Magnetostatic Spin Waves

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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1 Problem

Magnetostatics can be defined as the regime in which the magnetic fields $\mathbf{B}$ and $\mathbf{H}$ have no time dependence, and "of course" the electric fields $\mathbf{D}$ and $\mathbf{E}$ have no time dependence either. In this case, the divergence of the fourth Maxwell equation,

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_{\rm free} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\,, \tag{1}$$

(in Gaussian units) implies that,

$$\boldsymbol{\nabla} \cdot \mathbf{J}_{\rm free} = 0, \tag{2}$$

i.e., that the free currents flow in closed loops. Likewise, the time derivative of the fourth Maxwell equation implies that $\mathbf{J}_{\rm free}$ has no time dependence in magnetostatics.

Often, magnetostatics is taken to be the situation in which $\boldsymbol{\nabla} \cdot \mathbf{J}_{\rm free} = 0$ and $\mathbf{D}$, $\mathbf{E}$ and $\mathbf{J}_{\rm free}$ have no time dependence, without explicit assumption that $\mathbf{B}$ and $\mathbf{H}$ also have no time dependence. Discuss the possibility of waves of $\mathbf{B}$ and $\mathbf{H}$, consistent with the latter definition of magnetostatics [1].

Consider two specific examples of "magnetostatic" waves in which $\mathbf{J}_{\rm free} = 0$:

1. Ferromagnetic spin waves in a medium subject to zero external field, but which has a uniform static magnetization that is large compared to that of the wave. That is, $\mathbf{M} = M_0\,\hat{\mathbf{z}} + \mathbf{m}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, where $m \ll M_0$. Here, the quantum-mechanical exchange interaction is the dominant self interaction of the wave, which leads to an effective magnetic field in the sample given by $\mathbf{B}_{\rm eff} = \alpha\boldsymbol{\nabla}^2\mathbf{m}$, where α is a constant of the medium.

2. Waves in a ferrite cylinder in a uniform external magnetic field parallel to its axis, supposing the spatial variation of the wave is slight, so the exchange interaction may be ignored. Again, the time-depedendent part of the magnetization is assumed small compared to the static part. Show that the waves consist of transverse, magnetostatic fields that rotate with a "resonant" angular velocity about the axis.

In practice, the spin waves are usually excited by an external rf field, which is to be neglected here.

2 Solution

2.1 General Remarks

In both definitions of magnetostatics the electric field $\mathbf{E}$ has no time dependence, $\partial\mathbf{E}/\partial t = 0$, so the magnetic field $\mathbf{B}$ obeys $\partial^2\mathbf{B}/\partial t^2 = 0$, as follows on taking the time derivative of

Faraday's law,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

(in Gaussian units). In principle, this is consistent with a magnetic field that varies linearly with time, $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r})t$. However, this leads to arbitrarily large magnetic fields at early and late times, and is excluded on physical grounds. Hence, any magnetic field $\mathbf{B}$ that coexists with only static electric fields is also static.

There remains the possibility of a "magnetostatic wave" in a magnetic medium that involves the magnetic field $\mathbf{H}_{\rm wave}$ and magnetization density $\mathbf{M}_{\rm wave}$ that are related by,

$$0 = \mathbf{B}_{\rm wave} = \mathbf{H}_{\rm wave} + 4\pi \mathbf{M}_{\rm wave}. \tag{4}$$

If there are no free currents in the medium, and any electric field is static, then the fourth Maxwell equation is simply,

$$\boldsymbol{\nabla} \times \mathbf{H} = 0, \tag{5}$$

which defines a subset of magnetostatic phenomena.

2.2 Ferromagnetic Spin Waves

Consider a ferromagnetic material that consists of a single macroscopic domain with magnetization density $\mathbf{M} = M_0\,\hat{\mathbf{z}} + \mathbf{m}(\mathbf{r}, t)$, where M_0 is constant and $m \ll M_0$. We suppose there are no external electromagnetic fields. Associated with the magnetization density $\mathbf{M}$ are magnetic fields $\mathbf{B}$ and $\mathbf{H}$ whose values depend on the geometry of the sample. We suppose that the weak time-dependent magnetic fields due to $\mathbf{m}$ lead to even weaker time-dependent electric fields, such that the situation is essentially magnetostatic. The consistency of this assumption will be confirmed at the end of the analysis.

