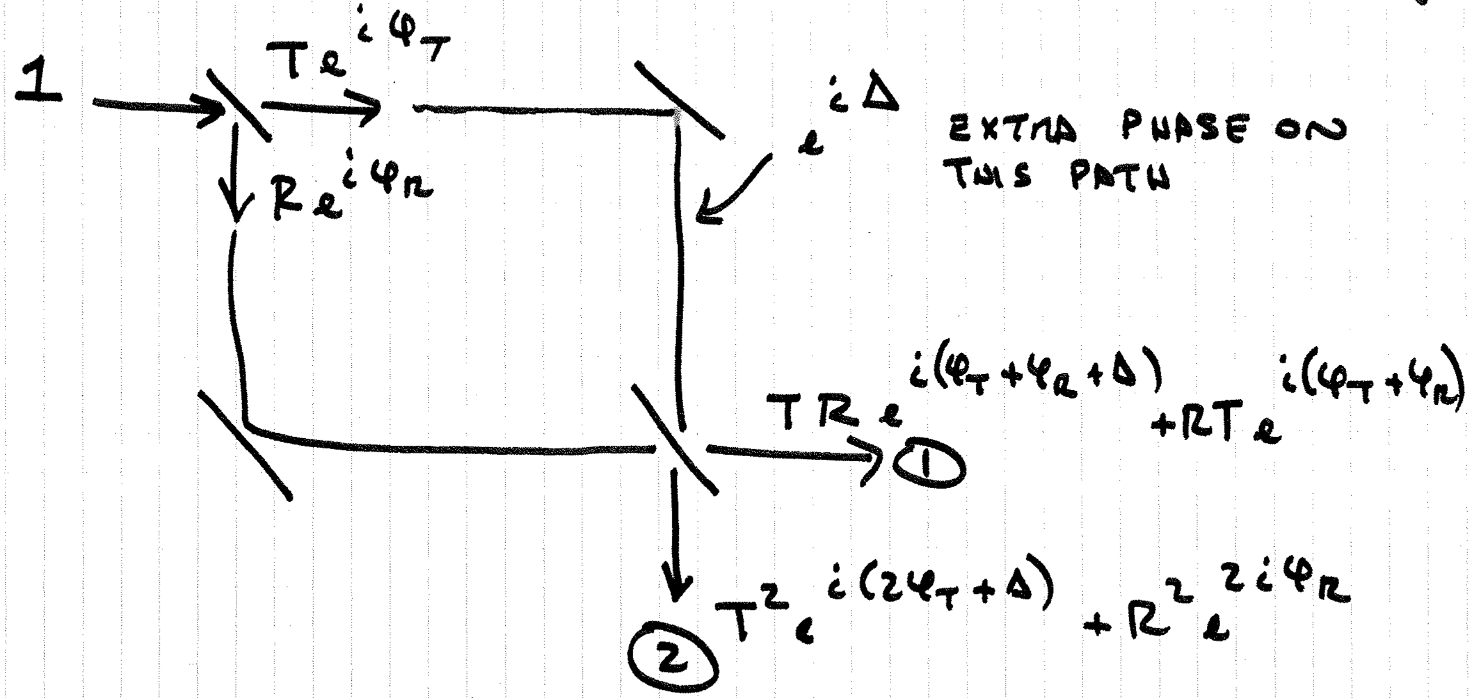
# BEAM-SPLITTER PARADOX    KTM    12/1/78






$$I_1 = 2T^2R^2\,(1+\cos\Delta)$$

$$I_2 = T^4 + R^4 + 2T^2R^2\cos\left[2(\varphi_T-\varphi_R)+\Delta\right]$$

$$I_1+I_2 = (T^2+R^2)^2 + 2T^2R^2\left\{\cos\Delta + \cos\left[2(\varphi_T-\varphi_R)+\Delta\right]\right\}$$

CASE A. LOSSLESS SPLITTER $\Leftrightarrow$ $T^2+R^2=1$

SET $\Delta = 0$

THEN $I_1 + I_2 = 1 + 2T^2R^2\left(1+\cos 2(\varphi_T-\varphi_R)\right) \overset{\text{LOSSLESS}}{=} 1$

$\therefore\ \varphi_T - \varphi_R = 90^\circ$ !!!

EVEN IF ITS NOT CLEAR THAT $I_1+I_2=1$, CERTAINLY $I_1+I_2 \le 1$, BUT $2T^2R^2\left(1+\cos 2(\varphi_T-\varphi_R)\right) \ge 0$

SO ONLY $\varphi_T-\varphi_R = 90^\circ$ IS CONSISTENT

THEN, $I_1+I_2=1$ NO MATTER WHAT $\Delta$ IS!

