Tipping a Bowl

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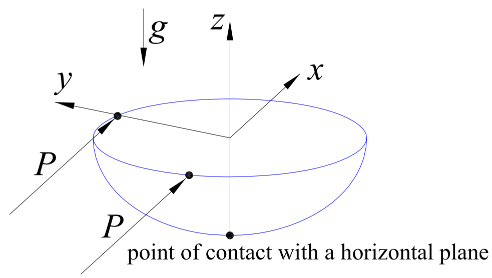
1 Problem

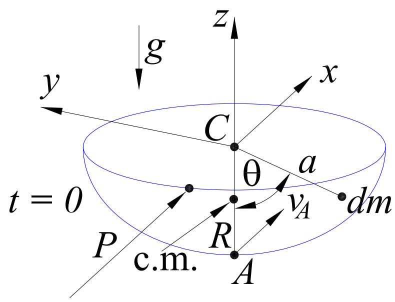
Show that a thin hemispherical bowl, initially at rest on a frictionless, horizontal surface, can be tipped over by a horizontal impulse to its rim if the impulse directed to the center of the bowl, but not if the impulse is (somehow) applied tangentially to the rim.



2 Solution

The bowl will, in general, tip over if a point on its rim touches the horizontal surface during the motion after the impulse, at which time a diameter of the bowl would be vertical.

2.1 Impulse toward the Center of the Bowl



For time $t > 0$, after the horizontal impulse $\mathbf{P} = p\,\hat{\mathbf{x}}$ at time $t = 0$, the only forces (ignoring friction) on the bowl are vertical: gravity and the normal force of the horizontal surface on the bowl. Hence, the horizontal velocity of the center of mass is constant for $t > 0$, and has value,

$$v_{x,\mathrm{cm}} = \frac{P}{m}\,, \tag{1}$$

for a bowl of mass m and radius a, as sketched in the figure above.

We now consider the limiting case that the bowl rotates until a diameter is vertical at some time $T > 0$, and the vertical velocity of the center of mass approaches zero at this time, after which the bowl tips over. Then, the translational kinetic energy of the bowl at time T is the same as at time $t = 0^+$, just after the impulse, while the rotational kinetic energy has dropped to zero and the gravitational potential energy of the bowl has increased by $mg\,|z_{\rm cm}(0)|$. This is possible if,

$$\mathrm{KE}_{\rm rot}(0^+) = \frac{I_{y,{\rm cm}}(0)\,\omega_y^2(0^+)}{2} = mg\,|z_{\rm cm}(0)|\,. \tag{2}$$

The angular velocity $\boldsymbol{\omega}(0^+)$ just after the impulse is related to the angular momentum $\mathbf{L}_{\rm cm}(0^+)$ about the center of mass, imparted by the impulse $\mathbf{P}$,

$$L_{x,{\rm cm}}(0^+) = 0 = I_{x,{\rm cm}}(0)\,\omega_x(0^+), \qquad L_{y,{\rm cm}}(0^+) = P\,|z_{\rm cm}(0)| = I_{y,{\rm cm}}(0)\,\omega_y(0^+),$$
$$L_{z,{\rm cm}}(0^+) = 0 = I_{z,{\rm cm}}(0)\,\omega_z(0^+), \tag{3}$$

where $I_{j,{\rm cm}}$ is the (principal) moment of inertia of the shell about an axis parallel to coordinate axis j and through the center of mass. Hence,

$$\boldsymbol{\omega}(0^+) = \left(0, \frac{P\,|z_{\rm cm}(0)|}{I_{y,{\rm cm}}(0)}, 0\right). \tag{4}$$

We now need the values of $z_{\rm cm}(0)$ and $I_{y,{\rm cm}}(0)$.

