Electromagnetic Plane Wave Expansion of Transition Radiation

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1 Problem

Analyze the electromagnetic fields of a charge q that moves with speed $v < c$, where c is the speed of light in vacuum, with velocity $\mathbf{v}$ perpendicular to the interface between media of indices of refraction n_1 and n_2. The accelerated motion of the bound charge distributions in these media leads to the phenomenon of transition radiation, first described (but not named) in sec. 7 of [1] for the case of conducting media.

Use the decomposition of fields into electromagnetic plane waves (with time dependence $e^{-i\omega t}$) advocated by Booker and Clemmow [2, 3, 4],[1]

$$\mathbf{B}_\omega(\mathbf{x}) = -\frac{4\pi^2\mu}{c}\int\int \frac{\mathbf{k}^\pm \times \mathbf{J}_{\omega,\mathbf{k}^\pm}}{k_z}\, e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\, dk_x\, dk_y\,, \tag{2}$$

$$\mathbf{E}_\omega(\mathbf{x}) = \frac{4\pi^2\mu}{c\,kn}\int\int \frac{\mathbf{k}^\pm \times (\mathbf{k}^\pm \times \mathbf{J}_{\omega,\mathbf{k}^\pm})}{k_z}\, e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\, dk_x\, dk_y\,, \tag{3}$$

in Gaussian units, where the $\pm$ sign holds for $z \gtrless 0$, μ is the relative permeability at $\mathbf{x}$, and

$$k = \frac{\omega c}{n}\,, \tag{4}$$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} = \begin{cases} \sqrt{k^2 - k_x^2 - k_y^2} & \text{if } k_x^2 + k_y^2 \le k^2, \\ i\sqrt{k_x^2 + k_y^2 - k^2} & \text{if } k_x^2 + k_y^2 > k^2, \end{cases} \tag{5}$$

and[2]

$$\mathbf{k}^\pm = \begin{cases} (k_x, k_y, k_z) & \text{if } z \ge 0, \\ (k_x, k_y, -k_z) & \text{if } z < 0. \end{cases} \tag{6}$$

The plane waves are homogeneous when $k_x^2 + k_y^2 \le k^2$, but they are inhomogeneous (evanescent, and significant only close to the plane $z = 0$) otherwise. The plane-wave decomposition (2)-(3) is not spherically symmetric, which is a reminder that all plane waves (and especially evanescent plane waves = "classical virtual photons") are convenient mathematical fictions, rather than entities with crisp physical reality.

[1] This method builds on the spirit of Weyl's representation [5] of a scalar spherical wave in terms of scalar plane waves,

$$\frac{e^{ikr}}{r} = \frac{i}{2\pi}\int\int \frac{e^{i(k_x x + k_y y + k_z |z|)}}{k_z}\, dk_x\, dk_y = \frac{i}{2\pi}\int\int \frac{e^{i\mathbf{k}^\pm\cdot\mathbf{x}}}{k_z}\, dk_x\, dk_y, \tag{1}$$

[2] The notation $\mathbf{k}^\pm$ follows [6].

2 Solution

We take the charge q to move along the x-axis and the interface between the two media to be the plane $x = 0$, with index n' for $x < 0$ and n for $x > 0$. For simplicity, we suppose that $\mu = 1$ in both media.

The solution parallels that of sec. 2.7 of [4], taking into account the different indices of refraction in the two half spaces. The extra complexity here is that the Fourier transform $\mathbf{J}_{\omega,\mathbf{k}}$ of the current density $\mathbf{J}(\mathbf{x})$ involves plane waves for which the wave vector $\mathbf{k}(x > 0)$ is different from $\mathbf{k}'(x < 0)$. Recall that continuity of a plane wave $e^{i\mathbf{x}\cdot\mathbf{x}}$ at the interface $x = 0$ requires that

$$k'_y = k_y, \qquad k'_z = k_z, \tag{7}$$

and recalling eq. (4),

$$k'_x = \sqrt{k'^2 - {k'_y}^2 - {k'_z}^2} = \sqrt{{k'_x}^2 + k'^2 - k^2} = \sqrt{k_x^2 + (n'^2 - n^2)\omega^2/c^2}. \tag{8}$$

The current density $\mathbf{J}$ can be written as

$$\mathbf{J} = qv\,\delta(x - vt)\,\delta(y)\,\delta(z)\,\hat{\mathbf{x}}. \tag{9}$$