I.E. LOSSLESS IS LOSS LESS.

CASE B. SPLITTER IS LOSSY: $T^2+R^2 = 1-\epsilon$ $\quad 0<\epsilon\le 1$

NOTE: $T^2R^2\big|_{MAX} = \frac{(1-\epsilon)^2}{4}$ IN THIS CASE.

$$I_1+I_2 = (1-\epsilon)^2 + 2T^2R^2\left(\cos\Delta + \cos\left(2(\varphi_T-\varphi_R)+\Delta\right)\right)$$

CAN $I_1+I_2 = 0$ ?

ONLY HOPE IS TO TAKE $T^2R^2 = T^2R^2\big|_{MAX}$

AND $(\cos\Delta + \ldots) = -2$

THEN $\Delta = \pi$, $2(\varphi_T-\varphi_R) = 2\pi$

$\varphi_T - \varphi_R = 180^\circ$

SO IT IS POSSIBLE FOR BEAMS TO VANISH.

<u>BUT</u>, NOTE THAT $I_1+I_2 < 1-\epsilon$ SINCE

THE INTENSITY ON 2ND SPLITTER IS ~~$I_1$~~. $T^2+R^2$

SUPPOSE $T^2R^2 = T^2R^2_{MAX}$, $\varphi_T-\varphi_R = 180^\circ$, BUT NOW

WE CHOOSE $\Delta = 0$.

THEN $I_1+I_2 = (1-\epsilon)^2 + \frac{(1-\epsilon)^2}{2}(1+1) = 2(1-\epsilon)^2 < 1-\epsilon$

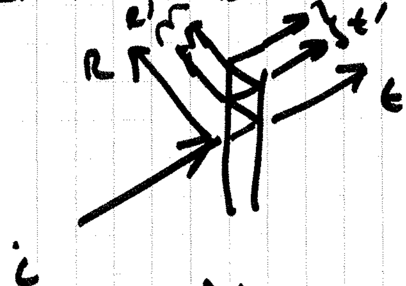
$$\Rightarrow \epsilon > 1/2$$

THUS BEAMS CAN VANISH ONLY IF 50% OF BEAM VANISHES IN THE 1ST SPLITTER. THEN IT'S NOT SO SURPRISING A SECOND, LOSSY SPLITTER CAN KILL THE REST.

PENCIL BEAM INCIDENT ON A SYMMETRIC SPLITTER.

IF SPLITTER IS SO THIN THAT MULTIPLE REFLECTIONS CAN BE NEGLECTED, THE ANALYSIS OF THE PLANE WAVE CASE STILL HOLDS & $\varphi_T - \varphi_R = 90^\circ$

IF MULTIPLE REFLECTIONS ARE IMPORTANT, THE INCIDENT ~~WAVE~~ BEAM INTERFERES ONLY WITH THE PRIMARY RADIATED BEAM (FOR NON-NORMAL INCIDENCE)




$A_i$ = INCIDENT

$$\underbrace{A_R e^{i\varphi}}_{\text{PRIMARY}} + \underbrace{A_R' e^{i\varphi_R'}}_{\text{HIGHER ORDER}} = \text{REFLECTED BEAMS}$$

$$A_t = \underbrace{A_i + A_R e^{i\varphi}}_{\text{INTERFERE}} + \underbrace{A_R' e^{i\varphi_R'}}_{\text{HIGHER ORDER.}} = \text{TRANSMITTED}$$

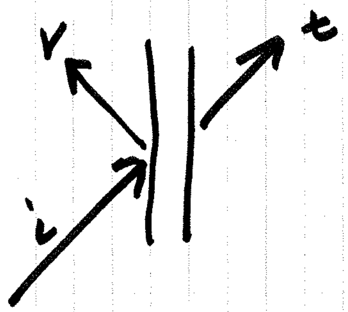
ENERGY CONSERVATION $\Rightarrow A_i^2 = A_R^2 + A_R'^2 + A_i^2 + A_R^2 + 2A_iA_R\cos\varphi + A_R'^2$

OR $\cos\varphi = -\dfrac{A_R}{A_i} - \dfrac{A_R'^2}{A_R A_i}$

AND WE GET NO SPECIAL CONSTRAINT ON THE PHASE RELATION BETWEEN THE PRIMARY REFLECTED & TRANSMITTED BEAMS.