The ferromagnetism is due to electron spins, whose dominant interaction is the quantum mechanical exchange interaction, in the absence of external fields. For a weak perturbation $\mathbf{m}$ of the magnetization, the exchange interaction preserves the magnitude of the magnetization, so its time evolution has the form of a precession [2],

$$\frac{d\mathbf{M}}{dt} = \boldsymbol{\Omega} \times \mathbf{M}. \tag{6}$$

As this is the same form as the precession of a magnetic moment in an external magnetic field [3], the precession vector $\boldsymbol{\Omega}$ is often written as a gyromagnetic factor $\Gamma = e/2m_e c \approx 10^7$ Hz/gauss times an effective magnetic field $\mathbf{B}_{\rm eff}$ (or $\mathbf{H}_{\rm eff}$). Here, $e > 0$ and m_e are the charge and mass of the electron, and c is the speed of light. For a weak perturbation in an isotropic medium [2],

$$\mathbf{B}_{\rm eff} = \alpha \boldsymbol{\nabla}^2 \mathbf{m}, \tag{7}$$

where α is a constant of the medium.

Then, the equation of motion of the magnetization $\mathbf{m}$ is,

$$\frac{d\mathbf{m}}{dt} = \alpha \Gamma \boldsymbol{\nabla}^2 \mathbf{m} \times \mathbf{M}. \tag{8}$$

For a plane-wave perturbation, whose phase factor is $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, the equation of motion (8) becomes,

$$i\,\omega\,\mathbf{m} = \alpha\Gamma k^2\,\mathbf{m}\times M_0\,\hat{\mathbf{z}}. \tag{9}$$

This is satisfied by a circularly polarized wave,

$$\mathbf{m} = m\,(\hat{\mathbf{x}} + i\,\hat{\mathbf{y}})\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{10}$$

that obeys the quadratic dispersion relation [4],

$$\omega = \alpha\Gamma M_0 k^2, \tag{11}$$

which implies that $\omega \ll ck$ in physical materials, where c is the speed of light. Hence, the electric fields are much smaller than the magnetic fields associated with the time-dependent magnetization $\mathbf{m}$, so that $\boldsymbol{\nabla}\times\mathbf{H} = 0$ to a good approximation, and we may use the term "magnetostatic" to describe the waves. These waves of magnetization are, however, better termed "spin waves", whose quanta are called "magnons".

2.3 Rotating Magnetostatic Modes in a Ferrite Cylinder

In the magnetostatic approximation the fields $\mathbf{B}$ and $\mathbf{H}$ obey,

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \boldsymbol{\nabla}\cdot(\mathbf{H}+4\pi\mathbf{M}) = 0, \qquad \boldsymbol{\nabla}\times\mathbf{H} = 0, \tag{12}$$

where the field $\mathbf{B}$ but not $\mathbf{H}$ and $\mathbf{M}$ must be static (or at least so slowly varying in time that the resulting electric field is small compared to $\mathbf{B}$). We first consider a ferrite of arbitrary shape of characteristic length a in a uniform external magnetic field $\mathbf{B}_{\rm ext} = \mathbf{H}_{\rm ext} = H_0\,\hat{\mathbf{z}}$. We suppose that this field is strong enough to induce a uniform magnetization $M_0\,\hat{\mathbf{z}}$ throughout the sample.

For waves with weak spatial dependence as we shall assume, the exchange interaction is negligible, since it varies as the second spatial derivative of $\mathbf{M}$. Then, the spins interact primarily with the local magnetic field $\mathbf{B}$ according to,

$$\frac{d\mathbf{M}}{dt} = \Gamma\,\mathbf{B}\times\mathbf{M} = \Gamma\,\mathbf{H}\times\mathbf{M}. \tag{13}$$

We consider a perturbation $\mathbf{m}$ to the magnetization that has frequency ω and wavelength large compared to the size of the the sample. Then, the total magnetization can be written as,

$$\mathbf{M} = M_0\,\hat{\mathbf{z}} + \mathbf{m}\,e^{-i\omega t}, \tag{14}$$

where $m \ll M_0$. Similarly, we write the magnetic field inside the sample as,

$$\mathbf{B} = B_z\,\hat{\mathbf{z}} + \mathbf{b}\,e^{-i\omega t}, \qquad \mathbf{H} = H_z\,\hat{\mathbf{z}} + \mathbf{h}\,e^{-i\omega t}, \tag{15}$$

where $B_z = H_z + 4\pi M_0$ and $H_z = H_0 - 4\pi N_z M_0$ are the sums of the external field and that due to the uniform magnetization $M_0\,\hat{\mathbf{z}}$, and so are also uniform for spheroidal (and cylindrical) samples whose axis is the z axis [5]. The "demagnetization" factor N_z varies between 1 for a disk and 0 for a cylinder. The perturbation $\mathbf{m}$ exists only inside the sample, but the corresponding perturbations $\mathbf{b}$ and $\mathbf{h}$ exist outside the sample as well.

