Does Centrifugal Force Affect Conduction Electrons?

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1 Problem

Taking the answer to be YES, consider a long cylindrical conductor of radius a that rotates about its axis with (small) constant angular velocity ω in negligible external electromagnetic fields. What is the surface electric charge density, and the electric potential difference between the axis and the surface of the cylinder, in the steady state?[1]

You may assume that the expansion of the cylinder due to its rotation is negligible.[2]

2 Solution

The use of the term "centrifugal force" implies an analysis in a rotating frame. This is complicated for electrodynamics, as reviewed in [1], so we work in the (inertial) lab frame, where, in the steady state, a conduction electron of charge $-e$ at radius r in a cylindrical coordinate system (r, ϕ, z) must experience a centripetal force $F_r = -m\omega^2 r$, if the electron is at rest with respect to the rotating cylinder.

However, if the mean free path for conduction electrons is large compared to their typical separation, then motion of a conduction electron is a better considered to be a sequence of straight-line segments, and no centripetal force is required. In particular, conduction electrons in perfect conductors and superconductors cannot be said to experience centrifugal force.

For the case of lower conductivity, where conduction electrons are approximately at rest with respect to the rotating cylinder, the needed centripetal force on them is due to a radial electric field E_r inside the cylinder, which exerts force $F_r = -eE_r$ on a conduction electron there.[3] Hence,

$$E_r(r < a) = \frac{m\omega^2 r}{e}\,. \tag{1}$$

From the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$ (in Gaussian units) we find,

$$\rho = \rho_+ + \rho_- = \frac{1}{4\pi r}\frac{\partial}{\partial r} r E_r = \frac{m\omega^2}{2\pi e}\,. \tag{2}$$

[1]This example exhibits (very small) unipolar induction, which is a correction to that found in the more usual cases of a rotating conductor in an external magnetic field (studied by Faraday in Arts. 84-100 of [2]), and a rotating magnetized cylinder (studied by Faraday in Arts. 3084-3122 of [3]; see also[4]).

[2]See [5] for discussion of the case of high angular velocity such that the rotating cylinder might pull itself apart.

[3]In principle, the rotating charge distribution creates an axial magnetic field that exerts a radial force of order v^2/c^2, where c is the speed of light in vacuum, on the conduction electrons whose azimuthal velocity is $v = \omega r$. For small angular velocity ω we neglect this force.

We suppose that the conductor when at rest in zero external fields has uniform number density n_0 of conduction electrons, in which case $\rho_+ = en_0$ while $\rho_- = -en_0$. When the conductor is rotating we suppose that ρ_+ is unchanged, ignoring possibly deformation of the rotating cylinder. Then, for the rotating cylinder,

$$\rho_- = \frac{m\omega^2}{2\pi e} - \rho_+ = \frac{m\omega^2}{2\pi e} - en_0 = -en_0\left(1 - \frac{mc^2\omega^2}{e^2 n_0 c^2}\right) = -en_0\left(1 - \frac{\omega^2}{2\pi n_0 r_0 c^2}\right), \tag{3}$$

where $r_0 = e^2 m/c^2$ is the classical electron radius. The magnitude of ρ_- inside the rotating cylinder is less than that when it is rest, so, by conservation of charge, there is an accumulation negative charge on the outer surface of the rotating cylinder. This can be described as the effect of centrifugal force on the conduction electrons. However, the effect is extremely small; for $\omega = 1$ rad/s, $|\rho_-|$ is less than en_0 by one part in 10^{12} in a copper cylinder, where $n_0 \approx 10^{23}/\mathrm{cm}^3$.

If the rotating cylinder is isolated and remains electrically neutral overall, then its cylindrical surface supports a uniform electric charge density σ given by,

$$\sigma = -\frac{\rho\,\mathrm{Volume}}{\mathrm{Area}} = -\frac{m\omega^2}{2\pi e}\frac{a}{2}. \tag{4}$$

The electric potential difference between the axis of the cylinder and its surface is,

$$\Delta V = \int_0^a E_r\, dr = \frac{m\omega^2 a^2}{2e}. \tag{5}$$

3 Comments

While the preceding analysis seems elementary, it would not have been endorsed by Maxwell, who did not consider that electric charge was carried by particles with a fixed ratio m/e of mass to charge.

Only after J.J. Thomson (1897) [6] measured that ratio for "cathode rays" (electrons) did the notion of (elementary) charged particles become generally accepted.[4] Prior to this, the influence of Faraday remained strong, that while electric charge was transported during the process of electrolysis, where *for this case of electro-chemical decomposition...the* chemical power*...is in direct proportion to the absolute quantity of electricity which passes,*[5] neither Faraday nor Maxwell would identify chemical power with mass (as is now the accepted interpretation).[6]

Some of Maxwell's comments on this theme appear in Arts. 568-577 of [11] (1873). Faraday's studies of electrolysis are mentioned in Art. 569, and the issue of whether magnetic energy includes mechanical kinetic energy was addressed in Arts. 574-577, including reports of experiments with null results (see also [12]).

