Currents in a Center-Fed Linear Dipole Antenna

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544

(June 27, 2007; updated June 3, 2009)

1 Problem

Deduce the approximate form of the current in a center-fed linear dipole antenna of half height h when excited by a voltage source $V_0\, e^{i\omega t}$ across the antenna terminals, whose gap width d is small compared to h. From this, give an expression for the impedance of the antenna (whose real part is the so-called radiation resistance).

You may assume that the antenna conductors have radius a small compared to the height h, and that they are perfect conductors.

2 Solution

This problem continues the use of techniques (discussed in [1]) that are inspired by Pocklington [2], who extended the insights of Lorenz [3] and Hertz [4], that electromagnetic fields can be deduced from the retarded vector potential, by consideration of the boundary condition that the tangential component of the electric field must vanish at the surface of a good/perfect conductor. Furthermore, Pocklington noted that to a first approximation for conductors that are thin wires, the vector potential at the surface of a wire depends only on the current in the wire at that point. Pocklington deduced an integral equation for the currents in the conductors, which is the basis of numerical electromagnetic codes such as NEC4 [14]. Semi-analytic analyses are more often based on variants of Pocklington's equation, as developed by L.V. King [5], E. Hallén [6] and R.W.P. King [7]-[10]. See also [15], on which this solution is based. A different analytic method, based on expansion of the fields of a biconical antenna in modes, has been pursued by Schelkunoff [11].[1]

As for the case of a receiving antenna [1], we suppose the antenna is excited by a specified input electric field $\mathbf{E}_{\rm in}$. For a transmitting antenna this is taken to be the internal field,

$$\mathbf{E}_{\rm in} = \begin{cases} \frac{V_0}{d}\, e^{i\omega t}\, \hat{\mathbf{z}} & (\rho < a, \phi, |z| < d/2), \\ 0 & (\text{elsewhere}), \end{cases} \tag{1}$$

of an rf generator that is located in the gap of width d between the terminals of the antenna, whose conductors of radius a lie along the z axis of a cylindrical coordinate system (ρ, ϕ, z) with its origin at the center of the antenna. This internal field is the negative of the (response) field in the gap in the more realistic case that the rf generator is located some distance from the antenna and connected to it via a transmission line. The incident electric field is zero outside of the gap at the antenna terminals. In particular, it is zero elsewhere on the conductors of the antenna.

[1] As more powerful computers have become available, the so called finite-difference time domain (FDTD) method has become more practical for antenna modeling. See, for example, [12, 13].

The input electromagnetic field excites an oscillating current distribution $\mathbf{J}(\mathbf{r},t) = \mathbf{J}(\mathbf{r})\, e^{i\omega t}$ in the conductors of the receiving antenna. If this current distribution is known, then the retarded vector potential $\mathbf{A}(\mathbf{r},t) = \mathbf{A}(\mathbf{r})\, e^{i\omega t}$ of the response fields can be calculated as,

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t' = t - \mathcal{R}/c)}{\mathcal{R}}\, d\mathrm{Vol}' = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')\, \frac{e^{-ik\mathcal{R}}}{\mathcal{R}}\, d\mathrm{Vol}'\, e^{i\omega t} = \mathbf{A}(\mathbf{r})\, e^{i\omega t}, \tag{2}$$

where $\mathcal{R} = |\mathbf{r} - \mathbf{r}'|$, c is the speed of light, ω is the angular frequency, $k = \omega/c$ is the wave number, and the medium outside the conductors is vacuum (with permittivity μ_0). In the present example the conductors are thin wires along the z axis, and we suppose that the current density $\mathbf{J}(\mathbf{r})$ is independent of azimuth in a cylindrical coordinate system (ρ, ϕ, z) and is well approximated by a current $I(z)$, which is symmetric in z,

$$I(-z) = I(z), \tag{3}$$

for a symmetric, center-fed linear dipole antenna. Then, the vector potential has only a z component,

$$A_z(\mathbf{r}) = \frac{\mu_0}{4\pi}\int I(z')\, \frac{e^{-ik\mathcal{R}(z,z')}}{\mathcal{R}(z,z')}\, dz'. \tag{4}$$

We work in the Lorenz gauge [3], where,

$$\boldsymbol{\nabla}\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0, \tag{5}$$

so the scalar potential $V(\mathbf{r},t) = V(\mathbf{r})\, e^{i\omega t}$ of the response fields is related to the vector potential according to,

$$V(\mathbf{r}) = \frac{ic}{k}\frac{\partial A_z(\mathbf{r})}{\partial z} \equiv \frac{ic}{k}\partial_z A_z(\mathbf{r}). \tag{6}$$

The response fields $\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r})\, e^{i\omega t}$ and $\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r})\, e^{i\omega t}$ can then be calculated from the vector potential $A_z(\mathbf{r})$ as,

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}V(\mathbf{r}) - i\omega\mathbf{A}(\mathbf{r}) = -\frac{ic}{k}[\partial^2_{rz}A_z(\mathbf{r})\,\hat{\boldsymbol{\rho}} + (\partial^2_z + k^2)A_z(\mathbf{r})\,\hat{\mathbf{z}}], \tag{7}$$

$$\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r}) = -\partial_\rho A_z(\mathbf{r})\,\hat{\boldsymbol{\phi}}. \tag{8}$$

2.1 The Thin-Wire Approximation

The key relation between the input electric field $\mathbf{E}_{\rm in}$ and the response field $\mathbf{E}$ is that the tangential component of the total electric field $\mathbf{E}_{\rm in} + \mathbf{E}$ must vanish at the surface of the conductors. In the thin-wire approximation, the only tangential component of interest is the z component, and for wire radius a much less that the antenna half height h, the constraint is essentially on the electric field on the z axis,

$$E_z(0,0,z) = -E_{\rm in} = 0 \qquad (d/2 < |z| < h), \tag{9}$$

since the input field (1) vanishes outside the gap between the terminals of the antenna. We also know that the response field is the negative of the input field inside the (narrow) gap, so we can write,

$$E_z(0,0,z) \approx -V_0\delta(z) \qquad (|z| < h). \tag{10}$$

From eq. (7), we obtain a differential equation for the vector potential,

$$(\partial_z^2 + k^2)A_z(0,0,z) = -\frac{ik}{c}E_{\rm in}(0,0,z) = \frac{ik}{c}E_z(0,0,z) \approx -\frac{ik}{c}V_0\delta(z) \qquad (|z| < h). \qquad (11)$$

Two solutions to the homogeneous differential equation $(\partial_z^2+k^2)A_z(0,0,z) = 0$ are, of course, $\cos kz$ and $\sin kz$. Since the righthand side of eq. (11) is symmetric in z, the vector potential will be also, and the function $\sin kz$ does not appear in it. A solution to the particular equation is $-iV_0 \sin k\,|z|\,/2c$, noting that $\int_{-\epsilon}^{\epsilon} \partial_z^2 \sin k\,|z|\; dz = \partial_z \sin k\,|z|\,|_{-\epsilon}^{\epsilon} = 2k$. Hence, a general solution to eq. (11) for $|z| < h$ can be written as,

$$A_z(0,0,z) \approx -\frac{i}{2c}(C\cos kz + V_0 \sin k\,|z|) \qquad (|z| < h). \qquad (12)$$

To evaluate the constant of integration C we need an additional condition on the system. In particular, we note that the current $I(z)$ must vanish at the ends of the antenna, $z = \pm h$. In the thin-wire approximation, the vector potential on the wire is proportional to the current in the wire at that point, because of the $1/\mathcal{R}$ dependence in eq. (4). In this approximation, the needed condition on the vector potential is that it also vanishes at the ends of the conductors. From this we find,

$$C = -V_0 \frac{\sin kh}{\cos kh}\,, \qquad (13)$$

and so the vector potential along the antenna is,

$$A_z(0,0,z) \approx \frac{iV_0}{2c}\frac{\sin kh \cos kz - \cos kh \sin k\,|z|}{\cos kh} = \frac{iV_0}{2c}\frac{\sin[k(h - |z|)]}{\cos kh} \qquad (|z| < h). \qquad (14)$$

Since the current is proportional to the vector potential on the axis in this approximation, we have,

$$I(z) \approx I_0 \frac{\sin[k(h - |z|)]}{\sin kh} \qquad (|z| < h), \qquad (15)$$

where $I_0 = I(0)$ is the current at the terminals of the antenna.

Thus, the thin-wire approximation (first advocated by Pocklington [2]) leads to the sinusoidal-current approximation, which latter approximation is often invoked without as detailed a justification as given above.[2]

The electromagnetic fields can now be calculated analytically everywhere from the current distribution (15). See, for example, [16]. However, the tangential component of the electric field so calculated does not vanish along the conductors of the antenna. The thin-wire approximation is not consistent with the perfect-conductor boundary condition on the electric field.