FOR A DIELECTRIC SPLITTER, ONE HAS $\varphi = 180^\circ$, ~~[illegible]~~ $180^\circ$ AND ALSO $180^\circ$ PHASE BETWEEN PRIMARY TRANSMITTED & REFLECTED BEAMS. BUT IT IS IMPOSSIBLE FOR THIS TO HAPPEN WITH $A_R' = 0$, I.E. WITH NO ENERGY IN HIGHER ORDERS.

PLANE WAVE INCIDENT ON A SYMMETRIC SPLITTER SUCH THAT INCIDENT, REFLECTED & TRANSMITTED WAVES ARE ALL IN THE SAME MEDIUM.




WE ASSUME MIRROR REFLECTION & TRANSMITTED ~~BEAM~~ WAVE HAS SAME DIRECTION AS INCIDENT WAVE.

LET $A_i$ = AMP OF INCIDENT WAVE $A_i$ REAL

$A_R e^{i\varphi}$ = AMP OF REFLECTED WAVE $A_R$ REAL

THEN $A_t = A_i + A_R e^{i\varphi}$ IF THE SPLITTER IS SYMMETRIC SO THAT THE RADIATED WAVES ARE EQUAL & IN PHASE ON BOTH SIDES OF SPLITTER. EXAMPLES: A SINGLE DIELECTRIC SHEET, OR A THIN METAL LAYER.

IF ENERGY IS CONSERVED : $|A_i|^2 = |A_R|^2 + |A_i + A_R e^{i\varphi}|^2$

$= |A_R|^2 + |A_i|^2 + |A_R^2| + 2A_iA_R\cos\varphi$

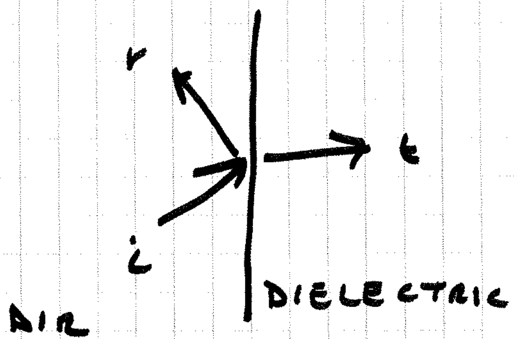
OR $\cos\varphi = -A_R/A_i$

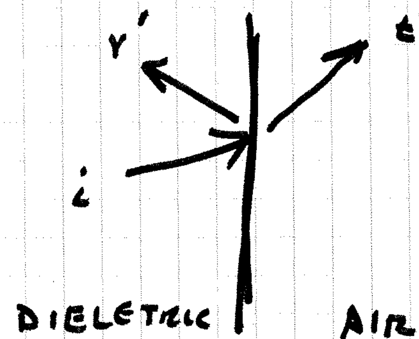
PHASE OF $A_t$ IS $\tan\theta = \dfrac{A_R \sin\varphi}{A_i + A_R\cos\varphi} = \dfrac{\sin\varphi}{-\frac{1}{\cos\varphi} + \cos\varphi} = -\cot\varphi$

$= \tan(\varphi + \pi/2)$

$\therefore \theta = \varphi + \pi/2$ & THERE IS $90^\circ$ DIFFERENCE BETWEEN $A_R$ & $A_T$ !

## MULTIPLE REFLECTIONS IN A SYMMETRIC DIELECTRIC SPLITTER



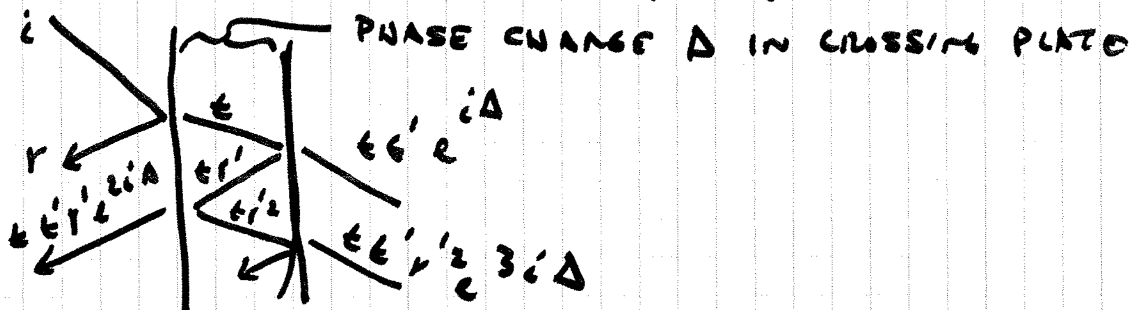




THE RULES ARE: $r$ REAL, PHASE = $180^\circ$

$r' = -r$, REAL, PHASE = 0

$t, t'$ COMPLEX, BUT $|t| = |t'|$, OR $t' = t^*$

IF ENERGY IS CONSERVED: $1 = r^2 + t^2 = r'^2 + t'^2$




$$\text{REFLECTED} = r + tt'r'e^{2i\Delta}\sum_{n=0}^{\infty} r'^{2n}e^{2ni\Delta}$$