Inserting eqs. (14) and (15) in the equation of motion (13), we keep only the first-order terms to find,

$$-i\,\omega\,\mathbf{m} = \Gamma\,\hat{\mathbf{z}} \times (M_0\,\mathbf{h} - H_z\,\mathbf{m}), \tag{16}$$

whose components are,

$$\begin{aligned} m_x &= i\frac{\Gamma}{\omega}(M_0\,h_y - H_z m_y), \\ m_y &= -i\frac{\Gamma}{\omega}(M_0, h_x - H_z m_x), \qquad (17) \\ m_z &= 0. \end{aligned}$$
$$\tag{18}$$

We solve for $\mathbf{m}$ in terms of $\mathbf{h}$ as,

$$\begin{aligned} m_x &= \alpha h_x - i\beta h_y, \\ m_y &= i\beta h_x + \alpha h_y, \end{aligned} \tag{19}$$

where,

$$\alpha = \frac{\Gamma^2 H_z M_0}{\Gamma^2 H_z^2 - \omega^2}\,, \qquad \beta = \frac{\Gamma M_0 \omega}{\Gamma^2 H_z^2 - \omega^2}\,. \tag{20}$$

For later use, we note that in cylindrical coordinates, (r, θ, z), eq. (19) becomes,

$$\begin{aligned} m_r &= \alpha h_r - i\beta h_\theta, \\ m_\theta &= i\beta h_r + \alpha h_\theta. \end{aligned} \tag{21}$$

As we are working in the magnetostatic limit (12), we also have that,

$$\boldsymbol{\nabla} \cdot \mathbf{b} = \boldsymbol{\nabla} \cdot (\mathbf{h} + 4\pi\mathbf{m}) = 0, \qquad \boldsymbol{\nabla} \times \mathbf{h} = 0. \tag{22}$$

Hence, the perturbation $\mathbf{h}$ can be derived from a scalar potential,

$$\mathbf{h} = -\boldsymbol{\nabla}\phi, \tag{23}$$

and so,

$$\boldsymbol{\nabla}^2\phi = 4\pi\boldsymbol{\nabla} \cdot \mathbf{m}. \tag{24}$$

Outside the sample the potential obeys Laplace's equation,

$$\boldsymbol{\nabla}^2\phi = 0 \qquad \text{(outside)}, \tag{25}$$

while inside the sample we find, using eq. (19),

$$(1 + 4\pi\alpha)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{(inside)}. \tag{26}$$

The case of an oblate or prolate spheroid with axis along the external field has been solved with great virtuosity by Walker [6], following the realization that higher-order modes deserved discussion [7]. Here, we content ourselves with the much simpler case of a long cylinder whose axis is along the external field, for which the lowest-order spatial mode was first discussed by Kittel [8]. We consider only the case of waves with no spatial dependence along the axis of the cylinder.

With these restrictions, both eqs. (25) and (26) reduce to Laplace's equation in two dimensions. We can now work in a cylindrical coordinate system (r, θ, z), where appropriate 2-D solutions to Laplace's equation have the form,

$$\phi(r < a, \theta) = \sum \frac{r^n}{a^n}(A_n\, e^{in\theta} + B_n\, e^{-in\theta}), \tag{27}$$

$$\phi(r > a, \theta) = \sum \frac{a^n}{r^n}(A_n\, e^{in\theta} + B_n\, e^{-in\theta}), \tag{28}$$

which is finite at $r = 0$ and ∞, has period 2π in θ, and is continuous at the boundary $r = a$.