The fields calculated in the far zone of the antenna from eq. (15) are quite accurate, but the fields calculated in the near zone are badly misestimated. A consequence is that the

[2]See, for example, sec. 9.4A of [18]. There it is also implied that radiation damping affects the current distributions. However, the oscillating charges on the surface of perfect conductors emit no net radiation, and experience no radiation damping/reaction force. Only currents in the power source experience radiation damping, which effect appears in the analysis as the radiation resistance of the antenna that is perceived by the power source.

reactance of the antenna at its terminals is not well determined in the thin-wire approximation. The thin-wire approximation to eq. (4) is that $A_z(0,0,z) \approx \mu_0 I(z)/4\pi$, so that eq. (14) evaluated at $z = 0$ implies that the current (15) is 90° out of phase with the drive voltage. Thus, writing $V_0 = I_0 Z$ we find $Z \approx -iZ_0/2\pi \tan kh$, where $Z_0 = \sqrt{\mu_0/\epsilon_0} = 377\ \Omega$, as also found for a receiving antenna in the thin-wire approximation [1]. This correctly indicates that the reactance of a short linear antenna is capacitive and that the reactance vanishes for $h \approx \lambda/2$, but the result differs from that of a simple model (see the Appendix) by the absence of a factor of $\ln(h/a)$. Furthermore, the real part of the current, and also of the impedance, is neglected in the thin-wire approximation, so the antenna does not appear to consume any energy from the rf power source.[3]

2.2 Solution via Pocklington's Integral Equation

If we do not assume that the vector potential at the conductors is proportional to the currents they carry, we can proceed by combining equations (4) and (11) into an integral equation that relates the incident electric field at the conductors to the response currents in those conductors (noting that ∂_z acts only on $\mathcal{R}(z,z')$),

$$\int_{-h}^{h} I(z')(\partial_z^2 + k^2)\,\frac{e^{-ik\mathcal{R}(z,z')}}{-ik\mathcal{R}(z,z')}\,dz' = \frac{4\pi}{Z_0}E_{\rm in}(z) \qquad (|z| < h). \tag{16}$$

This is Pocklington's integral equation [2], whose solution is implemented numerically in codes such as NEC4 [14]. See also [15]. The integral equation (16) is readily generalized to any case where the conductors of the antenna are piecewise linear, *i.e.*, to essentially all cases of practical interest. Some comments on technical difficulties in numerical solutions to these integral equations are given in [17].

The nonzero radius a of the conductors can be taken into account by using,

$$\mathcal{R}(z,z') = \sqrt{(z-z')^2 + a^2}, \tag{17}$$

rather than $\mathcal{R} = |z - z'|$.

Recalling from eqs. (9)-(10) that the input electric field for a thin, center-fed dipole antenna can be written $E_{\rm in}(z) \approx V_0\delta(z)$, Pocklington's integral equation (16) is, for $|z| < h$,

$$\begin{aligned}
\frac{4\pi}{Z_0}V_0\,\delta(z) &\approx \int_{-h}^{h} I(z')(k^2 + \partial_z^2)\,\frac{e^{-ik\mathcal{R}}}{-ik\mathcal{R}}\,dz' = \int_{-h}^{h} I(z')(k^2 + \partial_{z'}^2)\,\frac{e^{-ik\mathcal{R}}}{-ik\mathcal{R}}\,dz' \\
&= \int_{-h}^{h} [k^2 I(z') - I'(z')\partial_{z'}]\,\frac{e^{-ik\mathcal{R}}}{-ik\mathcal{R}}\,dz' \\
&= \int_{-h}^{h} [k^2 I(z') + I''(z')]\,\frac{e^{-ik\mathcal{R}}}{-ik\mathcal{R}}\,dz' - I'(h)\left(\frac{e^{-ik\mathcal{R}(z,h)}}{-ik\mathcal{R}(z,h)} + \frac{e^{-ik\mathcal{R}(z,-h)}}{-ik\mathcal{R}(z,-h)}\right),
\end{aligned} \tag{18}$$

[3] The real part of the antenna impedance, the so-called radiation resistance $R_{\rm rad}$, can be well calculated from the far-zone fields generated by the current distribution of eq. (15) using the relation $\langle P\rangle = I_0^2 R_{\rm rad}/2$, where $\langle P\rangle$ is the time-average radiated power, with the result $R_{\rm rad} = Z_0(kh)^2/6\pi = 20(kh)^2$ Ohms. See, for example, sec. 2.6 of [16].

where we note that $\partial_z^2 f[\mathcal{R}(z,z')] = \partial_{z'}^2 f[\mathcal{R}]$, so that we can integrate twice by parts, using the boundary condition that $I(\pm h) = 0$ and the fact $dI/dz \equiv I'(z) = -I'(-z)$ according to the symmetry condition (3). For a short linear antenna ($kh \ll 1$) eq. (18) can be approximated as,

$$\begin{aligned}
\frac{4\pi}{Z_0} V_0\, \delta(z) &\approx \int_{-h}^{h} [k^2 I(z') + I''(z')] \left(1 + \frac{i}{k\mathcal{R}(z,z')}\right) dz' - I'(h) \left(2 + \frac{i}{k\mathcal{R}(z,h)} + \frac{i}{k\mathcal{R}(z,-h)}\right) \\
&= \int_{-h}^{h} \left[k^2 I(z') + i\frac{k^2 I(z') + I''(z')}{k\mathcal{R}(z,z')}\right] dz' - \frac{iI'(h)}{k} \left(\frac{1}{\mathcal{R}(z,h)} + \frac{1}{\mathcal{R}(z,-h)}\right). \qquad (19)
\end{aligned}$$

In the thin-wire approximation, the current distribution in a short linear antenna follows from eq. (15) as,

$$I(z) = \begin{cases} I_0 \left(1 - \frac{|z|}{h}\right) & (|z| < h), \\ 0 & (|z| > h), \end{cases} \qquad (20)$$

whose derivatives are,

$$I'(z) = \frac{I_0}{h} \begin{cases} 1 & (-h < z < 0), \\ -1 & (0 < z < h), \\ 0 & (|z| > h), \end{cases} \qquad (21)$$

and,

$$I''(z) = -\frac{2I_0}{h}\delta(z) + \frac{I_0}{h}[\delta(z-h) + \delta(z+h)]. \qquad (22)$$

To use the forms (20)-(22) in eq. (19), we first note that,

$$\begin{aligned}
\int_{-h}^{h} \frac{I(z')}{\mathcal{R}(z,z')}\, dz' &= I_0 \int_{-h}^{h} \left(1 - \frac{|z'|}{h}\right) \frac{dz'}{\sqrt{(z'-z)^2 + a^2}} = I_0 \int_{-h-z}^{h-z} \left(1 - \frac{|z+x|}{h}\right) \frac{dx}{\sqrt{x^2+a^2}} \\
&= I_0 \int_{-h-z}^{h-z} \frac{dx}{\sqrt{x^2+a^2}} + \frac{I_0}{h} \int_{-h-z}^{-z} \frac{z+x}{\sqrt{x^2+a^2}}\, dx - \frac{I_0}{h} \int_{-z}^{h-z} \frac{z+x}{\sqrt{x^2+a^2}}\, dx \\
&= I_0 \ln \frac{\sqrt{(h-z)^2+a^2} + h - z}{\sqrt{(h+z)^2+a^2} - (h+z)} \\
&\quad + \frac{I_0 z}{h} \ln \frac{(\sqrt{z^2+a^2} - z)^2}{\left(\sqrt{(h+z)^2+a^2} - (h+z)\right)\left(\sqrt{(h-z)^2+a^2} + h - z\right)} \\
&\quad + \frac{I_0}{h} \left(2\sqrt{z^2+a^2} - \sqrt{(h+z)^2+a^2} - \sqrt{(h-z)^2+a^2}\right). \qquad (23)
\end{aligned}$$

Then, we find,

$$\frac{4\pi}{Z_0} V_0\, \delta(z) \approx k^2 h I_0 + ik \int_{-h}^{h} \frac{I(z')}{\mathcal{R}(z,z')}\, dz' - \frac{2iI_0}{kh} \left(\frac{1}{\mathcal{R}(z,0)} - \frac{1}{\mathcal{R}(z,h)} - \frac{1}{\mathcal{R}(z,-h)}\right). \qquad (24)$$