Then, its temporal Fourier transform is,

$$\mathbf{J}_\omega = \frac{q\,\hat{\mathbf{x}}}{2\pi}\,\delta(y)\,\delta(z)\int \delta(x - vt)e^{i\omega t}\,v\,dt = \frac{q\,e^{i\omega x/v}\,\hat{\mathbf{x}}}{2\pi}\,\delta(y)\,\delta(z). \tag{10}$$

We will deduce the fields only in the region $x > 0$, where the plane waves have wave vector $\mathbf{k}$. Here, the spatial Fourier transform of $\mathbf{J}_\omega$ can be written as

$$\begin{aligned}\mathbf{J}_{\omega,\mathbf{k}} &= \frac{q\,\hat{\mathbf{x}}}{(2\pi)^4}\int\!\!\int\!\!\int e^{i\omega x/v}\,\delta(y)\,\delta(z)\,e^{-i\mathbf{k}\cdot\mathbf{x}}d^3\mathbf{x} \\ &= \frac{q\,\hat{\mathbf{x}}}{(2\pi)^4}\left(\int_{-\infty}^{0} e^{-i(k'_x-\omega/v)x}\,dx + \int_0^\infty e^{-i(k_x-\omega/v)\,x}\,dx\right).\end{aligned} \tag{11}$$

Now,

$$\begin{aligned}\int_0^\infty e^{-iax}\,dx &= \int_0^\infty \cos ax\,dx - i\int_0^\infty \sin ax\,dx \qquad (12)\\ &= \int_0^\infty \cos ax\,dx - i\int_{\pi/2a}^\infty \sin ax\,dx - i\int_0^{\pi/2a} \sin ax\,dx \\ &= (1-i)\int_0^\infty \cos ax\,dx - \frac{i}{a} = \frac{1-i}{2}\int_{-\infty}^\infty \cos ax\,dx - \frac{i}{a} = \pi(1-i)\delta(a) - \frac{i}{a}\,,\end{aligned}$$

noting that $\delta(a) = \int e^{-iax}\,dx/2\pi = \int \cos ax\,dx/2\pi$, and similarly,

$$\begin{aligned}\int_{-\infty}^0 e^{-ibx}\,dx &= \int_{-\infty}^0 \cos bx\,dx - i\int_{-\infty}^0 \sin bx\,dx \qquad (13)\\ &= \int_{-\infty}^0 \cos bx\,dx - i\int_{-\infty}^{\pi/2b} \sin ax + i\int_0^{\pi/2b} \sin ax\,dx \\ &= (1+i)\int_{-\infty}^0 \cos bx\,dx + \frac{i}{b} = \frac{1+i}{2}\int_{-\infty}^\infty \cos bx\,dx + \frac{i}{b} = \pi(1+i)\delta(b) + \frac{i}{b}\,.\end{aligned}$$

Thus,

$$\mathbf{J}_{\omega,\mathbf{k}} = \frac{q\,\hat{\mathbf{x}}}{(2\pi)^4}\left\{\pi\left[(1-i)\delta(k_x-\omega/v)+(1+i)\delta(k'_x-\omega/v)\right]-\frac{i}{k_x-\omega/v}+\frac{i}{k'_x-\omega/v}\right\} \tag{14}$$

Using this in eq. (3), the temporal Fourier components of the magnetic field for $x > 0$ are given by

$$\begin{aligned}\mathbf{B}_\omega(\mathbf{x}) &= -\frac{4\pi^2}{c}\int\int J_{\omega,\mathbf{k}}\frac{\mathbf{k}^\pm\times\hat{\mathbf{x}}}{k_z}\,e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\,dk_x\,dk_y \\ &= -\frac{(1-i)q}{4\pi c}\int\int\delta(k_x-\omega/v)\frac{\pm k_z\,\hat{\mathbf{y}}-k_y\,\hat{\mathbf{z}}}{k_z}\,e^{i(k_x x+k_y y\pm k_z z)}\,dk_x\,dk_y \\ &\quad -\frac{(1+i)q}{4\pi c}\int\int\delta(k'_x-\omega/v)\frac{\pm k_z\,\hat{\mathbf{y}}-k_y\,\hat{\mathbf{z}}}{k_z}\,e^{i(k_x x+k_y y\pm k_z z)}\,dk_x\,dk_y \\ &\quad +\frac{iq}{4\pi^2 c}\int\int\left(\frac{1}{k_x-\omega/v}-\frac{1}{k'_x-\omega/v}\right)\frac{\pm k_z\,\hat{\mathbf{y}}-k_y\,\hat{\mathbf{z}}}{k_z}\,e^{i(k_x x+k_y y\pm k_z z)}\,dk_x\,dk_y\end{aligned} \tag{15}$$