The boundary conditions at $r = a$ in the magnetostatic limit (22) are that b_r and h_θ are continuous. The latter condition is already satisfied, since $h_\theta = -(1/r)\,\partial\phi/\partial\theta$. We note that,

$$b_r = h_r + 4\pi m_r = (1 + 4\pi\alpha)h_r - 4\pi i\beta h_\theta, \tag{29}$$

recalling eq. (21). Using eqs. (27) and (28), we find that continuity of b_r at $r = a$ requires,

$$\sum \frac{n}{a}\left[(1 + 2\pi\alpha + 2\pi\beta)A_n\, e^{in\theta} + (1 + 2\pi\alpha - 2\pi\beta)B_n\, e^{-in\theta}\right] = 0. \tag{30}$$

Nontrivial solutions are possible only if $2\pi(\alpha \pm \beta) = -1$, in either of which case there is an infinite set of modes that are degenerate in frequency. Using eq. (20), we find the "resonance" frequency to be,

$$\omega = \pm\Gamma(H_0 + 2\pi M_0), \tag{31}$$

noting that for a cylinder the demagnetization factor is $N_z = 0$, so that $H_z = H_0$, as is readily deduced by elementary arguments. Since we consider frequency ω to be positive, we see that the two solutions (31) correspond to two signs of H_0, and are essentially identical.

For spheroidal samples, the modes are enumerated with two integer indices, and are not all degenerate in frequency, as discussed in [6].

We close our discussion by showing that the electric field of the wave is much smaller than the magnetic field. The scalar potential for mode n is,

$$\phi_n(r < a) = \frac{r^n}{a^n}\, e^{i(n\theta - \omega t)}, \qquad \phi_n(r > a) = \frac{a^n}{r^n}\, e^{i(n\theta - \omega t)}. \tag{32}$$

We see that for $n > 0$ the potential rotates with angular velocity $\Omega_n = \omega/n$ about the z-axis. The potential is maximal at $r = a$, so consistency with special relativity requires that,

$$v(r = a) = \frac{a\omega}{n} \ll c, \tag{33}$$

which appears to have been (barely) satisfied in typical experiments [8]. We also see that for high mode number the spatial variation of the wave becomes rapid, and the neglect of the exchange interaction is no longer justified.

The magnetic field $\mathbf{h} = -\boldsymbol{\nabla}\phi$ of mode n has components,

$$h_r(r < a) = -n\frac{r^{n-1}}{a^n}\, e^{i(n\theta-\omega t)} = -\frac{n}{r}\phi_n, \qquad h_r(r > a) = n\frac{a^n}{r^{n+1}}\, e^{i(n\theta-\omega t)} = \frac{n}{r}\phi_n, \tag{34}$$

$$h_\theta(r < a) = ih_r(r < a), \qquad h_\theta(r > a) = -ih_r(r > a). \tag{35}$$

The monopole mode, $n = 0$, does not exist. The lowest mode is $n = 1$, which corresponds to a uniform, transverse field $\mathbf{h}$ that rotates about the z axis with angular velocity ω.

From eq. (21) we find the magnetization to be,

$$\mathbf{m} = -\frac{\mathbf{h}}{2\pi} \tag{36}$$

for all modes (for $r < a$ only, of course), so the magnetization of mode n also rotates with angular velocity ω/n.

The magnetic field $\mathbf{b} = \mathbf{h} + 4\pi\,\mathbf{m}$ is then,

$$\mathbf{b}(r < a) = -\mathbf{h}(r < a), \qquad \mathbf{b}(r > a) = \mathbf{h}(r > a). \tag{37}$$

Using either the r or θ component of Faraday's law, we find that the associated electric field $\mathbf{e}$ has only a z component,

$$\mathbf{e} = \frac{\omega}{c}\phi\,\hat{\mathbf{z}}\,, \tag{38}$$

both inside and outside the cylinder (consistent with continuity of the tangential component of the electric field at a boundary, and with $\boldsymbol{\nabla}\cdot\mathbf{e} = 0$). The ratio of the electric to the magnetic field of mode n at $r = a$ is,

$$\left|\frac{e_z}{b_r}\right| = \frac{a\,\omega}{nc}\,, \tag{39}$$

which is small so long as condition (33) is satisfied. Hence, the condition that $a\,\omega/c \ll 1$ is doubly necessary for the validity of this analysis.

Because the magnetization $\mathbf{m}$ is moving (rotating), there is an associated electric polarization $\mathbf{p}$ according to special relativity [9],

$$\mathbf{p} = \gamma\frac{\mathbf{v}}{c}\times\mathbf{m}. \tag{40}$$

For mode n we have $\mathbf{v} = \omega r\,\hat{\boldsymbol{\theta}}/n \ll c$, so $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1$, and,

$$\mathbf{p} = -\frac{\omega r m_r}{nc}\,\hat{\mathbf{z}} = \frac{\omega r h_r}{2\pi nc}\,\hat{\mathbf{z}} = -\frac{\omega}{2\pi c}\phi\,\hat{\mathbf{z}} = -\frac{\mathbf{e}}{2\pi}\,. \tag{41}$$