Of the various terms in eqs. (23)-(24), the one most like the delta-function $\delta(z)$ is $1/\mathcal{R}(z,0)$. So, we make the somewhat bold approximation of ignoring all other imaginary terms in eq. (24), and we write it as,

$$\frac{4\pi}{Z_0} V_0 \, \delta(z) \approx k^2 h I_0 - \frac{2iI_0}{kh\mathcal{R}(z,0)} \qquad (|z| < h). \tag{25}$$

Next, we integrate from $-h$ to h to evaluate the parameter I_0,

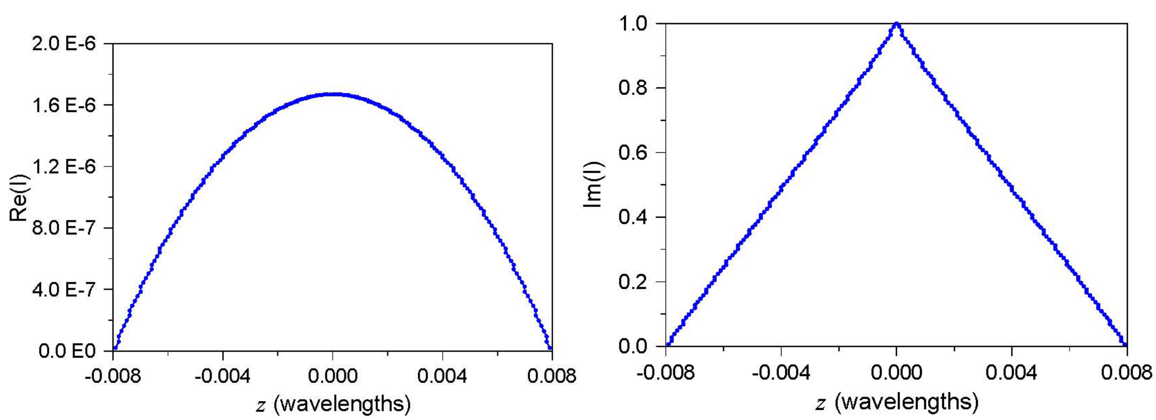
$$\begin{aligned} \frac{4\pi}{Z_0} V_0 &\approx 2k^2h^2 I_0 - \frac{2iI_0}{kh} \int_{-h}^{h} \frac{dz}{\sqrt{z^2+a^2}} = 2k^2h^2 I_0 - \frac{2iI_0}{kh} \ln \frac{\sqrt{h^2+a^2}+h}{\sqrt{h^2+a^2}-h} \\ &\approx 2k^2h^2 I_0 - \frac{4iI_0}{kh} \ln \frac{h}{a} \,. \end{aligned} \tag{26}$$

The terminal impedance Z of the antenna is then given by,

$$Z = \frac{V_0}{I_0} \approx \frac{Z_0 k^2 h^2}{2\pi} - \frac{iZ_0}{\pi kh} \ln \frac{h}{a} \,. \tag{27}$$

The solution to Pocklington's integral equation (16) based on the simple form (20) of the current distribution gives an estimate of $Z_0(kh)^2/2\pi$ for the (very small) radiation resistance of a short, linear dipole antenna, which result is a factor of three larger than nominal. Further, the estimate $-Z_0 \ln(h/a)/\pi kh$ of the (capacitive) reactance equals the value found in the Appendix by an electrostatic calculation.

For comparison, a NEC4 [14] calculation of a short linear antenna with $kh = 0.05$ and $h/a = 5 \times 10^5$ predicts a radiation resistance within 3% of $Z_0(kh)^2/6\pi$ and a reactance within 10% of $-(Z_0/\pi kh) \ln h/a$. In that calculation, the antenna was divided into 201 segments, with simple forms for the current distribution used for each segment. The predicted real and imaginary parts of the current distribution are shown in the figures below: the (larger) imaginary part is very close to the form (20), while the (very small) real part has the form $1 - z^2/h^2$ and a peak value close to $(kh)^3/6 \ln(h/a)$ times the peak value of the imaginary part of the current.



While the success of the approximate analytic solution (26)-(27) to Pocklington's integral equation is (to this author) remarkable, it may be of interest to explore in secs. 2.3-4 an analytic solution via another integral equation that has often been considered in the literature.

2.3 Solution via Hallén's Integral Equation

For purely linear antennas of lengths up to 1-2 wavelengths, one can obtain rather good solutions for the currents using eq. (4) together with the approximate solution (12) for the vector potential along the conductors,

$$\frac{Z_0}{2\pi}\int_{-kh}^{kh} I(z')\,\frac{e^{-ik\mathcal{R}}}{-ik\mathcal{R}}\,d(kz') = \frac{iZ_0}{2\pi}\frac{4\pi}{\mu_0}A_z(0,0,z) \approx V_0\sin|kz| + C\cos kz \qquad (|z|<h), \quad (28)$$

with $\mathcal{R}(z,z')$ again given by eq. (17). This integral equation is due to Hallén [6], whose method of solution was improved by King [8, 9, 10]. The current $I(z)$ must still vanish at the ends of the conductors, *i.e.*, $z=\pm h$, but the vector potential need not vanish there.

The simplest analytic approximation to the integral equation (28) is that the integrand is significant only when $\mathcal{R}$ is very small, *i.e.*, when $z \approx z'$, and hence,

$$I(z) \approx -\frac{2\pi i}{Z_0}\left[V_0\sin|kz| + C\cos kz\right] \qquad (|z|<h). \quad (29)$$

Setting $I(h)=0$ we recover the thin-wire solution of sec. 2.1.

To do better, we would like the impedance $Z = R + iX$ calculated from $V_0 = I_0 Z$ to include a real part (*i.e.*, to acknowledge the existence of radiation!). This implies that we cannot approximate the factor $e^{-ik\mathcal{R}}/-ik\mathcal{R}$ in eq. (28) as being purely imaginary. Now,

$$\frac{e^{-ik\mathcal{R}}}{-ik\mathcal{R}} = \frac{\sin k\mathcal{R}}{k\mathcal{R}} + i\frac{\cos k\mathcal{R}}{k\mathcal{R}} = \frac{\sin(k\mathcal{R}/2)\cos(k\mathcal{R}/2)}{k\mathcal{R}/2} + i\frac{\cos k\mathcal{R}}{k\mathcal{R}} \approx \cos\frac{k\mathcal{R}}{2} + \frac{i}{k\mathcal{R}}\,, \quad (30)$$

where the approximation holds for $k\mathcal{R} \lesssim 1$. Using this approximation in eq. (28), the real part of the integral is,

$$\begin{aligned}\frac{Z_0}{2\pi}\int_{-kh}^{kh} I(z')\cos\frac{k\mathcal{R}}{2}\,d(kz') &\approx \frac{Z_0}{2\pi}\int_0^{kh} I(z')\left(\cos\frac{k\,|z-z'|}{2} + \cos\frac{k\,|z+z'|}{2}\right)d(kz') \\ &= \frac{Z_0}{\pi}\cos\frac{kz}{2}\int_0^{kh} I(z')\cos\frac{kz'}{2}\,d(kz') \equiv R\cos\frac{kz}{2}\,, \quad (31)\end{aligned}$$

and the imaginary part of the integral is,

$$\frac{Z_0}{2\pi}\int_{-kh}^{kh}\frac{I(z')}{k\mathcal{R}}\,d(kz') \approx XI(z), \quad (32)$$

where R and X are real constants, with dimensions of impedance, that are to be determined. The approximate solution to eq. (28) can be written as,

$$iXI(z) \approx V_0\sin|kz| + C\cos kz - R\cos\frac{kz}{2}\,. \quad (33)$$

This suggests that the desired better approximation to the current is a linear combination of terms in $\sin|kz|$, $\cos kz$ and $\cos(kz/2)$. A form of the current based on these terms which satisfies the condition that $I(\pm h)=0$ is,

$$I(z) = I_{01}I_1(z) + I_{02}I_2(z) + I_{03}I_3(z), \quad (34)$$

where I_{01}, I_{02} and I_{03} are complex constants and the dimensionless currents $I_j,\ j = 1, 2, 3$, have the forms,

$$I_1(z) = \frac{\sin kh - \sin|kz|}{\sin kh} = \frac{\sin kh - \sin|kz|}{J_1}, \tag{35}$$

$$I_2(z) = \frac{\cos kz - \cos kh}{1 - \cos kh} = \frac{\cos kz - \cos kh}{J_2}, \tag{36}$$

$$I_3(z) = \frac{\cos(kz/2) - \cos(kh/2)}{1 - \cos(kh/2)} = \frac{\cos(kz/2) - \cos(kh/2)}{J_3}, \tag{37}$$

where,

$$J_1 = \sin kh, \qquad J_2 = 1 - \cos kh, \qquad \text{and} \qquad J_3 = 1 - \cos(kh/2), \tag{38}$$

such that the I_j are normalized to 1 at $z = 0$.