The term in eq. (15) proportional to $1-i$ is of the form of eq. (49) of [4], and correspond to the nonradiative field of the moving charge when $v < c/n$ and to its Čerenkov radiation when $v > c/n$. Similarly, the term in eq. (15) proportional to $1+i$ corresponds to the nonradiative field of the image charge[3] $q' = -q(\epsilon-1)/(\epsilon+1)$ when $v < c/n'$ and to the Čerenkov radiation of the charge q when $v > c/n'$ and the charge was at negative x. The last term in eq. (15) can have real $k_z = \sqrt{k^2-k_x^2-k_y^2} = \sqrt{\omega^2/n^2c^2-k_x^2-k_y^2}$ and therefore represents an additional form of radiation associated with the moving charge, which exists because of the transition between the different media at negative and positive x.

The field $\mathbf{B}$ is axially symmetric, and azimuthal, with respect to the axis of motion, which is the x-axis here. The azimuthal field $B_{\omega,\phi}(\mathbf{x})$ (about the x-axis) at distance $r_\perp$ from the x-axis can be evaluated as $-B_{\omega,\phi}(x,0,r_\perp)$ using eq. (15), In particular, the azimuthal magnetic field of the transition radiation is

$$B_{TR,\omega,\phi} = \frac{iq}{4\pi^2 c}\int\int_{\sqrt{k_x^2+k_y^2}<\omega/nc}\left(\frac{1}{k_x-\omega/v}-\frac{1}{k'_x-\omega/v}\right)e^{i(k_x x+k_z r_\perp)}\,dk_x\,dk_y. \tag{16}$$

where the region of integration is such that k_z is real.

2.1 Metal-Vacuum Transition

In the rest of this note we restrict our attention to the case of vacuum for $x > 0$ and a perfect conductor for $x < 0$. This corresponds to $n = 1$ and $n' = \infty$, in which case $k'_x = \infty$ according to eq. (8).

When the speed of the charge is less than the speed of light c/n in the surrounding medium, k_z is purely imaginary according to eq. (5), and all plane waves in the expansion (2) are evanescent. No radiation (to "infinity" [21]) is emitted by a charge moving uniformly

[3]See, for example, [7].

at sublight speed. Replacing (k_x, k_y) by polar coordinates (k_ρ, θ), the magnetic field of the transition radiation is

$$\begin{aligned} B_{TR,\omega,\phi} &= \frac{iq}{4\pi^2 c}\int_0^{\omega/c} k_\rho\, dk_\rho \int_0^{2\pi} d\theta \frac{e^{i\left(k_\rho x\cos\theta + \sqrt{\omega^2/c^2 - k_\rho^2}\, r_\perp\right)}}{k_\rho\cos\theta - \omega/v} \\ &= \frac{iq}{4\pi^2 c}\int_0^{\omega/c} e^{i\sqrt{\omega^2/c^2-k_\rho^2}\, r_\perp)}\, dk_\rho \int_0^{2\pi} d\theta \frac{e^{ik_\rho x\cos\theta}}{\cos\theta - \omega/vk_\rho} \end{aligned} \tag{17}$$

The field $\mathbf{B}$ is axially symmetric, and azimuthal, with respect to the axis of motion, which is the x-axis here. The azimuthal field $B_\phi(\mathbf{x}, t)$ (about the x-axis) at distance $r_\perp$ from the x-axis can be evaluated as $-B_y(x, 0, r_\perp, t)$ using eq. (2),

