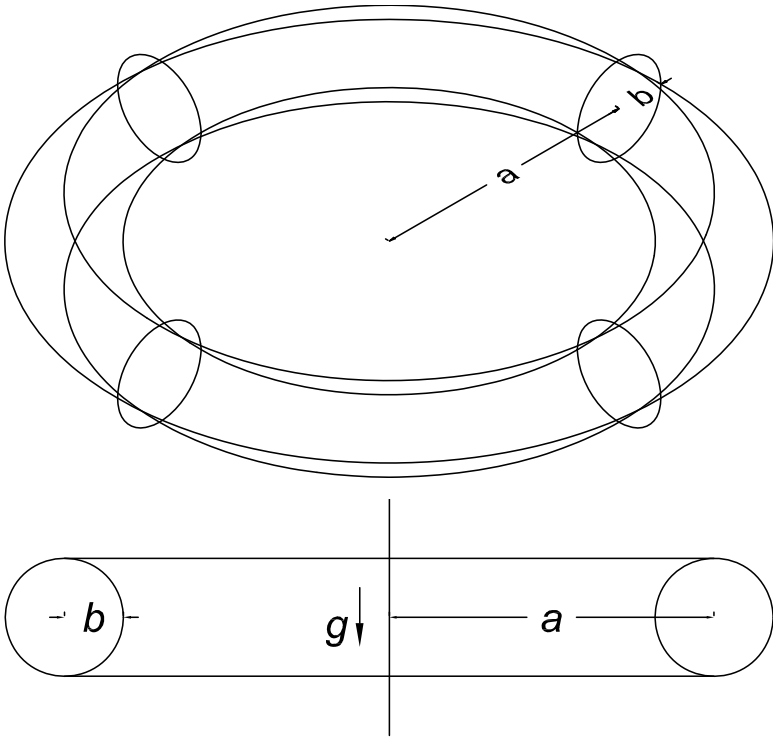
Motion on a Torus

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(October 21, 2000)

1 Problem

Find the frequency of small oscillations about uniform circular motion of a point mass that is constrained to move on the surface of a torus (donut) of major radius a and minor radius b whose axis is vertical.[1]



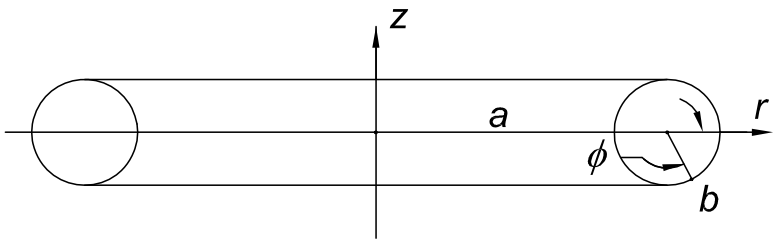
2 Solution

2.1 Attempt at a Quick Solution

Circular orbits are possible in both horizontal and vertical planes, but in the presence of gravity, motion in vertical orbits will be at a nonuniform velocity. Hence, we restrict our attention to orbits in horizontal planes.

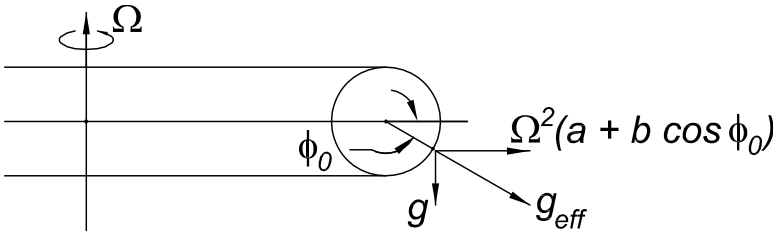
We use a cylindrical coordinate system (r, θ, z), with the origin at the center of the torus and the z axis vertically upwards, as shown below.

[1] That is, there exists gravity, whose direction is perpendicular to the midplane of the torus.