Using this form for the current in the integral equation (28), we have,

$$I_{01}Z_1(z) + I_{02}Z_2(z) + I_{03}Z_3(z) = V_0 \sin|kz| + C\cos kz \qquad (|z| < h), \tag{39}$$

where the forms,

$$Z_j(z) = \frac{Z_0}{2\pi}\int_{-kh}^{kh} I_j(z')\,\frac{e^{-ik\mathcal{R}}}{-ik\mathcal{R}}\,d(kz'), \qquad j = 1, 2, 3, \tag{40}$$

are computable functions of z using $\mathcal{R}(z, z')$ from eq. (17).

The four unknowns, C, I_{01}, I_{02} and I_{03} in eq. (39) could in principle be determined by evaluating this equation at four different z. However, we follow the recommendation of King [9, 10] that we only make an evaluation at $z = h$, and seek three additional constraints on these unknowns elsewhere.

At $z = h$, eq. (39) is just,

$$I_{01}Z_1(h) + I_{02}Z_2(h) + I_{03}Z_3(h) = V_0 \sin kh + C\cos kh. \tag{41}$$

Subtracting eq. (41) from eq. (39), we find,

$$I_{01}Z_{h1}(z) + I_{02}Z_{h2}(z) + I_{03}Z_{h3}(z) = -V_0 I_1(z)J_1 + C I_2(z)J_2 \qquad (|z| < h), \tag{42}$$

where,

$$Z_{hj}(z) \equiv Z_j(z) - Z_j(h) = \frac{Z_0}{2\pi}\int_{-kh}^{kh} I_j(z')\left(\frac{e^{-ik\mathcal{R}}}{-ik\mathcal{R}} - \frac{e^{-ik\mathcal{R}_h}}{-ik\mathcal{R}_h}\right) d(kz'), \qquad j = 1, 2, 3, \tag{43}$$

and,

$$\mathcal{R}_h = \mathcal{R}(h, z') = \sqrt{(h - z')^2 + a^2}. \tag{44}$$

We now apply the approximations of eqs. (31)-(33) to the functions $Z_{hj}(z)$, and write,

$$Z_{hj}(z) \approx R_j(\cos kz/2 - \cos kh/2) + iX_j I_j(z)J_j = R_j I_3(z)J_3 + iX_j I_j(z)J_j, \quad j = 1, 2, 3, \tag{45}$$

where the R_j and X_j are new unknown, real constants. These constants can be determined by numerical evaluation of the functions $Z_{hj}(z)$ at $z = 0$ where the currents $I_i(z)$ are maximal (or, in case $\lambda/4 < h < 5\lambda/8$, I_1 should be evaluated at $z = h - \lambda/4$). Thus, for $h < \lambda/4$,

$$Z_{hj}(0) = R_j I_3(0) J_3 + iX_j I_j(0) J_j, \qquad j = 1, 2, 3, \tag{46}$$

and,

$$R_j = \frac{Re[Z_{hj}(0)]}{I_3(0)J_3}\,, \qquad X_j = \frac{Im[Z_{hj}(0)]}{I_j(0)J_j}\,, \qquad j = 1, 2, 3. \tag{47}$$

Using the approximation (45) in eq. (42) we have,

$$\begin{aligned} &I_{01}[R_1 I_3(z) J_3 + iX_1 I_1(z) J_1] + I_{02}[R_2 I_3(z) J_3 + iX_2 I_2(z) J_2] + I_{03}(R_3 + iX_3) I_3(z) J_3 \\ &\approx -V_0 I_1(z) J_1 + C I_2(z) J_2 \qquad (|z| < h), \end{aligned} \tag{48}$$

Equating separately the coefficients of the functions $I_1(z)$, $I_2(z)$ and $I_3(z)$ we obtain the needed three additional constraints on the unknowns C, I_{01}, I_{02} and I_{03},

$$I_{01} \approx i\frac{V_0}{X_1}\,, \tag{49}$$

$$C \approx iX_2 I_{02}, \tag{50}$$

$$R_2 I_{02} + (R_3 + iX_3) I_{03} \approx -R_1 I_{01}. \tag{51}$$

Using these we can rewrite the fourth constraint, eq. (41), as,

$$[Z_2(h) - iX_2 \cos kh] I_{02} + Z_3(h) I_{03} = -I_{01} Z_1(h) + V_0 \sin kh \approx -I_{01}[Z_1(h) + iX_1 \sin kh]. \tag{52}$$

Solving eqs. (51)-(52) for I_{02} and I_{03} we find,

$$I_{02} \approx I_{01}\frac{(R_3 + iX_3)[Z_1(h) + iX_1 \sin kh] - R_1 Z_3(h)}{R_2 Z_3(h) - (R_3 + iX_3)[Z_2(h) - iX_2 \cos kh]}\,, \tag{53}$$

$$I_{03} \approx -I_{01}\frac{R_2[Z_1(h) + iX_1 \sin kh] - R_1[Z_2(h) - iX_2 \cos kh]}{R_2 Z_3(h) - (R_3 + iX_3)[Z_2(h) - iX_2 \cos kh]}\,. \tag{54}$$

The current $I_0 = I(0)$ at the terminals of the antenna now follows from eq. (34) as,

$$I_0 = I_{01} + I_{02} + I_{03} \equiv \frac{V_0}{Z}\,, \tag{55}$$

so the input impedance Z of the antenna is also determined.

Numerical examples of these approximations for $h = \lambda/2$, λ and $3\lambda/2$ are given in [10, 15]. *So much numerical computation is involved for long antennas in the present semi-analytic approach that in practice it's better to use fully numerical codes such as [14].*

2.4 Hallén's Integral Equation for a Short Linear Antenna

To obtain a more complete analytic approximation, we restrict our attention to a short linear antenna, for which $kh \ll 1$.

In this limit, the currents $I_2(z)$ and $I_3(z)$ of eqs. (36)-(37) are proportional to one another, so we consider only two forms of the (normalized, dimensionless) current,

$$
\begin{aligned}
I_1(z) &= 1 - \frac{|z|}{h}\,, && (56)\\
I_2(z) &= 1 - \frac{z^2}{h^2}\,, && (57)
\end{aligned}
$$

which are the limiting forms of the currents (35)-(36) when $kh \ll 1$. Then, the total current is given by,

$$
I(z) = I_{01} I_1(z) + I_{02} I_2(z), \qquad (58)
$$

where I_{01} and I_{02} are complex constants to be determined.

Using this form for the current in the integral equation (28), we have,

$$
I_{01} Z_1(z) + I_{02} Z_2(z) = V_0 \sin|kz| + C\cos kz, \qquad (|z| < h), \qquad (59)
$$

where,

$$
Z_j(z) = \frac{Z_0}{2\pi}\int_{-kh}^{kh} I_j(z')\,\frac{e^{-ik\mathcal{R}}}{-ik\mathcal{R}}\,d(kz') \approx \frac{Z_0}{2\pi}\int_{-kh}^{kh} I_j(z')\left(1 + \frac{i}{k\mathcal{R}}\right) d(kz'), \qquad j = 1, 2. \quad (60)
$$

In particular,

$$
\begin{aligned}
Z_1(z) &\approx \frac{kZ_0}{2\pi}\int_{-h}^{h}\left(1 - \frac{|z'|}{h}\right)\left(1 + \frac{i}{k\mathcal{R}}\right) dz' \\
&= \frac{khZ_0}{2\pi} + \frac{iZ_0}{2\pi}\int_{-h}^{h}\left(1 - \frac{|z'|}{h}\right)\frac{dz'}{\sqrt{(z'-z)^2 + a^2}} \\
&= \frac{khZ_0}{2\pi} + \frac{iZ_0}{2\pi}\int_{-h-z}^{h-z}\left(1 - \frac{|z+x|}{h}\right)\frac{dx}{\sqrt{x^2 + a^2}} \\
&= \frac{khZ_0}{2\pi} + \frac{iZ_0}{2\pi}\int_{-h-z}^{h-z}\frac{dx}{\sqrt{x^2 + a^2}} + \frac{iZ_0}{2\pi h}\int_{-h-z}^{-z}\frac{z+x}{\sqrt{x^2 + a^2}}\,dx \\
&\quad -\frac{iZ_0}{2\pi h}\int_{-z}^{h-z}\frac{z+x}{\sqrt{x^2 + a^2}}\,dx \\
&= \frac{khZ_0}{2\pi} + \frac{iZ_0}{2\pi}\ln\frac{\sqrt{(h-z)^2 + a^2} + h - z}{\sqrt{(h+z)^2 + a^2} - (h+z)} \\
&\quad +\frac{izZ_0}{2\pi h}\ln\frac{(\sqrt{z^2 + a^2} - z)^2}{\left(\sqrt{(h+z)^2 + a^2} - (h+z)\right)\left(\sqrt{(h-z)^2 + a^2} + (h-z)\right)} \\
&\quad +\frac{iZ_0}{2\pi h}\left(2\sqrt{z^2 + a^2} - \sqrt{(h+z)^2 + a^2} - \sqrt{(h-z)^2 + a^2}\right). \qquad (61)
\end{aligned}
$$

