The Charge Distribution on the Cathode of a Straw Tube Chamber

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1 Problem

A straw tube chamber is a low-cost version of a gas-filled proportional counter, which latter is a descendent of the Geiger counter.[1] These devices consist of a pair of coaxial conducting cylinders with the region between the cylinders is filled with a gas such as argon. The inner cylinder of radius a is the anode, and is held at potential V; the outer cylinder of radius b is the cathode, and is grounded.

If a penetrating charged particle passes through the chamber, it will ionize about two gas molecules per mm of path length. The ionization electrons are pulled by the electric field towards the anode. Close to the anode, the field is strong enough that the electrons gain enough energy during one mean free path to ionize the molecule they hit next, liberating one or more additional electrons. In a proportional chamber, the field is kept low enough that the resulting Townsend avalanche[2] involves 10^4-10^6 molecules.

What is the time dependence, $I(t)$, of the current that flows off the anode due to the avalanche of a single initial electron?

What is the spatial dependence, $q(z)$ of the charge distribution induced on the anode during the time when the current is large, where the z axis is the chamber axis? You may restrict your attention to values of z far from the ends of the tube of length l.

Measurement of the charge distribution via a segmented cathode permits localization in z of the ionization, and hence, of the initiating charged particle.[3]

You may ignore the tiny current that flows while the electron drifts towards the anode. The avalanche takes place so close to the anode, that the small remaining drift time for the electrons to reach the anode may also be ignored. In this approximation, the situation at $t = 0$ is that electrons of total charge $-q_0$ reside on the anode in close proximity to positive ions of total charge $+q_0$. Current flows off the anode only when some of the field lines from the positive ions detach from the electrons on the anode, and extend to the cathode where charge is induced to terminate these field lines. This occurs only as the positive ions move

[1] E. Rutherford and H. Geiger, *An Electrical Method of Counting the Number of α-Particles from Radioactive Substances*, Proc. Roy. Soc. London A **81**, 141 (1908),
http://physics.princeton.edu/~mcdonald/examples/detectors/rutherford_prsla_81_141_08.pdf
H. Geiger and W. Müller, *Elektronenzählrohr zur Messung schwächster Aktivit/"aten*, Naturw. **16**, 617 (1928), http://physics.princeton.edu/~mcdonald/examples/detectors/geiger_naturw_16_617_28.pdf

[2] J.S. Townsend, *The conductivity in Gases produced by Motion of Negatively-charged Ions*, Nature **62**, 340 (1900), http://physics.princeton.edu/~mcdonald/examples/detectors/townsend_nature_62_340_00.pdf
Phil. Mag. **6**, 198 (1901), http://physics.princeton.edu/~mcdonald/examples/detectors/townsend_pm_6_198_01.pdf

[3] C. Leonidopoulos, C. Lu and A.J. Schwartz, *Development of a Straw Tube Chamber with Pickup-Pad Readout*, Nucl. Instr. and Meth. A427, 465 (1999),
http://physics.princeton.edu/~mcdonald/examples/detectors/leonidopoulos_nim_a427_465_99.pdf

away from the anode, with velocity related by

$$v = \mu E, \tag{1}$$

where μ is the positive-ion mobility.

2 Solution

2.1 Elementary Solution for $I(t)$

The form of the current $I(t)$ in a cylindrical "straw tube" can be found by an elementary method, so we illustrate this first.

The current that flows off the anode is equal to minus the rate of change of the charge $q(t) < 0$ that remains on the anode as the positive ions of total charge q_0 move outward according to $r(t)$.

The key to an elementary solution is that although the positive ions occupy a very small volume around the point $(r, \theta, z) = (r(t), 0, 0)$ in cylindrical coordinates, the charge they induce on the cathode is exactly the same as if those ions were uniformly spread out over a cylinder of radius r.