A point on the surface of the torus can also be described by two angular coordinates, one of which is the azimuth θ in the cylindrical coordinate system. The other angle we define as ϕ measured with respect to the plane $z = 0$ in a vertical plane that contains the point as well as the axis, as also shown in the figure above.

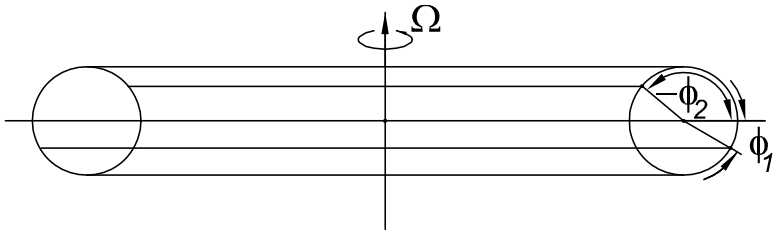
We seek motion at constant angular velocity Ω about the z axis, which suggests that we consider a frame that rotates with this angular velocity. In this frame, the particle (whose mass we take to be unity) is at rest at angle ϕ_0, and is subject to the downward force of gravity g and the outward centrifugal force $\Omega^2(a + b\cos\phi_0)$, as shown in the figure below.



The resultant force, which we call $g_{\rm eff}$, must be perpendicular to the surface of the torus. Hence, the angle ϕ_0 of the steady circular orbits must obey,

$$\tan\phi_0 = \frac{g}{\Omega^2(a + b\cos\phi_0)}. \tag{1}$$

Since the right hand side of eq. (1) is positive, we see that there are two solutions, one at angle ϕ_1 in the first quadrant, and another at angle ϕ_2 in the third quadrant, as shown in the figure below.



It seems "obvious" that only the motion at angle ϕ_1 in the first quadrant is stable, in the sense of supporting small oscillations about the steady motion when perturbed slightly.

It is tempting to analyze these oscillations in the rotating frame as simple pendulum motion subject to an effective gravity $g_{\rm eff}$. This would imply that the frequency ω of the small oscillations is,

$$\omega = \sqrt{\frac{g_{\rm eff}}{b}} = \sqrt{\frac{\sqrt{g^2 + \Omega^4(a + b\cos\phi_0)^2}}{b}}. \tag{2}$$

However, during such simple pendulum motion, the particle would have a velocity $\mathbf{v}$ with a component perpendicular to the z axis, the axis of rotation, and so there exists a Coriolis

force $-2\Omega\hat{\mathbf{z}} \times \mathbf{v}$. This Coriolis force is perpendicular to the plane of the assumed simple pendulum motion, and therefore inconsistent with that assumption.

While one could pursue the solution including the Coriolis force, it is also appropriate to use a Lagrangian approach.

2.2 Solution via the Lagrangian

We take θ and ϕ as the two independent coordinates. The r and z coordinates of the particle are given in terms of θ and ϕ as,

$$r = a + b\cos\phi, \qquad z = -b\sin\phi. \tag{3}$$

Hence, the components of the velocity in cylindrical coordinates are,

$$v_r = -b\,\dot{\phi}\sin\phi, \qquad v_\theta = r\dot{\theta}, \qquad v_z = -b\,\dot{\phi}\cos\phi. \tag{4}$$

The kinetic energy of the unit-mass particle is,

$$T = \frac{1}{2}[(a + b\cos\phi)^2\dot{\theta}^2 + b^2\dot{\phi}^2], \tag{5}$$

and the potential energy is,

$$V = -b\,g\sin\phi. \tag{6}$$

The Lagrangian $\mathcal{L} = T - V$ does not depend on θ, so the angular momentum,

$$L_z = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = (a + b\cos\phi)^2\dot{\theta} \tag{7}$$

is a constant of the motion.[2]

The ϕ equation of motion is given by,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \frac{\partial\mathcal{L}}{\partial\phi} = b^2\ddot{\phi} = -b\,(a + b\cos\phi)\dot{\theta}^2\sin\phi + b\,g\cos\phi. \tag{8}$$

For steady motion in a horizontal circle at angle ϕ_0, $\dot{\theta} \equiv \Omega$, and eq. (8) yields the condition (1).