$$\text{TRANSMITTED} = tt'e^{i\Delta}\sum_{n=0}^{\infty} r'^{2n}e^{2ni\Delta}$$

IF PENCIL BEAMS, THE TERMS DON'T INTERFERE

IF PLANE WAVES, THERE IS INTERFERENCE

$$R = r + \frac{tt'r'e^{2i\Delta}}{1 - r'^2e^{2i\Delta}}$$

$$T = \frac{tt'e^{i\Delta}}{1 - r'^2e^{2i\Delta}}$$

NOW $r' = -r$ AND $t t' = |t|^2 = 1 - r^2$

$$R = r\left(1 - \frac{(1-r^2)\,e^{2i\Delta}}{1-r^2 e^{2i\Delta}}\right) = r\,\frac{1-e^{2i\Delta}}{1-r^2 e^{2i\Delta}}$$

$$T = \frac{(1-r^2)\,e^{i\Delta}}{1-r^2 e^{2i\Delta}}$$

$$\frac{T}{R} = \frac{1-r^2}{r}\,\frac{e^{i\Delta}}{1-e^{2i\Delta}} = \frac{1-r^2}{r}\,\frac{e^{i\Delta}-e^{-i\Delta}}{2} = \left(\frac{1-r^2}{r}\right) i\sin\Delta$$

$\therefore$ PHASE OF $T/R = 90^\circ$

OR PHASE DIF BETWEEN $T$ & $R$ IS 90°.

---

IF PENCIL BEAMS, AND NO INTERFERENCE, WE CAN CHECK CONS. OF ENERGY:

$$R^2 = r^2 + t^2 t'^2 r'^2 \sum r'^{4n} = r^2\left(1 + \frac{t^4}{1-r^4}\right) = r^2\left(1 + \frac{t^2}{1+r^2}\right) = \frac{2r^2}{1+r^2}$$

USING $r^2 + t^2 = 1$, $t'^2 = t^2$ AND $r'^2 = r^2$

$$T^2 = t^2 t'^2 \sum r'^{4n} = \frac{t^4}{1-r^4} = \frac{t^2}{1+r^2}$$

$$R^2 + T^2 = \frac{2r^2 + t^2}{1+r^2} = 1\,!$$

---

LIKEWISE, WITH SOME EFFORT, CAN VERIFY THAT $R^2 + T^2 = 1$ FOR PLANE WAVES.

MORE ON PENCIL BEAM IN SYMMETRIC DIELECTRIC SPLITTER.

SUPPOSE WANT PRIMARY REFLECTED & TRANSMITTED BEAMS TO HAVE THE SAME INTENSITY.

$R_0 = r \qquad I_{R_0} \sim r^2$

$T_0 = t t' e^{i\Delta} \qquad I_{T_0} \sim t^2 t'^2 = t^4$

$\therefore \ r^2 = t^4$

BUT $r^2 + t^2 = 1$

OR $t^4 + t^2 - 1 = 0$

$$t^2 = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{\sqrt{5}-1}{2} = .62$$

$$t^4 = .38 = r^2$$

$I_{R_0} + I_{T_0} = .76$

$\Rightarrow$ 24% OF ENERGY IN HIGHER ORDERS

USE THIS SPLITTER IN INTERFEROMETER AS ON p 1, 2

$\epsilon = .24$ IF ALL HIGHER ORDER BEAMS ARE BLOCKED.

$T^2 = R^2 = .38$, $\varphi_T - \varphi_R$ CAN BE MADE ARBITRARY.