For later use we need the values of Z_1 at $z = 0$ and h,

$$
\begin{aligned}
Z_1(0) &\approx \frac{khZ_0}{2\pi} + \frac{iZ_0}{2\pi}\ln\frac{\sqrt{h^2 + a^2} + h}{\sqrt{h^2 + a^2} - h} - \frac{iZ_0}{\pi h}\left(\sqrt{h^2 + a^2} - a\right) \\
&\approx \frac{khZ_0}{2\pi} + \frac{iZ_0}{\pi}\left(\ln\frac{h}{a} - 1\right) \approx \frac{khZ_0}{2\pi} + \frac{iZ_0}{\pi}\ln\frac{h}{a}\,, \qquad (62)
\end{aligned}
$$

and,

$$
\begin{aligned}
Z_1(h) &\approx \frac{khZ_0}{2\pi} + \frac{iZ_0}{2\pi} \ln \frac{a}{\sqrt{(2h)^2+a^2}-2h} + \frac{iZ_0}{2\pi} \ln \frac{(\sqrt{h^2+a^2}-h)^2}{a\left(\sqrt{(2h)^2+a^2}-2h\right)} \\
&\quad + \frac{iZ_0}{2\pi h}\left(2\sqrt{h^2+a^2}-\sqrt{(2h)^2+a^2}-a\right) \\
&\approx \frac{khZ_0}{2\pi} + \frac{iZ_0}{2\pi} \ln \frac{4h}{a} + \frac{iZ_0}{2\pi} \ln \frac{a}{h} - \frac{iaZ_0}{2\pi h} \approx \frac{khZ_0}{2\pi} + \frac{iZ_0}{\pi} \ln 2, \qquad (63)
\end{aligned}
$$

noting that $a/h \ll 1$. Similarly,

$$
\begin{aligned}
Z_2(z) &\approx \frac{kZ_0}{2\pi} \int_{-h}^{h} \left(1-\frac{z'^2}{h^2}\right)\left(1+\frac{i}{k\mathcal{R}}\right) dz' \\
&= \frac{khZ_0}{3\pi} + \frac{iZ_0}{2\pi} \int_{-h}^{h} \left(1-\frac{z'^2}{h^2}\right) \frac{dz'}{\sqrt{(z'-z)^2+a^2}} \\
&= \frac{khZ_0}{3\pi} + \frac{iZ_0}{2\pi} \int_{-h-z}^{h-z} \left(1-\frac{(z+x)^2}{h^2}\right) \frac{dx}{\sqrt{x^2+a^2}} \\
&= \frac{Z_0}{3\pi} + \frac{iZ_0(h^2-z^2)}{2\pi h^2} \int_{-h-z}^{h-z} \frac{dx}{\sqrt{x^2+a^2}} - \frac{izZ_0}{\pi h^2} \int_{-h-z}^{h-z} \frac{x}{\sqrt{x^2+a^2}}\, dx \\
&\quad - \frac{iZ_0}{2\pi h^2} \int_{-h-z}^{h-z} \frac{x^2}{\sqrt{x^2+a^2}}\, dx \\
&= \frac{khZ_0}{3\pi} + \frac{iZ_0(h^2-z^2)}{2\pi h^2} \ln \frac{\sqrt{(h-z)^2+a^2}+h-z}{\sqrt{(h+z)^2+a^2}-(h+z)} \\
&\quad + \frac{izZ_0}{\pi h^2}\left(\sqrt{(h+z)^2+a^2}-\sqrt{(h-z)^2+a^2}\right) \\
&\quad - \frac{iZ_0}{4\pi h^2}\left((h+z)\sqrt{(h+z)^2+a^2}+(h-z)\sqrt{(h-z)^2+a^2}\right) \\
&\quad + \frac{ia^2Z_0}{4\pi h^2} \ln \frac{\sqrt{(h-z)^2+a^2}+h-z}{\sqrt{(h+z)^2+a^2}-(h+z)}\,. \qquad (64)
\end{aligned}
$$

For later use we record the values of Z_2 at $z = 0$ and h,

$$
\begin{aligned}
Z_2(0) &\approx \frac{khZ_0}{3\pi} + \frac{iZ_0}{2\pi} \ln \frac{\sqrt{h^2+a^2}+h}{\sqrt{h^2+a^2}-h} - \frac{iZ_0}{4\pi h}\sqrt{h^2+a^2} + \frac{ia^2Z_0}{4\pi h^2} \ln \frac{\sqrt{h^2+a^2}+h}{\sqrt{h^2+a^2}-h} \\
&\approx \frac{khZ_0}{3\pi} + \frac{iZ_0}{\pi}\left(\ln\frac{h}{a}-\frac{1}{2}\right) \approx \frac{khZ_0}{3\pi} + \frac{iZ_0}{\pi} \ln \frac{h}{a}\,, \qquad (65)
\end{aligned}
$$

and,

$$
\begin{aligned}
Z_2(h) &\approx \frac{khZ_0}{3\pi} + \frac{iZ_0}{2\pi h}\left(\sqrt{(2h)^2+a^2}-2a\right) + \frac{ia^2Z_0}{4\pi h^2} \ln \frac{a}{\sqrt{(2h)^2+a^2}-2h} \\
&\approx \frac{khZ_0}{3\pi} + \frac{iZ_0}{\pi} + \frac{ia^2Z_0}{2\pi h^2} \ln \frac{2h}{a} \approx \frac{khZ_0}{3\pi} + \frac{iZ_0}{\pi}. \qquad (66)
\end{aligned}
$$

As before, we find one constraint on the unknown coefficients C, I_{01} and I_{02} by evaluating eq. (59) at $z = h$,

$$I_{01}Z_1(h) + I_{02}Z_2(h) = V_0 \sin kh + C \cos kh. \tag{67}$$

Subtracting eq. (67) from eq. (59), we find (recalling eqs. (35)-(36) and (43)),

$$I_{01}Z_{h1}(z) + I_{02}Z_{h2}(z) = -V_0 I_1(z)J_1 + C I_2(z)J_2 \qquad (|z| < h), \tag{68}$$

where now,

$$J_1 = kh, \qquad \text{and} \qquad J_2 = \frac{k^2h^2}{2}\,. \tag{69}$$

We again apply the approximations of eqs. (31)-(33) to the functions $Z_{hj}(z)$, and write,

$$Z_{hj}(z) \approx R_j(\cos(kz/2) - \cos(kh/2)) + iX_j I_j(z)J_j \approx R_j \frac{I_2(z)J_2}{4} + iX_j I_j(z)J_j, \quad j = 1, 2, \tag{70}$$

noting that $\cos(kz/2) - \cos(kh/2) \approx (1 - z^2/h^2)k^2h^2/8 = I_2(z)J_2/4$ for $kh \ll 1$. The constants R_j and X_j can be determined by evaluation of the functions $Z_{hj}(z)$ at $z = 0$ where the currents $I_i(z)$ are maximal. Thus,

$$Z_{hj}(0) = R_j \frac{I_2(0)J_2}{4} + iX_j I_j(0)J_j = \frac{k^2h^2R_j}{8} + iX_j I_j(0)J_j, \qquad j = 1, 2 \tag{71}$$

such that,

$$R_j = \frac{8Re[Z_{hj}(0)]}{k^2h^2} = \frac{8Re[Z_j(0) - Z_j(d)]}{k^2h^2}, \quad X_j = \frac{Im[Z_{hj}(0)]}{I_j(0)J_j} = \frac{Im[Z_j(0) - Z_j(d)]}{I_j(0)J_j}\,. \tag{72}$$

Referring to eqs. (56)-(57), (62)-(63), (65)-(66) and (69) we find,

$$R_1 \approx 0 \approx R_2, \qquad X_1 \approx \frac{Z_0}{\pi kh} \ln \frac{h}{a}\,, \qquad X_2 \approx \frac{2Z_0}{\pi k^2h^2} \ln \frac{h}{a}\,. \tag{73}$$