Because the superposition principle holds in electrostatics, the problem of the chamber with voltage V on the anode plus ions at a fixed position between the anode and cathode can be separated into two parts. First, an empty chamber with voltage V on the anode, and second, a grounded chamber with positive ions inside. [That is, we decompose the problem into cases A and B of the discussion of the reciprocation theorem, even though we won't use that theorem here.]

For the second part, the radial electric field in the region $a < r < r(t)$ can be calculated from the charge q on the anode as

$$E(r) = \frac{2q(t)}{rl}\,, \tag{2}$$

using Gauss' Law, where $l \gg b$ is the length of the cylinder. Similarly, the electric field in the region $r(t) < r < b$ is

$$E(r) = \frac{2(q_0 + q(t))}{rl}\,. \tag{3}$$

The potential difference between the inner and outer cylinder must be zero. Hence,

$$0 = \frac{2q(t)}{l}\int_a^{r(t)} \frac{dr}{r} + \frac{2(q_0 + q(t))}{l}\int_{r(t)}^b \frac{dr}{r} = \frac{2q_0}{l}\ln\frac{b}{r(t)} + \frac{2q(t)}{l}\ln\frac{b}{a}\,, \tag{4}$$

and so

$$q(t) = -q_0\frac{\ln(b/r(t))}{\ln(b/a)}. \tag{5}$$

The current is

$$I(t) = -\dot{q}(t) = -\frac{q_0}{\ln(b/a)}\frac{v(t)}{r(t)}. \tag{6}$$

To calculate the dynamical quantities $r(t)$ and $v(t)$, we must return to the full problem of the ions in a chamber with voltage V. The electric field in the chamber is only slightly perturbed by the presence of the ions, and so is given by

$$E(r) = \frac{V}{r \ln(b/a)}\,. \tag{7}$$

According to (1), the positive ions have velocity

$$v(r) = \frac{\mu V}{r \ln(b/a)}\,, \tag{8}$$

which integrates to give

$$r^2(t) = a^2 + \frac{2\mu V}{\ln(b/a)} t. \tag{9}$$

Inserting (8-9) in (6), we find

$$I(t) = -\frac{q_0}{2t_0 \ln(b/a)} \frac{1}{1 + t/t_0}\,, \tag{10}$$

where

$$t_0 = \frac{a^2 \ln(b/a)}{2\mu V}\,. \tag{11}$$

The idealized current pulse has a very sharp rise, and falls off rapidly over characteristic time t_0, which is about 20 nsec in typical straw tube chambers.

2.2 Green's Reciprocation Theorem and Weighting Fields

This problem, and problems with more complex geometry, can be solved by an application of Green's reciprocation theorem, which states that if a set of fixed conductors is at potentials V_i when carrying charges Q_i, and at potentials V_i' when carrying charges Q_i', then

$$\sum_i V_i Q_i' = \sum_i V_i' Q_i. \tag{12}$$

To see this, we label the 3-dimensional potential distribution associated with charges Q_i by $\phi(\mathbf{r})$, and that associated with charges Q_i' by ϕ'. The space outside the conductors is charge free and with dielectric constant $\epsilon = 1$. Then, $\nabla^2\phi = 0 = \nabla^2\phi'$ outside the conductors.

We invoke Green's theorem,

$$\int (\phi\nabla^2\phi' - \phi'\nabla^2\phi) d\text{vol} = \oint (\phi\boldsymbol{\nabla}\phi' - \phi'\boldsymbol{\nabla}\phi) \cdot d\mathbf{S}, \tag{13}$$

where we take the bounding surface S to be that of the set of conductors. Hence,[4]

$$0 = \sum_i \oint (V_i \boldsymbol{\nabla}\phi_i' - V_i' \boldsymbol{\nabla}\phi_i) \cdot d\mathbf{S}_i = -4\pi \sum_i (V_i Q_i' - V_i' Q_i), \tag{14}$$

using Gauss' Law (in Gaussian units) that

$$4\pi Q_i = \oint \mathbf{E}_i \cdot d\mathbf{S}_i = -\oint \boldsymbol{\nabla}\phi_i \cdot d\mathbf{S}_i. \tag{15}$$