PATH DIF BETWEEN 2 ARMS

$$I_1 + I_2 = .58 + .29\left(\cos\delta + \cos\left(\delta + 2(\varphi_T - \varphi_R)\right)\right)$$

CHOOSE $\varphi_T - \varphi_R = \begin{matrix} 0 \Rightarrow \text{PLATE } \tfrac{1}{2} \text{ WAVE THICK} \\ \pi \Rightarrow \text{1 WAVE} \end{matrix}$

AND $\delta = \pi$

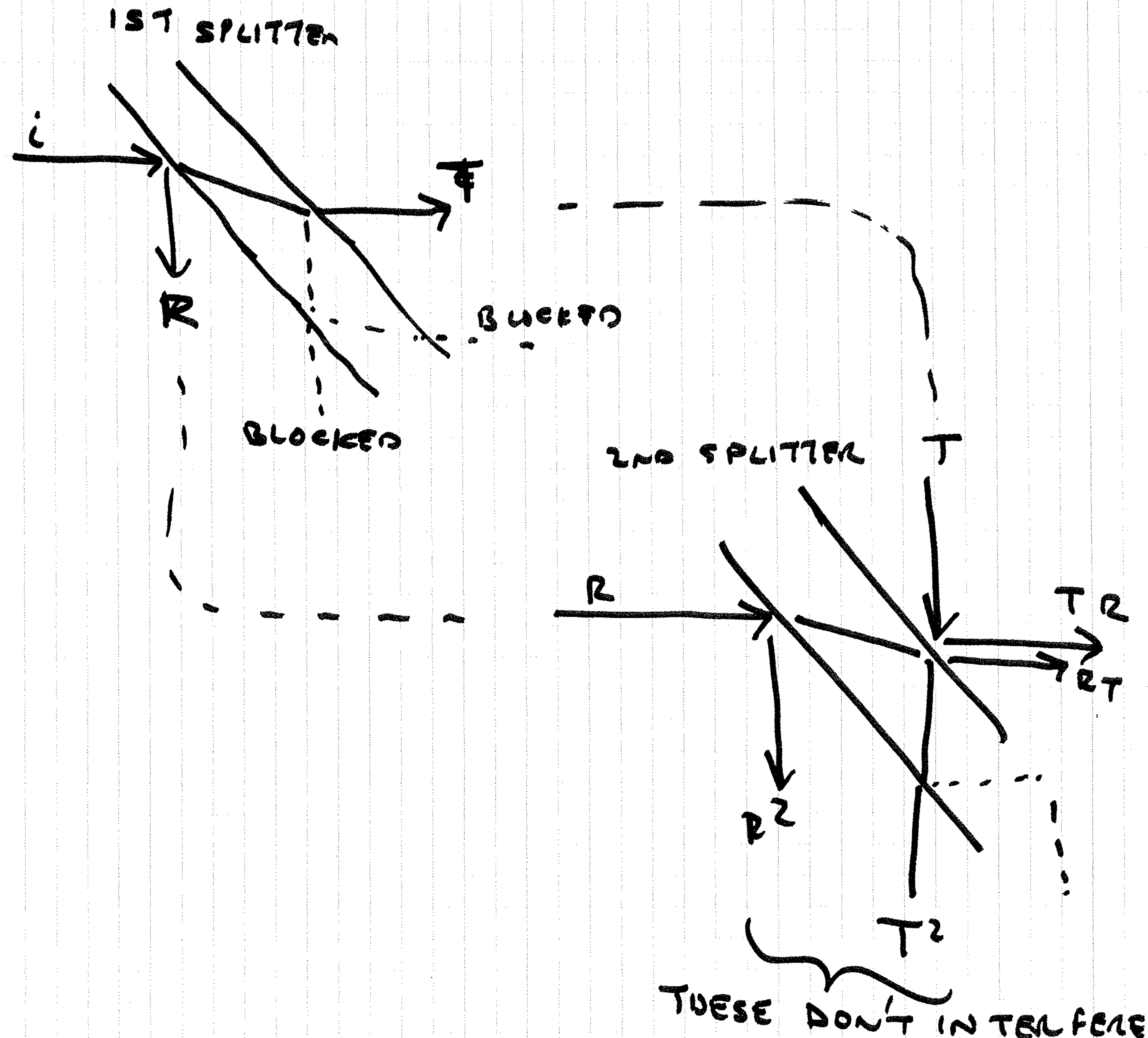
THEN $I_1 + I_2 = 0$ BUT WAIT . . . . .

PARADOX ?

CHOOSE $\delta = 0$

$$I_1 + I_2 = .58 + 2(.29) = 1.16 > 1 \;?$$

POSSIBLE RESOLUTION: IF YOU BLOCK HIGHER ORDER BEAMS BETWEEN SPLITTERS, YOU CANNOT COMPLETELY RECOMBINE THE BEAMS AGAIN!