Using the approximation (70) in eq. (68) we have,

$$I_{01}\left(\frac{R_1}{4} I_2(z)J_2 + iX_1 I_1(z)J_1\right) + I_{02}\left(\frac{R_2}{4} + iX_2\right) I_2(z)J_2 \approx -V_0 I_1(z)J_1 + C I_2(z)J_2, \tag{74}$$

Equating separately the coefficients of the functions $I_1(z)$ and $I_2(z)$ we obtain the needed two additional constraints on the unknowns C, I_{01} and I_{02},

$$I_{01} \approx i\frac{V_0}{X_1} \approx i\frac{\pi khV_0}{Z_0 \ln(h/a)}\,, \tag{75}$$

$$C \approx \frac{R_1}{4} I_{01} + \left(\frac{R_2}{4} + iX_2\right) I_{02} \approx iX_2 I_{02}. \tag{76}$$

Using these we can solve the first constraint, eq. (67), for I_{02},

$$I_{02} \approx \frac{khV_0 - I_{01}Z_1(h)}{Z_2(h) - iX_2} \approx i\frac{\pi k^3h^3V_0}{2Z_0 \ln(h/a)} + \frac{\pi k^4h^4V_0}{4Z_0 \ln^2(h/a)}\,. \tag{77}$$

The (complicated) impedance function $Z(z)$ is now given by,

$$Z(z) = \frac{I_{01}I_1(z)Z_1(z) + I_{02}I_2(z)Z_2(z)}{I_{01}I_1(z) + I_{02}I_2(z)}\,, \tag{78}$$

using eqs. (56)-(57), (61), (64) and (76)-(77).

The input impedance Z of the antenna is determined from the current,

$$I(0) = I_{01}I_1(0) + I_{02}I_2(0) = I_{01} + I_{02} \approx i\frac{\pi khV_0}{Z_0\ln(h/a)} + \frac{\pi k^4h^4V_0}{4Z_0\ln^2(h/a)} \tag{79}$$

at the terminals according to,

$$Z = \frac{V_0}{I(0)} \approx \frac{Z_0k^2h^2}{4\pi} - i\frac{Z_0}{\pi kh}\ln\frac{h}{a}\,. \tag{80}$$

The real part of the impedance is closer to the radiation resistance $Z_0(kh)^2/6\pi$ of a short linear antenna than the approximation of sec. 2.2, and the imaginary part again equals the capacitive reactance of a short linear antenna as estimated in the Appendix.

Thus, with substantial effort, the solution to Hallén's integral equation based on a current distribution with two terms provides a small improvement over the approximate solution to Pocklington's integral equation with only one function for the current.

2.5 Fields Very Close to a Dipole Antenna

The approximate forms of the currents found in secs. 2.3-4 are fairly good over most of the length of the antenna, with the greatest error being close to the terminals at $z = 0$. The resulting estimates for antenna impedances, are reasonably accurate, the near electric and magnetic fields calculated from the approximate currents using the retarded vector potential (4) according to eqs. (7)-(8) are reasonably accurate except very close to the antenna conductors [9, 10]. But, even in the better approximations of secs. 2.2-4 the tangential electric field $E_z(a, \phi, |z| < h)$ is not zero.

It appears that analytic calculations based sums of sinusoidal currents, such as eqs. (34) and (58) are not sufficient for good accuracy of the currents very close to the antenna terminals and of the tangential electric field very close to the conductors. However, numerical analyses, such as NEC4 [14], in which the antenna is subdivided into segments, on which sinusoidal currents are assumed to flow, can achieve rather good accuracy for the currents near the antenna terminals (and hence for the antenna impedance), as well as satisfying the condition that the tangential electric field as calculated from the retarded vector potential vanish at the surface of the conductors.

For analyses such as those in secs. 2.2-4 in which the tangential electric field is not zero at the conductors when calculated via the retarded vector potential, a method for obtaining somewhat more realistic fields close to the conductors has been given by King and Wu [10].

Very close to the conductors, the radial electric field E_ρ is related to the charge distribution $q(z)\,e^{i\omega t}$ along the conductors by Gauss' law,

$$E_\rho(\rho \gtrsim a, \phi, |z| < h, t) \approx \frac{q(z)\,e^{i\omega t}}{2\pi\epsilon_0\rho}\,. \tag{81}$$

The charge distribution can be obtained from the current distribution $I(z)\,e^{i\omega t}$ via the continuity equation,

$$\frac{\partial I}{\partial z} = -\frac{\partial q}{\partial t} = -i\omega q, \tag{82}$$

so that,

$$q(z) = \frac{i}{\omega}\frac{dI(z)}{dz} = \frac{iI'(z)}{\omega}\,. \tag{83}$$

Thus,

$$E_\rho(\rho \gtrsim a, \phi, |z| < h, t) \approx \frac{iI'(z)}{2\pi\epsilon_0\omega\rho}\,e^{i\omega t} = \frac{iI'(z)Z_0}{2\pi k\rho}\,e^{i\omega t}. \tag{84}$$

The peak electric field at the surface of the conductor can be estimated using eq. (15) for the current distribution. The derivative $I'(z)$ is greatest at the end of the conductor, $|z| = h$, where $I'_{\rm max} = kI_0/\sin kh$. Thus, the peak electric field at the surface of the conductor, $\rho = a$, is,

$$E_{\rho,\rm max} \approx \frac{I_0 Z_0}{2\pi a \sin kh} = \frac{V_0}{2\pi a \sin kh}\frac{Z_0}{Z}\,. \tag{85}$$

For a half-wave antenna ($h \approx \lambda/4$) the antenna impedance $Z \approx 70\,\Omega$ is real, so,

$$E_{\rho,\rm max} \approx 0.86\frac{V_0}{a} \qquad \text{(half-wave antenna)}, \tag{86}$$

which is larger than the electric field V_0/d between the terminals of the antenna if $a < 1.17d$.

The scalar potential $V(\rho, z, t)$ close to the surface of the conductor can be obtained from eqs. (15) and (84) by integration,

$$V(\rho, z, t) = V(a, z, t) - \int_a^\rho E_\rho(\rho, z, t)\,d\rho = V(a, z, t) \pm i\frac{V_0\,e^{i\omega t}}{2}\frac{Z_0}{\pi Z}\frac{\cos[k(h - |z|)]}{\sin kh}\ln\frac{\rho}{a}. \tag{87}$$

The scalar potential $V(a, z, t)$ at the surface of the conductor can be estimated from the vector potential there using the gauge condition (5)-(6) and supposing that the thin-wire approximation (14) actually describes the vector potential $A_z(a, z, t)$ at the surface of the conductor. Then,

$$V(a, z, t) = \frac{ic}{k}\frac{\partial A_z(a, z, t)}{\partial z} = \pm\frac{V_0\,e^{i\omega t}}{2}\frac{\cos[k(h - |z|)]}{\cos kh} \qquad (|z| < h), \tag{88}$$

and,

$$V(\rho, z, t) = \pm\frac{V_0\,e^{i\omega t}}{2}\cos[k(h - |z|)]\left(\frac{1}{\cos kh} + \frac{i}{\sin kh}\frac{Z_0}{\pi Z}\ln\frac{\rho}{a}\right) \qquad (\rho \gtrsim a,\ |z| < h). \tag{89}$$

As h approaches $\lambda/4$ the term $1/\cos kh$ grows large and the scalar potential is very large near the tips of the antenna.[4] *However, this high voltage is not associated with a large electric field because the latter is determined by the derivative with respect to ρ of the other term in parentheses, in which term the factor $1/\sin kh$ is unity for $h = \lambda/4$. Hence, the*

[4]This result was anticipated qualitatively by Poincaré by analogy with acoustic resonators. See Fig. 81 of [19].

high voltage at the surface of the conductor is not relevant to the issue of possible electric discharge (corona) at the surface of the antenna.