The shell has area $2\pi a^2$, so the surface mass density is $\sigma = m/2\pi a^2$. The z-coordinate of the center of mass at time $t = 0$ is related by,

$$z_{\rm cm}\,m = \int z\,dm = \int_0^1 (-a\cos\theta)\,(2\pi a^2\sigma\,d\cos\theta) = -ma\int_0^1 \cos\theta\,d\cos\theta = -\frac{ma}{2}\,, \tag{5}$$

and thus,

$$z_{\rm cm}(0) = -\frac{a}{2}\,. \tag{6}$$

The moment of inertia $I_{y,{\rm cm}}(0)$ $(= I_{x,{\rm cm}}(0)$ by symmetry) is related to the moment of inertia about the center C of the shell by the parallel axis theorem, $I_{y,C}(0) = I_{y,{\rm cm}}(0) + m\,z_{\rm cm}^2(0)$, while $I_{y,C}(0)$ for a hemispherical shell has the same form as that for a complete spherical shell,

$$I_{y,C}(0) = m\left(\langle r_x^2\rangle + \langle r_z^2\rangle\right) = \frac{2}{3}m\left(\langle r_x^2\rangle + \langle r_y^2\rangle + \langle r_z^2\rangle\right) = \frac{2}{3}ma^2, \tag{7}$$

$$I_{y,{\rm cm}}(0) = I_{y,C}(0) - m\,z_{\rm cm}^2(0) = \frac{2}{3}ma^2 - m\frac{a^2}{4} = \frac{5ma^2}{12}\,. \tag{8}$$

We note also that,

$$I_{z,C}(0) = m\left(\langle r_x^2\rangle + \langle r_y^2\rangle\right) = I_{y,C} = \frac{2}{3}m\left(\langle r_x^2\rangle + \langle r_y^2\rangle + \langle r_z^2\rangle\right) = \frac{2}{3}ma^2$$
$$= I_{y,C}(0) = I_{z,{\rm cm}}(0). \tag{9}$$

We now have that,

$$\omega_y(0^+) = \frac{Pa/2}{\frac{5}{12}ma^2} = \frac{6P}{5ma}\,, \qquad \mathrm{KE}_{\mathrm{rot}}(0^+) = \frac{1}{2}\frac{5ma^2}{12}\left(\frac{6P}{5ma}\right)^2 = \frac{3P^2}{10m}\,, \tag{10}$$

so the minimum impulse that will tip the bowl over is,

$$P_{\min} = m\sqrt{\frac{5ga}{3}}\,. \tag{11}$$

2.1.1 Velocity of the Point of Contact

As a "sidelight," we can compute the velocity $\mathbf{v}_A(0^+)$ of the point A of contact of the bowl with the horizontal surface just after the impulse.

From Chasles' theorem, the velocity of a point A on a moving/rotating rigid body is related by,

$$\mathbf{v}_A = \mathbf{v}_{\mathrm{cm}} + \boldsymbol{\omega} \times \mathbf{R}, \tag{12}$$

where $\mathbf{v}_{\mathrm{cm}}$ is the velocity of the center of mass of the object, and $\mathbf{R}$ is the position vector from the center of mass to point A.

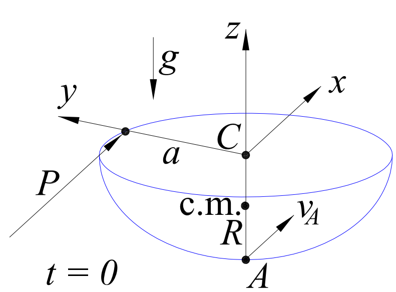
Just after the impulse, $\mathbf{v}_{\mathrm{cm}}(0^+) = (P/m, 0, 0)$, and $\mathbf{R}(0) = (0, 0, -a - z_{\mathrm{cm}}(0)) = (0, 0, -a/2)$,

$$\boldsymbol{\omega}(0^+) \times \mathbf{R}(0) = \left(0, \frac{6P}{5ma}, 0\right) \times (0, 0, -a/2) = \left(-\frac{3P}{5m}, 0, 0\right), \tag{13}$$

and,

$$\mathbf{v}_A(0^+) = \frac{2P}{5m}\,\hat{\mathbf{x}}. \tag{14}$$

2.2 Impulse Tangent to the Rim



For an impulse tangent to the rim, as sketched above, the resulting angular velocity has both $y-$ and z-components, and the subsequent motion is complex.