NO INTERFERENCE

$$R^2 = r^2 + t^2t'^2r'^2 \sum r'^{4n} = r^2\left(1+\frac{t^4}{1-r^4}\right)$$

$$T^2 = t^2t'^2 \sum r'^{4n} = \frac{t^4}{1-r^4}$$

$$R^2+T^2 = r^2 + t^4\frac{(1+r^2)}{1-r^4}$$

$$= r^2 + \frac{t^4}{1-r^2}$$

$$= r^2+t^2$$

$$= 1$$

BUT $r^2+t^2=1$

SO $t^4 = t^2(1-r^2)$

$$(1-A)(1-A^*) = 1+A^2-2\,\mathrm{Re}\,A$$

---

$$R^2 = r^2 + \frac{t^2t'^2r'^2}{1+r'^4-2r'^2\cos 2\Delta} + 2tt'rr'\,\mathrm{Re}\,\frac{e^{2i\Delta}}{1-r'^2e^{2i\Delta}}$$

$$\mathrm{Re}\,\frac{e^{2i\Delta}\left(1-r'^2e^{-2i\Delta}\right)}{1+r'^4-2r'^2\cos 2\Delta}$$

$$\mathrm{Re}\,\frac{\cos 2\Delta - r'^2}{\quad}$$

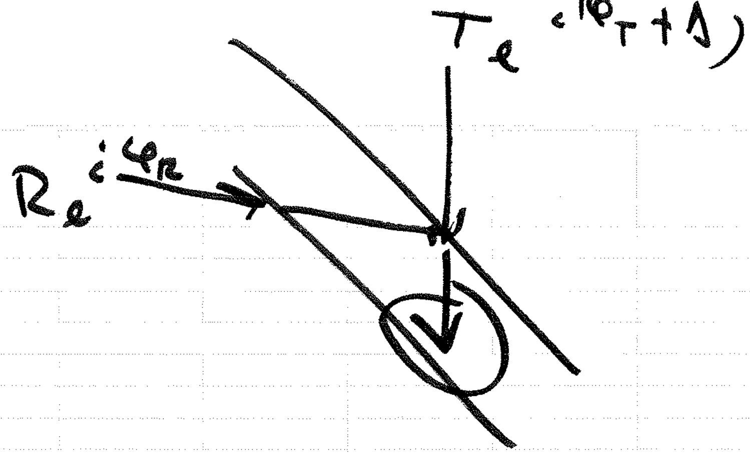
$$T^2 = \frac{t^2t'^2}{1-r'^4-2r'^2\cos 2\Delta}$$

$$R^2+T^2 = r^2 + \frac{t^2t'^2r'^2 + t^2t'^2 - 2tt'rr'(r^2-\cos 2\Delta)}{1+r'^4-2r'^2\cos 2\Delta}$$

$$+\frac{t^2\left(t^2r^2+t^2+2r^4-2r^2\cos 2\Delta\right)}{(\qquad)}$$

$$+t^2\left(\underbrace{\underbrace{t^2r^2+r^4}_{r^2}+t^2}_{1}+r^4-2r^2\cos 2\Delta\right)$$

$$= r^2+t^2 = 1$$




$$te^{i\delta}\,($$

$$Re^{i\varphi_R}\cdot te^{i\delta}\cdot r'e^{i\delta} + Te^{i(\varphi_T+\Delta)}\,te^{i\delta}$$

$$= te^{i\delta}\left[Re^{i\varphi_R}\,r'e^{i\delta} + Te^{i(\varphi_T+\Delta)}\right]$$