Similarly, close to the conductors the magnetic field follows from Ampère's law (as the displacement current can be neglected here compared to the conduction current),

$$B_\phi(\rho \gtrsim a, \phi, |z| < h, t) \approx \frac{\mu_0 I(z)}{2\pi\rho}\, e^{i\omega t}. \tag{90}$$

Finally, the electric field component E_z can be found from the ϕ component of Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$,

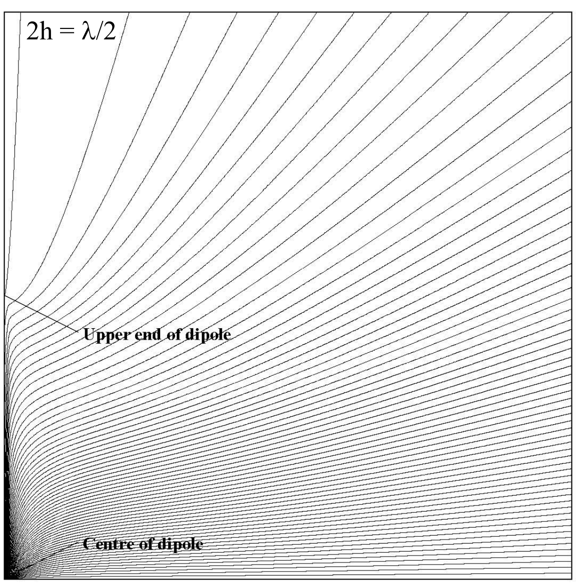
$$\frac{\partial E_z}{\partial \rho} = \frac{\partial E_p}{\partial z} + i\omega B_\phi \approx \frac{i}{2\pi\epsilon_0\omega\rho}\left[I''(z) + k^2 I(z)\right]\, e^{i\omega t}. \tag{91}$$

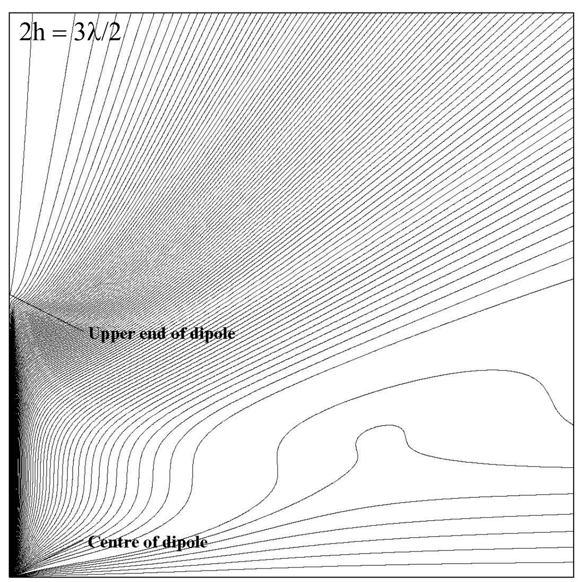
Integrating out from $\rho = a$ where $E_z = 0$ by assumption, we find,

$$E_z(\rho \gtrsim a, \phi, |z| < h, t) \approx \frac{i}{2\pi\epsilon_0\omega}\left[I''(z) + k^2 I(z)\right] \ln\frac{\rho}{a}\, e^{i\omega t}. \tag{92}$$

For $|z| > h$ and for ρ larger than a few times the radius a, the fields can be calculated from the retarded vector potential. The fields E_ρ and B_ϕ so calculated agree with the forms (84)-(90) close to the conductors, while eq. (92) is a better approximation for the axial field E_z close to the conductors.

Once good accuracy is obtained for the fields close to the conductors of the antenna, computation of the Poynting vector shows how power flows out from the feedpoint of the antenna and is guided by the conductors of the antenna into the far zone. Such computations were first made for antennas with conductors of nonzero radius by the Landstorfer group [20]. The figures below were produced by NEC4 (thanks to Alan Boswell) for dipole antennas of total lengths $\lambda/2$ and $3\lambda/2$.





An analytic calculation of the surface currents, of the near and far fields and of the Poynting vector field of a split-sphere antenna has been given recently by Jackson [21].

2.6 Approximation Relation between the Charge and Voltage Distributions

The charge distribution $\rho(z)$ in a linear antenna is related to the current distribution $I(z)$ by the continuity equation,

$$\frac{dI}{dz} = -\frac{d\rho\, e^{-i\omega t}}{dt} = i\omega\rho, \qquad \text{so that} \qquad \rho(z) = -\frac{i}{\omega}\frac{dI}{dz}\,. \tag{93}$$

In the thin-wire approximation, the vector potential $A_z(z)$ along the wire is proportional to the current distribution,

$$A_z(0,0,z) \approx \frac{iV_0 \tan kh}{2cI_0} I(z) = \frac{iZ \tan kh}{2c} I(z) \qquad (|z| < h), \tag{94}$$

according to eqs. (14)-(15), where $Z = V_0/I_0$ is the (complex) terminal impedance of the antenna. Then, the Lorenz gauge condition (6) relates the voltage distribution along the wire as,[5]

$$V(0,0,z) = \frac{ic}{k}\frac{\partial A_z(0,0,z)}{\partial z} \approx -\frac{Z \tan kh}{2k}\frac{dI}{dz} = -\frac{icZ \tan kh}{2}\rho(z) \qquad (|z| < h). \tag{95}$$

For a short linear antenna, the impedance is capacitive, $Z \approx -iZ_0/2\pi \tan kh = -2i/c \tan kh$, so $V(z) \approx \rho(z) = \pm V_0/2$, which is uniform along each arm of the antenna.

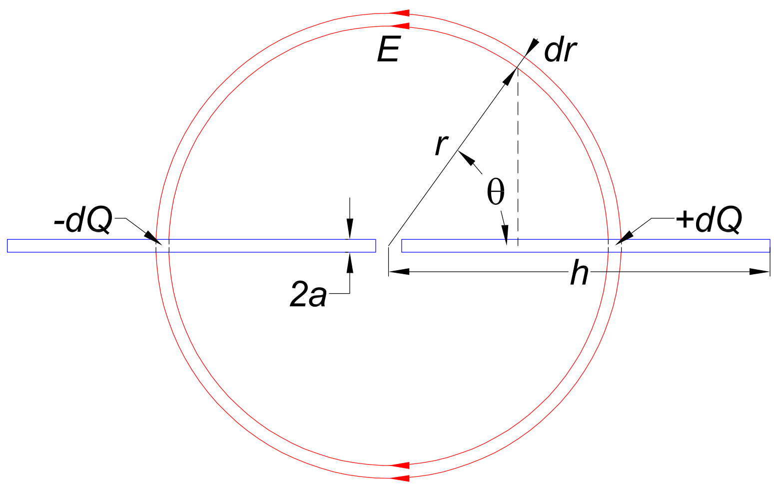
The approximation (95) is less accurate for "resonant" antennas, for which Z is real. For example, a half-wave antenna has $\rho(0) \approx 0$ but, as always, $V(0) = \pm V_0/2$. Away from the antenna terminals V and ρ are 90° out of phase, and the approximation (95) is fairly good.

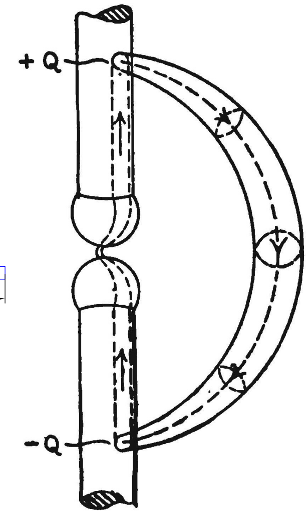
In Schelkunoff's analysis [11], the voltage is proportional to the charge density only for the principal mode. For antennas of length $h \gtrsim \lambda/4$ the higher modes becomes increasingly important, and the approximation (95) becomes less accurate.

A Appendix: Capacitance and Reactance of a Short Linear Dipole Antenna

We estimate the capacitance of a short linear antenna as suggested by Schelkunoff in sec. 10.3 of [24]. The key assumption is that the electric field lines from one arm of the dipole antenna to the other follow semicircular paths (the principal mode), as shown in the figure below.[6]

[5] This relation appears on p. 427 of [23].
[6] On the right is Fig. 86 from [19].





If so, all the field lines emanating from charge dQ in interval dr at distance r from the center of the antenna cross a surface of area $2\pi r\,dr\,\sin\theta$ that lies on a cone of half angle θ, so the electric field strength at (r,θ) is,

$$E = \frac{dQ/dr}{2\pi\epsilon_0 r\sin\theta}\,. \tag{96}$$

The voltage difference between the two arms of the antenna is,

$$\Delta V = 2\int_{\theta_{\min}}^{\pi/2} Er\,d\theta = \frac{dQ/dr}{\pi\epsilon_0}\int_{a/r}^{\pi/2}\frac{d\theta}{\sin\theta} = \frac{dQ/dr}{\pi\epsilon_0}\ln[\tan(\theta/2)]_{a/r}^{\pi/2} = \frac{dQ/dr}{\pi\epsilon_0}\ln(2r/a). \tag{97}$$

This voltage difference should be independent of position along the antenna. The charge distribution dQ/dr is indeed constant to a good approximation for short dipole antennas, but the factor $\ln(2r/a) = -\ln(\theta_{\min}/2)$ is constant only for a biconical dipole antenna (as much favored theoretically by Schelkunoff). A reasonable approximation for a linear dipole antenna is to use $r = h/2$ as a representative length in eq. (97), which leads to the estimate,

$$\Delta V \approx \frac{dQ/dr}{\pi\epsilon_0}\ln(h/a). \tag{98}$$

The corresponding capacitance per unit length along the antenna is,

$$\frac{dC}{dr} \approx \frac{\pi\epsilon_0}{\ln(h/a)}\,, \tag{99}$$

and the total capacitance is,

$$C \approx \frac{\pi\epsilon_0 h}{\ln(h/a)}\,. \tag{100}$$

This estimate ignores the contribution to the capacitance of roughly $\pi\epsilon_0 a^2/d$ associated with the electric field in the gap d between the terminals of the antenna, as is reasonable when $d \approx a$ since then $\ln(h/a) \ll h/a \approx dh/a^2$.