$$\begin{aligned} B_\phi &= -B_y(x,0,r_\perp,t) = -2Re\int_0^\infty B_{\omega,y}(x,0,r_\perp)\, e^{-i\omega t}\, d\omega \\ &= \frac{q\mu}{\pi c^2} Re\int dl_y \int_0^\infty \omega\, d\omega\, e^{i\omega x/v} e^{-(\omega r_\perp/c)\sqrt{c^2/v^2-n^2+l_y^2}} e^{-i\omega t} \quad \text{where} \quad l_y = ck_y/\omega \\ &= \frac{q\mu}{\pi r_\perp^2} Re\int \frac{dl_y}{\left[ic(x-vt)/r_\perp v - \sqrt{c^2/v^2-n^2+l_y^2}\right]^2} \\ &= \frac{q\mu}{\pi r_\perp^2}\int \frac{-[c(x-vt)/r_\perp v]^2 + c^2/v^2 - n^2 + l_y^2}{\left\{[c(x-vt)/r_\perp v]^2 + c^2/v^2 - n^2 + l_y^2\right\}^2}\, dl_y \\ &= \frac{q\mu}{\pi r_\perp^2} 2\pi i \frac{d}{dl_y} \left.\frac{-[c(x-vt)/r_\perp v]^2 + c^2/v^2 - n^2 + l_y^2}{\left[l_y + i\sqrt{[c(x-vt)/r_\perp v]^2 + c^2/v^2 - n^2}\right]^2}\right|_{l_y = i\sqrt{[c(x-vt)/r_\perp v]^2+c^2/v^2-n^2}} \\ &= \frac{q\mu v}{c}\frac{(1-n^2v^2/c^2)r_\perp}{[(x-vt)^2 + (1-n^2v^2/c^2)r_\perp^2]^{3/2}}, \end{aligned} \tag{18}$$

where the integral in l_y was evaluated by completing the contour at $+\infty$. This is the usual result for the magnetic field of a charge moving at constant, sublight speed.

For completeness, we also calculate the axially symmetric electric field $\mathbf{E}(x, 0, r_\perp, t)$ in the x-z plane. Comparing eqs. (2) and eq. (3), we infer that

$$E_z(x,0,r_\perp,t) = -\frac{c}{\epsilon\mu v}B_y(x,0,r_\perp,t) = \frac{q}{\epsilon}\frac{(1-n^2v^2/c^2)r_\perp}{[(x-vt)^2 + (1-n^2v^2/c^2)r_\perp^2]^{3/2}}. \tag{19}$$

The y-component of Faraday's law tells us that

$$\frac{\partial E_x(x,0,r_\perp,t)}{\partial r_\perp} = \frac{\partial E_z(x,0,r_\perp,t)}{\partial x} - \frac{1}{c}\frac{\partial B_y(x,0,r_\perp,t)}{\partial t} = -\frac{3q}{\epsilon}\frac{(1-n^2v^2/c^2)^2(x-vt)r_\perp}{[(x-vt)^2 + (1-n^2v^2/c^2)r_\perp^2]^{3/2}}, \tag{20}$$

which integrates to

$$E_x(x,0,r_\perp,t) = \frac{q}{\epsilon}\frac{(1-n^2v^2/c^2)(x-vt)}{[(x-vt)^2 + (1-n^2v^2/c^2)r_\perp^2]^{3/2}}. \tag{21}$$

Thus, the electric field is radial with respect to the position of the charge, $(x - vt, 0, 0)$, and is related to the magnetic field by

$$\mathbf{E} = \frac{n\mathbf{v}}{c} \times \mathbf{B}. \tag{22}$$

2.1.1 Čerenkov Radiation: $c/n < v < c$

When a charge q moves with speed v greater than that of light, c/n, in a medium (but with $v < c$, of course), the plane-wave expansion (2) contains both homogeneous and inhomogeneous waves, and radiation is therefore emitted. This is sometimes considered to be paradoxical in that the charge is not obviously accelerating. However, the radiation exists only when the charge moves through a medium with index of refraction greater than 1, in which case the charges in the medium are accelerated by the passing charge q, and we can say that is the medium, rather than the charge itself, which emits the radiation. Of course, the radiated energy must come from the charge itself, so there must be a (small) back reaction of the medium on the passing charge, which decelerates the latter.

The temporal expansion of the magnetic field is, from eq. (2),

$$\mathbf{B}_\omega(\mathbf{x}) = -\frac{q\mu\, e^{i\omega x/v}}{2\pi c} \int \frac{\pm k_z\,\hat{\mathbf{y}} - k_y\,\hat{\mathbf{z}}}{k_z}\, e^{i(k_y y \pm k_z z)}\, dk_y, \tag{23}$$

where

$$k_x = \frac{\omega}{v} = \frac{ck}{nv} \qquad \text{and} \qquad k_z = \sqrt{k^2(1 - c^2/n^2v^2) - k_y^2}. \tag{24}$$