In the present problem, we have a small charge q_0 at position $\mathbf{r}_0(t)$ that moves under the influence of the field due to conductors $i = 1, ..., n$ that are held at potentials V_i. The charges Q_i on the conductors obey $Q_i \gg q_0$, so the motion of charge q_0 is determined, to a very good approximation by the charges Q_i on the conductors when $q_0 = 0$. Hence, the problem can be considered as the superposition of two situations:

A: charge q_0 absent; conductors $i = 1, ...n$ at potentials V_i.

B: charge q_0 present; conductors $i = 1, ...n$ grounded, with charges ΔQ_i on them.

We are particularly interested in the charge on electrode 1, whose time rate of change is the desired current $I(t)$.

To use the reciprocation theorem, we suppose that in case B the charge resides on a tiny conductor at position $\mathbf{r}_0$ that is at the potential $V_0 = \phi_A(\mathbf{r}_0)$ obtained from case A. Then, the charges and potentials in case B can be summarized as

B: $\{q_0,\ V_0;\ \Delta Q_i,\ V_i = 0,\ i = 1, ..., n\}$.

We solve the electrostatics problem for a third case,

C: $\{q_0' = 0,\ V_0'(\mathbf{r}_0);\ Q_1,\ V_1' = 1;\ \Delta Q_i = 0,\ V_i' = 0,\ i = 2, ..., n.\}$,

in which conductor 1 is held at unit potential, the charges on all other conductors at zero, and all other conductors are grounded except for the tiny conductor at position $\mathbf{r}_0$. Again, we solve this problem as in case A, first ignoring the tiny conductor, then evaluating V_0' as $\phi_C(\mathbf{r}_0)$.

The reciprocation theorem (12) applied to cases B and C implies that

$$0 = q_0 V_0' + \Delta Q_1 \cdot 1. \tag{16}$$

The current that moves off electrode 1 in case B is therefore,

$$I_1 = -\frac{d\Delta Q_1}{dt} = q_0 \frac{dV_0'(\mathbf{r}_0)}{dt} = q_0 \boldsymbol{\nabla} V_0'(\mathbf{r}_0) \cdot \frac{d\mathbf{r_0}}{dt} = -q_0 \mathbf{E}_{\rm w} \cdot \mathbf{v}, \tag{17}$$

where the velocity $\mathbf{v}$ of the charge is determined using the fields from case A, and

$$\mathbf{E}_{\rm w} = -\boldsymbol{\nabla} V_0'(\mathbf{r}_0) = -\boldsymbol{\nabla}\phi_C(\mathbf{r}_0) \tag{18}$$

[4]Green's theorem (13) first appeared as eq. (12), p. 26 of G. Green, *Mathematical Papers* (1828), http://physics.princeton.edu/~mcdonald/examples/EM/green_papers.pdf.
The reciprocation theorem (13) was hinted at on pp. 33-39, but may have first been explicitly stated in Art. 85b, p. 105 of J.C Maxwell, *A Treatise on Electricity and Magnetism*, 2nd ed. (1878), http://physics.princeton.edu/~mcdonald/examples/EM/maxwell_treatise_v1_04.pdf.
That this theorem also holds for conductors in a medium of uniform dielectric constant is shown in sec. 3.09, p. 54 of W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill, 1968).

is called the weighting field.[5] For the case of two conductors (plus charge q_0) one of which is grounded, the weighting field is the same as the field from case A, but in general they are distinct.[6]

2.3 $I(t)$ via Reciprocity

Referring to the prescription in the statement of the problem, we first solve case C, in which the inner electrode is at unit potential and the outer electrode is grounded. We quickly find that

$$V_C(r) = \frac{\ln(b/r)}{\ln(b/a)}\,. \tag{19}$$

According to (17), the current off the inner electrode is therefore,

$$I(t) = -q_0 \frac{dV_C}{dr} v(r) = -\frac{q_0}{\ln(b/a)} \frac{v(t)}{r(t)}\,, \tag{20}$$

as previously found in (6). We again solve for v and $r(t)$ as in (7-9), which corresponds to the use of case A, to obtain the solution (10-11).