The (capacitive) reactance of the short dipole antenna is then estimated to be,

$$X = -\frac{1}{\omega C} = -\frac{1}{ckC} \approx -\frac{\ln(h/a)}{\pi\epsilon_0 ckh} = -\frac{Z_0}{\pi}\frac{\ln(h/a)}{kh}\,. \tag{101}$$

Acknowledgment

Thanks to Tim Hunt for reviving the author's interest in this problem, and to Bruce Cragin for spotting an error in a previous version of this note.

References

[1] K.T. McDonald, *Voltage Across the Terminals of a Receiving Antenna* (June 25, 2007), http://kirkmcd.princeton.edu/examples/receiver.pdf

[2] H.C. Pocklington, *Electrical Oscillations in Wires*, Proc. Camb. Phil. Soc. **9**, 324 (1897), http://kirkmcd.princeton.edu/examples/pocklington_camb_9_324_97.pdf

[3] L. Lorenz, *Ueber die Identität der Schwingungen des Lichts mit den elektrischen Strömen*, Ann. d. Phys. **207**, 243 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_ap_207_243_67.pdf
On the Identity of the Vibration of Light with Electrical Currents, Phil. Mag. **34**, 287 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_pm_34_287_67.pdf

[4] H. Hertz, *Die Kräfte electrischer Schwimgungen, behandelt much der Maxwellschen Theorie*, Ann. d. Phys. **36**, 1 (1889), http://kirkmcd.princeton.edu/examples/EM/hertz_ap_36_1_89.pdf
reprinted in chap. 9 of H. Hertz, *Electric Waves* (Dover, New York, 1962). A translation by O. Lodge appeared as *The Forces of Electrical Oscillations Treated According to Maxwell's Theory*, Nature **39**, 402 (1889), http://kirkmcd.princeton.edu/examples/ph501lecture16/hertz.pdf

[5] L.V. King, *On the Radiation Field of a Perfectly Conducting Base Insulated Cylindrical Antenna Over a Perfectly Conducting Plane Earth, and the Calculation of Radiation Resistance and Reactance*, Phil. Trans. Roy. Soc. London **A236**, 381 (1937), http://kirkmcd.princeton.edu/examples/EM/king_ptrsl_a236_381_37.pdf

[6] E. Hallén, *Theoretical Investigations into the Transmitting and Receiving Qualities of Antennas*, Nova Acta Regiae Soc. Sci. (Upsula) **11**, 3 (Nov. 1938), http://kirkmcd.princeton.edu/examples/EM/hallen_narssu_11_4_3_38.pdf
Electromagnetic Theory (Chapman and Hall, London, 1962).

[7] R. King and C.W. Harrison, Jr, *The Distribution of Current Along a Symmetrical Center-Driven Antenna*, Proc. Inst. Radio Eng. **31**, 548 (1943), http://kirkmcd.princeton.edu/examples/EM/king_procire_31_548_43.pdf

[8] R.W.P. King, *The Theory of Linear Antennas* (Harvard U. Press, Cambridge, MA, 1956), http://kirkmcd.princeton.edu/examples/EM/king_antennas_56.pdf
The Linear Antenna – Eighty Years of Progress, Proc. IEEE **55**, 2 (1967), http://kirkmcd.princeton.edu/examples/EM/king_pieee_55_2_67.pdf ,

[9] R. King, *Linear Arrays: Currents, Impedances, and Fields, I*, IRE Trans. Ant. Prop. **7**, S440 (1959), http://kirkmcd.princeton.edu/examples/EM/king_iretap_7_s440_59.pdf

[10] R.W.P. King and T.T. Wu, *Currents, Charges and Near Fields of Cylindrical Antennas*, Radio Science (J. Res. NBS/USNC-USRI) **69D**, 429 (1965),
http://kirkmcd.princeton.edu/examples/EM/king_rs_69d_429_65.pdf
The Electric Field Very Near a Driven Cylindrical Antenna, Radio Science **1**, 353 (1966),
http://kirkmcd.princeton.edu/examples/EM/king_rs_1_353_66.pdf

[11] S.A. Schelkunoff, *Theory of Antennas of Arbitary Size and Shape* Proc. IRE **29**, 493 (1941), http://kirkmcd.princeton.edu/examples/EM/schelkunoff_pire_29_493_41.pdf

[12] J.G. Maloney, G.S. Smith and W.R. Scott Jr, *Accurate computation of the radiation from simple antennas using the finite-difference time domain method*, IEEE Trans. Ant. Prop. **38**, 1059 (1990), http://kirkmcd.princeton.edu/examples/EM/maloney_ieeetap_38_1059_90.pdf

[13] R. Luebbers *et al.*, *FDTD calculation of radiation patterns, impedance, and gain for a monopole antenna on a conducting box*, IEEE Trans. Ant. Prop. **40**, 1577 (1999),
http://kirkmcd.princeton.edu/examples/EM/luebbers_ieeetap_40_1577_92.pdf

[14] G.J. Burke, *Numerical Electromagnetic Code – NEC4*, UCRL-MA-109338 (January, 1992), http://www.llnl.gov/eng/ee/erd/ceeta/emnec.html
http://kirkmcd.princeton.edu/examples/NEC_Manuals/NEC4TheoryMan.pdf
http://kirkmcd.princeton.edu/examples/NEC_Manuals/NEC4UsersMan.pdf

[15] S.J. Orfanidis, *Currents on Linear Antennas*, Chap. 21 of *Electromagnetic Waves and Antennas*, http://www.ece.rutgers.edu/~orfanidi/ewa/ch21.pdf

[16] K.T. McDonald, *Radiation in the Near Zone of a Center-Fed Linear Antenna* (June 21, 2004), http://kirkmcd.princeton.edu/examples/linearantenna.pdf

[17] G. Fikioris and T.T. Wu, *On the Application of Numerical Methods to Hallén's Equation*, IEEE Trans. Ant. Prop. **49**, 383 (2001),
http://kirkmcd.princeton.edu/examples/EM/fikioris_ieeetap_49_383_01.pdf

[18] J.D. Jackson, *Classical Electrodynamics*, 3$^{\rm rd}$ ed. (Wiley, 1999),
http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[19] H. Poincaré, *Maxwell's Theory and Wireless Telegraphy* (McGraw, 1904),
http://kirkmcd.princeton.edu/examples/EM/poincare_vreeland_04.pdf

[20] F.M. Landstorfer and H.H. Meinke, *Energy Transport in Electromagnetic Fields* (1975),
http://kirkmcd.princeton.edu/examples/EM/landstorfer_75c.pdf

[21] J.D. Jackson, *How an antenna launches its input power into radiation: The pattern of the Poynting vector at and near an antenna*, Am. J. Phys. **74**, 280 (2006),
http://kirkmcd.princeton.edu/examples/EM/jackson_ajp_74_280_06.pdf

[22] D.H. Johnson, *Origins of the Equivalent Circuit Concept: The Voltage-Source Equivalent*, Proc. IEEE **91**, 636 (2003),
http://kirkmcd.princeton.edu/examples/EM/johnson_pieee_91_636_03.pdf
Origins of the Equivalent Circuit Concept: The Current-Source Equivalent, Proc. IEEE **91**, 817 (2003), http://kirkmcd.princeton.edu/examples/EM/johnson_pieee_91_817_03.pdf

[23] A. Hund, *A Discussion on "Electrical Oscillations in Antennas and Induction Coils" By John M. Miller*, Proc. I.R.E. **8**, 424 (1920),
http://kirkmcd.princeton.edu/examples/EM/hund_pire_8_424_20.pdf

[24] S.A. Schelkunoff and H.T. Friis, *Antennas, Theory and Practice* (Wiley, New York, 1952), http://kirkmcd.princeton.edu/examples/EM/schelkunoff_friis_52.pdf