$\delta = \varphi_T = 0$ $\qquad T = \epsilon^2$

$r' = -r = R \qquad R > 0$

$\varphi_\pi = \pi$

$$= \epsilon\left(R^2 + Te^{i\Delta}\right)$$

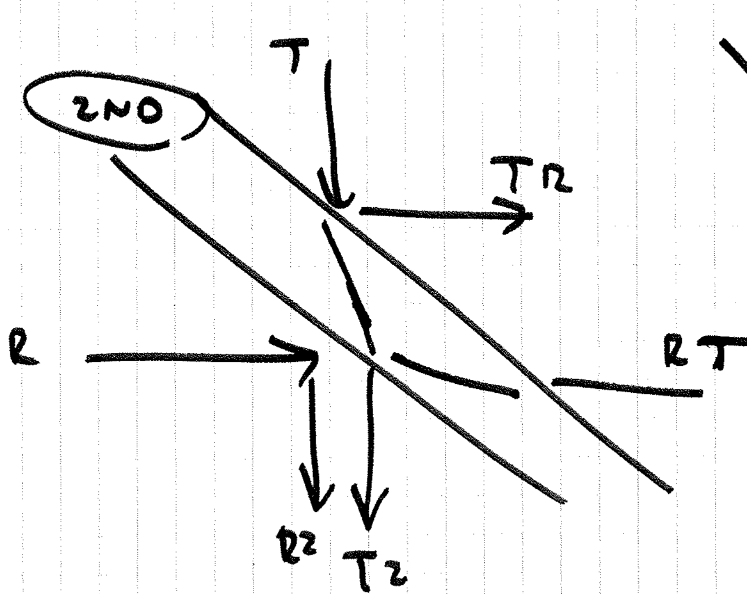
square $\quad t^2\left(R^4 + T^2 + 2TR^2\cos\Delta\right)$

$I_2 \quad R^4 + \left(TR^4 + T^3 + 2T^2R^2\cos\Delta\right) + 2T^2R^2(1+\cos\Delta)$

$$\left(\frac{1-\epsilon}{2}\right)^2 + \epsilon\,T\left[\left(\left(\frac{1-\epsilon}{2}\right)^2 + \left(\frac{1-\epsilon}{2}\right)\right]\right. + 2\left(\frac{1-\epsilon}{2}\right)^2 + 4\left(\frac{1-\epsilon}{2}\right)^2\cos\Delta$$

$$= \frac{3}{4}(1-\epsilon)^2 + \sqrt{\frac{1-\epsilon}{2}}\left(\frac{1-\epsilon}{2}\right)\left(\frac{1-\epsilon}{2}+1\right) + (1-\epsilon)^2\cos\Delta$$

OR YOU CAN TRY




AGAIN SOMETHING DOESN'T INTERFERE

DETAILS OF CASE ON p 8.

IGNORE HIGHER ORDERS BUT NOTE THAT BEAM $T^2$

INTERFERES WITH 2ND ORDER BEAM $R^2$!

IN NOTATION OF p1 & p5

$$I_1 = 2T^2R^2(1 + \cos\Delta)$$

$$I_2 = R^4 + \Big(T^2 e^{i(2\varphi_T + \Delta)} + \underbrace{tt'r'e^{2i\delta} \cdot Re^{i\varphi_R}}_{\text{2ND ORDER REFLECTED}}\Big)^2$$

NOW ~~$T = tt'e^{i\delta}$ $R =$~~

$$Te^{i\varphi_T} = tt'e^{i\delta} \qquad Re^{i\varphi_R} = re^{i\pi} = -$$

$$T = tt' \quad \varphi_T = \delta \qquad R = r \quad \varphi_R = \pi$$

AND WE HAVE CHOSEN $\varphi_T = 0$

$$I_2 = R^4 + (T^2 e^{i\Delta} \; TR^2)^2$$

$$= R^4 + T^4 + T^2R^4 - 2T^3R^2 \cos\Delta$$

THIS SIGN MUST BE −, BUT I GET + ?

SO $I_1 + I_2 = (R^2 + T^2)^2 + 2T^2R^2 \cos\Delta\,(1 - T) + T^2R^4$

$$= (1-\epsilon)^2 + \frac{(1-\epsilon)^2}{2}\cos\Delta\left(1 - \sqrt{\frac{1-\epsilon}{2}}\right) + \left(\frac{1-\epsilon}{2}\right)^3$$

$$< \frac{3}{2}(1-\epsilon)^2 - \frac{(1-\epsilon)^{5/2}}{2\sqrt{2}} + \left(\frac{1-\epsilon}{2}\right)^3$$

$$.87 - .18 + .05 = .74$$

THIS MUST BE LESS THAN $1 - \epsilon = .76$ TO CONSERVE ENERGY. SO IT'S O.K.

$$I_1 + I_2 \text{ MIN} = (1-\epsilon)^2 - \frac{(1-\epsilon)^2}{2}\left(1 - \sqrt{\frac{1-\epsilon}{2}}\right) + \left(\frac{1-\epsilon}{2}\right)^3$$

$$= \frac{1}{2}(1-\epsilon)^2 + \frac{(1-\epsilon)^{5/2}}{2\sqrt{2}} + \left(\frac{1-\epsilon}{2}\right)^3$$