The electric displacement is related by $\mathbf{d} = \mathbf{e} + 4\pi\,\mathbf{p}$, which has the value,

$$\mathbf{d}(r < a) = -\mathbf{e} = -\frac{\omega}{c}\phi\,\hat{\mathbf{z}}\,, \qquad \mathbf{d}(r > a) = \mathbf{e} = \frac{\omega}{c}\phi\,\hat{\mathbf{z}}\,, \tag{42}$$

The fourth Maxwell equation now implies that,

$$\boldsymbol{\nabla}\times\mathbf{h} = -\boldsymbol{\nabla}\times\boldsymbol{\nabla}\phi = \frac{1}{c}\frac{\partial\mathbf{d}}{\partial t} = \pm i\frac{\omega^2}{c^2}\phi\,\hat{\mathbf{z}} \tag{43}$$

Thus, there is a small violation of the magnetostatic conditions (22), but this is second order in the small quantity $a\,\omega/c$ (noting that $\boldsymbol{\nabla}\phi \approx \phi/a$).

References

[1] The regime in which electric fields have time dependence, but magnetic fields do not, is explored in K.T. McDonald, *An Electrostatic Wave* (July 28, 2002),
http://kirkmcd.princeton.edu/examples/bernstein.pdf

[2] See, for example, sec. 69 of E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics, Part 2*, 2nd ed. (Pergamon, 1980), http://kirkmcd.princeton.edu/examples/CM/landau_sp2_80.pdf

[3] For an example of this phenomenon, see K.T. McDonald, *Wave Amplification in a Magnetic Medium* (May 1, 1979),
http://kirkmcd.princeton.edu/examples/magnetic_waves.pdf

[4] F. Bloch, *Zur Theorie des Ferromagnetismus*, Z. Phys. **61**, 206 (1930),
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[5] The magnetic field $\mathbf{H}$ (and $\mathbf{B} = \mathbf{H} - 4\pi\mathbf{M}$) inside a spheroid with uniform magnetization $\mathbf{M} = M_0\,\hat{\mathbf{z}}$ along its axis can be deduced from chap. 5, probs. 80 and 82 of W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, 1968),
http://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf
If we denote the ratio of the axial length of the spheroid to its diameter by c, then $c = 0$ is a disk, $0 < c < 1$ is an oblate spheriod, $c = 1$ is a sphere, $1 < c < \infty$ is a prolate spheroid, and $c = \infty$ is a cylinder. For an oblate spheroid of aspect ratio c, the "radial" coordinate is $\varsigma = c/\sqrt{1 - c^2}$, and the magnetic field due to the uniform magnetization is,

$$\mathbf{H} = -4\pi\mathbf{M}\left\{1 - \varsigma\left[(1 + \varsigma^2)\cot^{-1}\varsigma - \varsigma\right]\right\}. \tag{44}$$

For example, a disk with $c = 0$ has $\varsigma = 0$ also, and $\mathbf{H} = -4\pi\mathbf{M}$, $\mathbf{B} = 0$. For a sphere, $c = 1$ and $\varsigma \to \infty$, in which limit $\cot^{-1}\varsigma \to 1/\varsigma - 1/3\varsigma^3$, so that $\mathbf{H} = -4\pi\mathbf{M}/3$ and $\mathbf{B} = 8\pi\mathbf{M}/3$. For a prolate spheroid of aspect ratio c, the "radial" coordinate is $\eta = c/\sqrt{c^2 - 1}$, and,

$$\mathbf{H} = -4\pi\mathbf{M}\left\{1 - \eta\left[(1 - \eta^2)\coth^{-1}\eta + \eta\right]\right\}. \tag{45}$$

For a cylinder with $c \to \infty$, we have $\eta = 1$, $\coth^{-1}\eta = 0$, and $\mathbf{H} = 0$, $\mathbf{B} = 4\pi\mathbf{M}$. The fields for a sphere can also be obtained from the limit $c \to 1$, $\eta \to \infty$ and $\coth^{-1}\eta \to 1/\eta + 1/3\eta^3$. The expressions in braces in eqs. (44) and (45) correspond to the demagnetization factor N_z introduced in the main text.

[6] L.R. Walker, *Magnetostatic Modes in Ferromagnetic Resonance*, Phys. Rev. **105**, 390 (1957), http://kirkmcd.princeton.edu/examples/EM/walker_pr_105_390_57.pdf
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[9] R. Becker, *Electromagnetic Fields and Interactions* (Dover, 1982), sec. 87,
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