The angular velocity of the bowl just after the impulse is now, recalling that $R = a/2$,

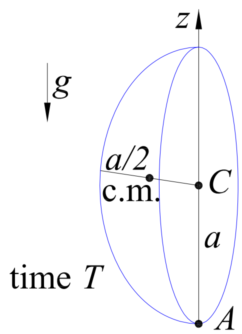
$$\boldsymbol{\omega}(0^+) = \left(0, \frac{Pa/2}{I_{y,\mathrm{cm}}(0)}, -\frac{Pa}{I_{z,\mathrm{cm}}(0)}\right), \quad \boldsymbol{\omega}(0^+) \times \mathbf{R}(0) = \left(-\frac{Pa/2}{I_{y,\mathrm{cm}}(0)}, 0, 0\right), \tag{15}$$

In addition to thinking about energy, it is useful to note another conserved quantity, the vertical component of angular momentum about the center of mass.

After the impulse, the only forces on the system (neglecting friction) are gravity, and the (vertical) normal force $\mathbf{N}$ due to the horizontal surface, acting at the point of contact. Hence, the torque about the center of mass, $\boldsymbol{\tau} = \mathbf{R} \times \mathbf{N}$, has no vertical component, and the angular momentum $L_{z,\mathrm{cm}}$ about the center of mass is constant for $t > 0$.

Suppose the rim of the bowl touches the horizontal surface at some time $T > 0$. Then, recalling eq. (3),

$$L_{z,\mathrm{cm}}(T) = I_{z,\mathrm{cm}}(T)\,\omega_z(T) = L_{z,\mathrm{cm}}(0^+) = Pa, \tag{16}$$



We see from the figure that the configuration of the shell when the rim touches the horizontal surface implies that the moment of inertia $I_{z,\mathrm{cm}}(T)$ is the same as the initial moment of inertia $I_{y,\mathrm{cm}}(0)$, namely $5ma^2/12$. Hence,

$$\omega_z(T) = \frac{Pa}{I_{y,\mathrm{cm}}(0)} = \frac{12P}{5ma}\,, \tag{17}$$

and the rotational kinetic energy about the center of mass at time T is,

$$\mathrm{KE}_{\mathrm{rot}}(T) = \sum_k \frac{I_{k,\mathrm{cm}}(T)\,\omega_{k,\mathrm{cm}}^2(T)}{2} > \frac{I_{z,\mathrm{cm}}(T)\,\omega_{z,\mathrm{cm}}^2(T)}{2} = \frac{1}{2}\frac{5ma^2}{12}\left(\frac{12P}{5ma}\right)^2 = \frac{6P^2}{5m}\,, \tag{18}$$

where index k labels a principal axis at time T, of which the z-axis is one.

However, the rotational kinetic just after the impulse is, recalling eqs. (8), (9) and (13),

$$\mathrm{KE}_{\mathrm{rot}}(0^+) = \sum_j \frac{I_{j,\mathrm{cm}}(0)\,\omega_{j,\mathrm{cm}}^2(0^+)}{2} = 0 + \frac{1}{2} I_{y,\mathrm{cm}}(0)\left(\frac{Pa/2}{I_{y,\mathrm{cm}}(0)}\right)^2 + \frac{1}{2} I_{z,\mathrm{cm}}(0)\left(\frac{-Pa}{I_{z,\mathrm{cm}}(0)}\right)^2$$
$$= \frac{Pa^2}{8\frac{5}{12}ma^2} + \frac{Pa^2}{2\frac{2}{3}ma^2} = \frac{21P}{20m} < \mathrm{KE}_{\mathrm{rot}}(T). \tag{19}$$

Furthermore, since the horizontal velocity of the center of mass is constant for $t > 0$, the translational kinetic energy at time T could be greater, but not less, than that at time $t = 0^+$. Hence, the total kinetic energy required for the rim to touch the horizontal surface is greater than the kinetic energy at time $t = 0^+$, such that the rim can never actually touch the surface for any $t > 0$.

2.2.1 Velocity of the Point of Contact

For completeness, the velocity of the point of contact with the horizontal surface just after the impulse is, recalling eqs. (1), (12) and (15),

$$\mathbf{v}_A(0^+) = \left(\frac{P}{m} - \frac{Pa/2}{I_{y,\mathrm{cm}}(0)}, 0, 0 \right) = \left(\frac{P}{m} - \frac{P(a/2)(a/2)}{\frac{5}{12}ma^2}, 0, 0 \right) = \frac{3P}{5m}\,\hat{\mathbf{x}}\,, \tag{20}$$

50% larger that for an impulse perpendicular to the rim, eq. (14).