$$= .29 + .18 + .05 = .52 \neq 0$$

IN THIS ARRANGEMENT, WITH $\Delta = \pi$,

$I_1 = 0$

$I_2$ HAS 2 PIECES $\begin{cases} R^4 = .14 \\ T^4 + \ldots = .38 \end{cases}$

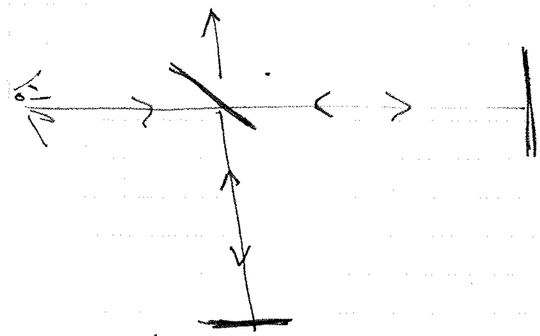
SO WHILE THE BEAMS ARE WEAK, THEY HAVE NOT VANISHED!

TO VANISH, WE MUST HAVE $\frac{3}{2}(1-\epsilon)^2 = \frac{(1-\epsilon)^{5/2}}{2\sqrt{2}}$

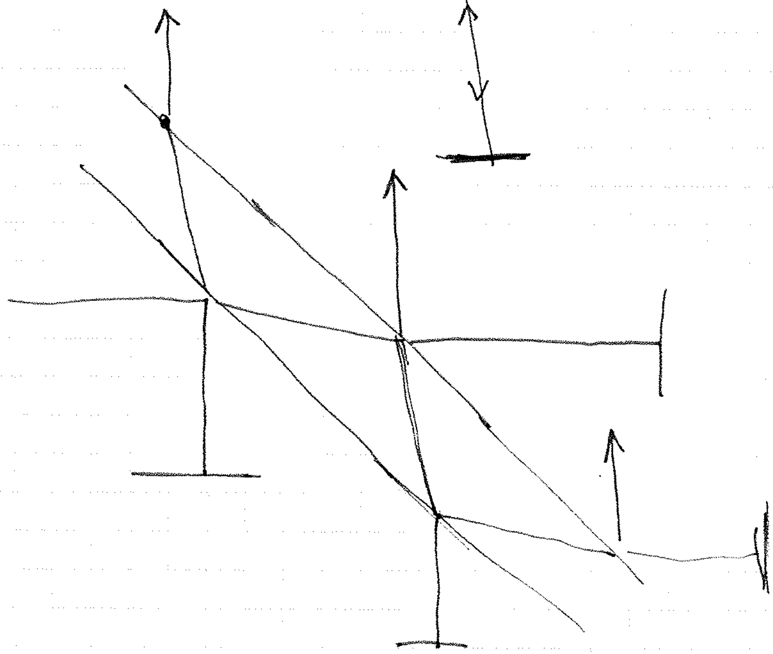
OR $\sqrt{1-\epsilon} = 3\sqrt{2}$ WHICH IS IMPOSSIBLE FOR ANY $\epsilon$. (OTHER THAN $\epsilon = 1$.)

6/30/81

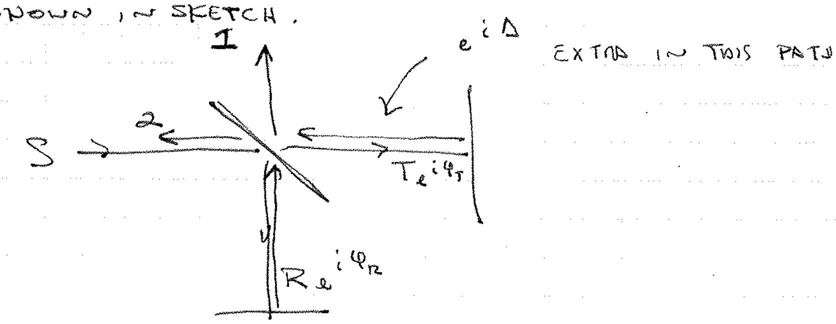
BDW'S ARRANGEMENT



BLOWUP:



**A**: SPLITTER IS ONLY A SINGLE LAYER, IGNORE HIGHER ORDER TERMS SHOWN IN SKETCH.




$$A_1 = TR\,e^{i(\varphi_T+\varphi_R+\Delta)} + RT\,e^{i(\varphi_T+\varphi_R)}$$

$$I_1 = 2T^2R^2(1+\cos\Delta)$$

$$A_2 = R^2 e^{2i\varphi_R} + T^2 e^{2i\varphi_T + i\Delta}$$

$$I_2 = T^4 + R^4 + 2T^2R^2\cos[2(\varphi_T-\varphi_R)+\Delta]$$

EXACTLY AS ON P1.