We consider the possibility of small oscillations about this steady motion of the form,

$$\phi = \phi_0 + \epsilon\cos\omega t, \qquad \dot{\theta} = \Omega + \delta\cos\omega t, \tag{9}$$

where ϵ and δ are small constants. To use these in the equations of motion (7)-(8) we need the relations,

$$\sin\phi \approx \sin\phi_0 + \epsilon\cos\phi_0\cos\omega t, \tag{10}$$

$$\cos\phi \approx \cos\phi_0 - \epsilon\sin\phi_0\cos\omega t, \tag{11}$$

$$\dot{\theta}^2 \approx \Omega^2 + 2\delta\,\Omega\cos\omega t, \tag{12}$$

[2]The energy $E = T + V$ is also a constant of the motion, but we will not use it in this analysis.

which are accurate to first order in ϵ and δ. Inserting these in eq. (6), we find that,

$$\begin{aligned} L_z &= (a + b\cos\phi_0 - b\,\epsilon\sin\phi_0\cos\omega t)(\Omega + \delta\cos\omega t) \\ &\approx (a + b\cos\phi_0)^2\Omega + \delta(a + b\cos\phi_0)^2\cos\omega t - 2\epsilon b\,\Omega(a + b\cos\phi_0)\cos\omega t, \end{aligned} \tag{13}$$

and equating the first-order terms in $\cos\omega t$ we learn that,

$$\delta = \frac{2\epsilon\, b\,\Omega\sin\phi_0}{a + b\cos\phi_0}. \tag{14}$$

Similarly, eq. (8) becomes,

$$\begin{aligned} -\epsilon b\,\omega^2\cos\omega t \;\approx\; & -(a + b\cos\phi_0 - b\,\epsilon\sin\phi_0\cos\omega t)(\Omega^2 + 2\delta\,\Omega\cos\omega t)(\sin\phi_0 + \epsilon\cos\phi_0\cos\omega t) \\ & + g(\sin\phi_0 + \epsilon\cos\phi_0\cos\omega t). \end{aligned} \tag{15}$$

Equating the first-order terms in $\cos\omega t$, we find,

$$\begin{aligned} b\,\omega^2 &= -b\,\Omega^2\sin^2\phi_0 + 2\frac{\delta}{\epsilon}\Omega(a + b\cos\phi_0)\sin\phi_0 + \Omega^2(a + b\cos\phi_0)\cos\phi_0 + g\sin\phi_0 \\ &= 3b\,\Omega^2\sin^2\phi_0 + \Omega^2(a + b\cos\phi_0)\cos\phi_0 + g\sin\phi_0 \\ &= 3b\,\Omega^2\sin^2\phi_0 + \frac{\Omega^2(a + b\cos\phi_0)}{\cos\phi_0} = 3b\,\Omega^2\sin^2\phi_0 + \frac{g}{\sin\phi_0}, \end{aligned} \tag{16}$$

where the second line follows using eq. (14), and the third line is obtained using eq. (1).

Thus, ω^2 is positive and the motion is stable for $\phi_0 = \phi_1$ in the first quadrant. For $\phi_0 = \phi_2$ in the third quadrant, the second term of the third line of eq (16) is negative, but it is not so "obvious" that ω^2 is negative and that the motion is unstable.

In the limit that $\Omega = 0$, then $\phi_0 \to 90^\circ$ and the particle rests on the bottom of the torus. The above analysis then gives $\omega = \sqrt{g/b}$ as for a simple pendulum of length b. This limit is formally stable against perturbation that conserve angular momentum, which is zero. However, a perturbation that results in a nonzero angular momentum results in (perturbed) circular motion that carries the point far from its original position.

In the limit that $g = 0$, $\phi_0 = 0$ and $\omega = \Omega\sqrt{1 + a/b}$.

In each of these limits, ω takes on the value obtained in the shortcut analysis that neglected the Coriolis force.