2.4 The Charge Distribution $q(z)$ on the Cathode

The more detailed question as to the longitudinal charge distribution on the cathode can be solved by the reciprocation method if we conceptually divide the cathode cylinder into a ring of length dz at position z_1 plus two cylinders that extends to $z = \pm l/2$ where l is the length of the cylinder. We label the ring as electrode 1, and calculate the charge $\Delta Q_1 = q(z)dz$ induced on this ring when the positive ion charge q_0 is at position $(r_0, 0, z_0)$ in cylindrical coordinates (r, θ, z).

According to the prescription (16) given in the statement of the problem,

$$\Delta Q_1 = -q_0 V_C(r_0, 0, z_0), \tag{21}$$

[5]The result (17) was first deduced by the present method by W. Shockley, *Currents to Conductors Induced by a Moving Point Charge*, J. Appl. Phys. **9**, 635 (1939),
http://physics.princeton.edu/~mcdonald/examples/EM/shockley_jap_9_635_39.pdf,
and by S. Ramo, *Currents Induced by Electron Motion*, Proc. I.R.E. **27**, 584 (1939),
http://physics.princeton.edu/~mcdonald/examples/EM/ramo_pire_27_584_39.pdf.
That this result also follows from an energy argument was pointed out by C.K. Jen, *On the Induced Current and Energy Balance in Electronics*, Proc. I.R.E. **29**, 349 (1941),
http://physics.princeton.edu/~mcdonald/examples/EM/jen_pire_29_345_41.pdf.
For discussion that the weighting-field method holds for multiple charges (space charge) and for a uniform dielectric medium, see, for example, L.A. Hamel and M. Julien, *Generalized demonstration of Ramo's theorem with space charge and polarization effects*, Nucl. Instr. Meth. A **597**, 207 (2008),
http://physics.princeton.edu/~mcdonald/examples/EM/hamel_nim_a597_207_08.pdf.
See also, K.T. McDonald, *Does Space Charge or the Dielectric Constant Affect Induced Charge in a Liquid Argon Detector?* (June 4, 2016), http://physics.princeton.edu/~mcdonald/examples/induced.pdf.

[6]Strictly, the weighting-field method has been deduced for electrostatic examples. It's validity for time-dependent cases as a consequence of so-called Lorentz reciprocity is discussed in W. Riegler and P. Windischhofer, *Signals induced on electrodes by moving charges, a general theorem for Maxwell's equations based on Lorentz-reciprocity* (Jan. 28, 2020), https://arxiv.org/abs/2001.10592

where case C now consists of a cylinder of radius b grounded except for the ring at position z_1 at unit potential, and a grounded cylinder at radius a. For z not close to the ends of the cylinder, the end surfaces $z = \pm l/2$ may be approximated as at ground potential.

This problem is very similar to that discussed in sec. 5.36 of W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (Mcgraw-Hill, New York, 1968).

Laplace's equation, $\nabla^2\phi_C(\mathbf{r}) = 0$ holds for the potential in the region $a < r < b$. The problem has azimuthal symmetry, so ϕ_C will be independent of θ. Since the planes $z = \pm l/2$ are grounded, the longitudinal functions in the Fourier series expansion,

$$\phi_C = \sum_n R_n(r) Z_n(z), \tag{22}$$

must have the form $Z_n = \sin 2n\pi z/l$. The equation for the radial functions $R_n(r)$ follows from Laplace's equation as

$$\frac{d^2 R_n}{dr^2} + \frac{1}{r}\frac{dR_n}{dr} - \left(\frac{2n\pi}{l}\right)^2 R_n = 0. \tag{23}$$

The solutions of this are the modified Bessel functions of order zero, $I_0(2n\pi r/l)$ and $K_0(2n\pi r/l)$. Both of these are finite on the interval $a < r < b$, so the expansion (22) will include them both.

