E. Gralla, A.I. Harte, and R.M. Wald, *Bobbing and kicks in electromagnetism and gravity*, Phys. Rev. D **81**, 104012 (2010),
http://kirkmcd.princeton.edu/examples/GR/gralla_prd_81_104012_10.pdf

[14] K.T. McDonald, *"Hidden" Momentum in a Linked Pair of Gyrostats* (Aug. 17, 2012),
http://kirkmcd.princeton.edu/examples/gyrostat.pdf