Helmholtz (1881) [13] came close to associating electric currents with charge carriers of definite mass, but still considered this as speculation.

[4]An extensive discussion of the historical context of Thomson's measurement is given in [7].
[5]From Art. 377 of [8], now known as Faraday's first law of electrolysis.
[6]For historical reviews, see [9, 10].

Lodge (1876) followed Maxwell with experiments searching for momentum transfer during electrolysis, but also with null results, sec. 16 of [14], and Art. 189 of [15]. Hertz (1881) [16] also reported a null experimental result. A positive result was reported by Colley (1882) [17], that when a vial of electrolyte was dropped, a brief electric potential difference between its top and bottom ends appeared at the moment of impact. Des Coudres conducted experiments in the 1890's [18, 19] in which an electric potential difference was detected between the center and outer edge of a rotating cylinder of electrolyte.

The studies of Lodge, Colley and Des Coudres concerned the masses of the ionic charge carriers in electrolytes. (Inconclusive) efforts were made by Nichols (1906) [20] to observe an electric potential difference in a rotating metal cylinder, where electrons are the charge carriers. This effect has still not been conclusively observed for electrons.

Tolman (1913-1926) [23]-[27] followed the suggestion of Maxwell in Art. 574 of [11] that if a current-carrying coil is mounted as a torsion pendulum, and the current is suddenly stopped, a small, transient deflection of the pendulum should be observable due to the torque associated with disappearance of the mechanical angular momentum of the current. This effect was observed, although not very accurately.[7]

The converse effect, also considered by Maxwell in Art. 574 of [11], that a (small) torque would be required to generate the mechanical angular momentum of a loop of electric current as the current increases, was finally observed by Barnett (1931) [29]. This was part of a larger effort [30] to measure the gyromagnetic ratio of the electron, $\Gamma = \mu/L$, where μ is the magnetic moment and L is the angular momentum. A classical view of an electron of charge e and mass m moving in a circle of radius r with velocity v is that $\mu = IA = (ev/2\pi r)(\pi r^2) = (e/2m)(mvr) = (e/2m)L$, such that $\Gamma = e/2m$, as perhaps first clearly noted in [31, 32]. The first reported measurement of Γ, by Einstein and de Haas (1915) [31], claimed the result to be the classical value, while Barnett (1915) [32] reported twice that, as eventually confirmed by him and others.

4 Current Loop (added Oct. 19, 2020)

We consider also the case of a loop (torus) of steady electric current that is at rest in the (inertial) lab frame. The torus has major radius a and, say, a rectangular cross section of width w in r and height h in z, where $w \ll a$. We use a cylindrical coordinate system (r, ϕ, z) with origin at the center of the torus and the z-axis also that of the torus.

As in sec. 2 above, centripetal force is only relevant for a resistive loop where the mean free path of conduction electrons is of the order of their typical spacing.

For a such resistive loop, in the approximation that the battery, of voltage V, which drives the electric current is very thin in azimuth, the azimuthal electric field inside the torus varies as,

$$E_\phi(r) = \frac{V}{2\pi r}, . \tag{6}$$

for some constant k.

[7] A later experiment by Kettering and Scott (1944) [28] reported more significant results.

The (azimuthal) current density inside the loop, of electrical conductivity ς, has the form,

$$J(r) = \varsigma E_\phi(r) = \frac{\varsigma V}{2\pi r} = v(r)\rho_-(r) \approx -en_0 v(r), \qquad v(r) \approx -\frac{\varsigma V}{2\pi e n_0 r}\,, \tag{7}$$

where v is the average (azimuthal) drift velocity of the conduction electrons (with charge density $\rho_- \approx -\rho_+ = -en_0$). This contrasts with the linear variation of v with radius for the case of a rotating conductor (with no applied electric field).

The centripetal force on a conduction electron, due to the radial electric field E_r, must have the form,

$$F_r = -eE_r = -\frac{mv^2}{r} = -\frac{\varsigma^2 m V^2}{4\pi^2 e^2 n_0^2 r^3}\,, \qquad E_r = \frac{\varsigma^2 m V^2}{4\pi^2 e^3 n_0^2 r^3}\,. \tag{8}$$

Then, from the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho$ we find, with initial neglect of the small spatial variation of $\rho_- \approx -en_0 = -\rho_+$.

$$\rho = \rho_- + en_0 \approx -\frac{\varsigma^2 m V^2}{8\pi^3 e^3 n_0^2 r^4}\,, \qquad \rho_- \approx -en_0 \left(1 + \frac{\varsigma^2 m V^2}{8\pi^3 n_0^3 e^4 r^4}\right). \tag{9}$$

Here, the negative charge density ρ_- is (slightly) larger in magnitude than for the case of zero current. Charge conservation then implies a small increase in the (positive) surface charge density on the loop, which is, however, much smaller than the surface charge density required to shape the electric field E_ϕ inside the to loop.[8]

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