The boundary condition that $\phi_C(a, \theta, z) = 0$ is satisfied by the expansion

$$\phi_C = \sum_n A_n \frac{\frac{I_0(2n\pi r/l)}{I_0(2n\pi a/l)} - \frac{K_0(2n\pi r/l)}{K_0(2n\pi a/l)}}{\frac{I_0(2n\pi b/l)}{I_0(2n\pi a/l)} - \frac{K_0(2n\pi b/l)}{K_0(2n\pi a/l)}} \sin\frac{2n\pi z}{l}\,, \tag{24}$$

where the form of the denominator is chosen to simplify the evaluation of the boundary condition at $r = b$. Here, $\phi_C = 0$, except of an interval dz long about z where it is unity. Hence, the Fourier coefficients are

$$A_n = \frac{2}{l}\sin\frac{2n\pi z_1}{l} dz. \tag{25}$$

In sum, the charge distribution q(z) on the cathode at radius b due to positive charge q_0 at $(r_0, 0, z_0)$ follows from (21) and (23-24) as

$$q(z) = -\frac{2q_0}{l}\sum_n \frac{\frac{I_0(2n\pi r_0/l)}{I_0(2n\pi a/l)} - \frac{K_0(2n\pi r_0/l)}{K_0(2n\pi a/l)}}{\frac{I_0(2n\pi b/l)}{I_0(2n\pi a/l)} - \frac{K_0(2n\pi b/l)}{K_0(2n\pi a/l)}} \sin\frac{2n\pi z}{l}\sin\frac{2n\pi z_0}{l}\,. \tag{26}$$

A numerical evaluation of (26) is illustrated in Fig. 1. As is to be expected, the induced charge distribution on the cathode has characteristic width of order $b - r_0$, the distance of the positive charge from the cathode.

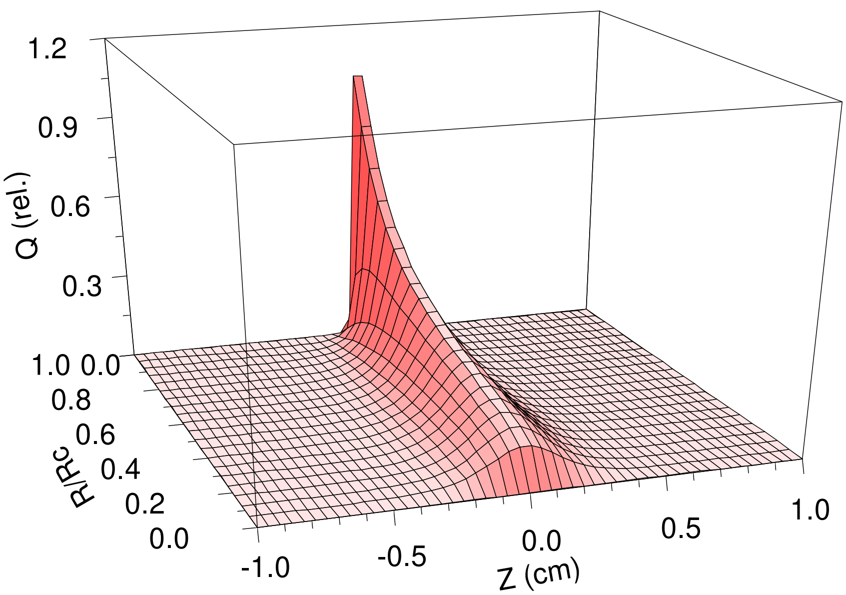

Figure 1: The induced charge distribution (26) on the cathode of a straw tube chamber of radius $R_C = 0.25$ cm due to positive ion charge at radius R.