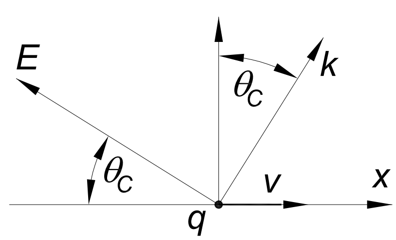
For plane waves in the x-y plane, $k_z = 0$ and $k_y = k\sqrt{1 - c^2/n^2v^2} = k_x(nv/c)\sqrt{1 - c^2/n^2v^2} = k_x\sqrt{n^2v^2/c^2 - 1}$, which is real, so these waves are homogeneous, and carry energy away from the charge q. Similarly, for plane waves in the x-z plane, $k_y = 0$ and $k_z = k_x\sqrt{n^2v^2/c^2 - 1}$. The wave vector $\mathbf{k}$ for the homogeneous waves (radiation field) does not have a continuous angular distribution, but always makes angle θ_C to the y-z plane, where

$$\tan\theta_C = \frac{k_x}{k_y(k_z = 0)} = \frac{k_x}{k_z(k_y = 0)} = \frac{1}{\sqrt{n^2v^2/c^2 - 1}}\,, \tag{25}$$

so that

$$\cos\theta_C = \frac{1}{\sqrt{1 + \tan^2\theta_C}} = \frac{c}{nv}\,. \tag{26}$$

The angle θ_C is the famous Čerenkov angle.



Since $\mathbf{k}^\pm \cdot \mathbf{E}_{\omega,\mathbf{k}^\pm} = 0$, the electric field points only in a single direction, namely at the Čerenkov angle θ_C to the negative x-axis (and the magnetic field circles about the x-axis). *This field configuration was first depicted by Heaviside [18].*

The temporal Fourier expansion of the electric field follows from eq. (3) as

$$\mathbf{E}_\omega(\mathbf{x}) = \frac{q\, e^{i\omega x/v}}{2\pi\epsilon v} \int \frac{-k_x(n^2v^2/c^2 - 1)\,\hat{\mathbf{x}} + k_y\,\hat{\mathbf{y}} \pm k_z\,\hat{\mathbf{z}}}{k_z}\, e^{i(k_y y \pm k_z z)}\, dk_y. \qquad (27)$$

The electric field in, say, the x-z plane for $z > 0$ consists of plane waves with $k_y = 0$, so we have that

$$\begin{aligned}
\mathbf{E}(x,0,z>0,t) &= 2Re\int_0^\infty \mathbf{E}_\omega(x,0,z>0)\, e^{-i\omega t}\, d\omega \\
&= \frac{q}{\pi\epsilon v}\, Re\int_0^\infty [-\tan\theta_C(n^2v^2/c^2-1)\,\hat{\mathbf{x}} + \hat{\mathbf{z}}]\, e^{i\omega[(x+z/\tan\theta_C)/v - t]}\, d\omega \\
&= -\frac{2q}{\epsilon v}\, [\tan\theta_C(n^2v^2/c^2-1)\,\hat{\mathbf{x}} - \hat{\mathbf{z}}]\, \delta\left(\frac{x+z/\tan\theta_C}{v} - t\right). \qquad (28)
\end{aligned}$$

At time $t = 0$ the electric field in the x-z plane for $z > 0$ is nonzero only along the line $z = -x\tan\theta_C$, as shown in the figure. By a similar argument the magnetic field in the z-z plane is nonzero only along this line. The electric and magnetic fields are is azimuthally symmetric, so the fields are nonzero only on the Čerenkov cone. The present argument predicts infinite fields on this cone, whereas in reality the index n exceeds unity for only a finite range of frequency, and the fields extend slightly outside the cone, and are finite.

To deduce the frequency spectrum of the radiated power, we first note that the total energy d^2U that crosses an area element $d\mathbf{Area}$, integrated over all time, is

$$\begin{aligned}
d^2U &= \int_{-\infty}^{\infty} \mathbf{S}\cdot d\mathbf{Area}\, dt = \frac{c}{4\pi\mu}\, d\mathbf{Area}\cdot\int_{-\infty}^{\infty} \mathbf{E}\times\mathbf{B}\, dt \\
&= \frac{c}{4\pi\mu}\, d\mathbf{Area}\cdot\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{E}_\omega\times\mathbf{B}\, e^{-i\omega t}\, d\omega\, dt \\
&= \frac{c}{2\mu}\, d\mathbf{Area}\cdot\int_{-\infty}^{\infty} \mathbf{E}_\omega\times\mathbf{B}_\omega^\star\, d\omega = \frac{c}{\mu}\, d\mathbf{Area}\cdot Re\int_0^\infty \mathbf{E}_\omega\times\mathbf{B}_\omega^\star\, d\omega, \qquad (29)
\end{aligned}$$

since $\mathbf{E}_\omega(-\omega) = \mathbf{E}_\omega^\star(\omega)$ and $\mathbf{B}_\omega(-\omega) = \mathbf{B}_\omega^\star(\omega)$. Equal amounts of energy cross any plane at $z > 0$ or at $z < 0$, so the total energy radiated is twice that which crosses a plane at $z > 0$,

$$U = \frac{2c}{\mu}\int\int dx\, dy\, \hat{\mathbf{z}}\cdot Re\int_0^\infty (\mathbf{E}_\omega\times\mathbf{B}_\omega^\star)_{z>0}\, d\omega. \qquad (30)$$

The energy radiated per unit frequency interval and per unit path length of the charge's motion along the x-axis is independent of x. Since $B_x = 0$, we have

$$\begin{aligned}
\frac{d^2U}{d\omega\, dx} &= \frac{2c}{\mu}\int dy\, Re(E_{\omega,x}B_{\omega,y}^\star)_{z>0} \\
&= \frac{2c}{\mu}\frac{q}{2\pi\epsilon v}\frac{q\mu}{2\pi c}\, Re\int\int\int \frac{k_x(n^2v^2/c^2-1)}{k_z}\, e^{i(k_y y + k_z z)} e^{-i(k'_y y + k'^\star_z z)}\, dy\, dk_y\, dk'_y \\
&= \frac{2c}{\mu}\frac{q\omega}{2\pi\epsilon v^2}\frac{q\mu}{2\pi c}(n^2v^2/c^2-1)\, Re\int\int 2\pi\delta(k_y - k'_y)\frac{e^{i(k_z - k'^\star_z)z}}{k_z}\, dk_y\, dk'_y
\end{aligned}$$

$$
\begin{aligned}
&= \frac{q^2\omega}{\pi\epsilon v^2}(n^2v^2/c^2 - 1)\, Re \int \frac{e^{-2Im(k_z)z}}{k_z}\, dk_y \\
&= \frac{q^2\omega n^2}{\pi\epsilon c^2}(1 - c^2/n^2v^2) \int_{-(\omega n/c)\sqrt{1-c^2/n^2v^2}}^{(\omega n/c)\sqrt{1-c^2/n^2v^2}} \frac{dk_y}{\sqrt{(\omega^2 n^2/c^2)(1 - c^2/n^2v^2) - k_y^2}} \\
&= \frac{q^2\mu\omega}{c^2}\left(1 - \frac{c^2}{n^2v^2}\right), \qquad (31)
\end{aligned}
$$

where we note that in the fourth line the integrand is real only when k_z is real. Equation (31) is the standard result for the energy spectrum of Čerenkov radiation [1], which has the surprising feature (of little practical import) that a magnetic medium of index n emits μ times as much Čerenkov radiation as does a dielectric medium of the same index. As usual, we note that the index n can be greater than unity for only a finite range of frequencies, so that the total power radiated over all frequencies is finite.

The x-component of the electric field at the charge is, using eq. (27),

$$
\begin{aligned}
E_x(vt,0,0,t) &= 2Re \int_0^\infty E_{\omega,x}(vt,0,0)\, e^{-i\omega t} d\omega \qquad (32) \\
&= -\int_0^\infty d\omega\, \frac{q\omega}{\pi\epsilon v^2}(n^2v^2/c^2 - 1)\, Re \int \frac{1}{k_z}\, dk_y = -\int_0^\infty d\omega\, \frac{q\mu\omega}{c^2}\left(1 - \frac{c^2}{n^2v^2}\right).
\end{aligned}
$$

This is a peculiar result in that we might have expected the electric field to diverge at the charge.[4] The field (32) acts opposite to the direction of the charge's velocity and decelerates it. The work done by the electron per unit path length is $-qE_x$, whose Fourier component at frequency ω equals the energy radiated per unit path length. That is, the work done by the electron on the Čerenkov field is transformed into the Čerenkov radiation.

For additional discussion of the relation of radiation by moving charges to the plane-wave decomposition of their fields, see [23].

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