A Tricky Tripos Problem

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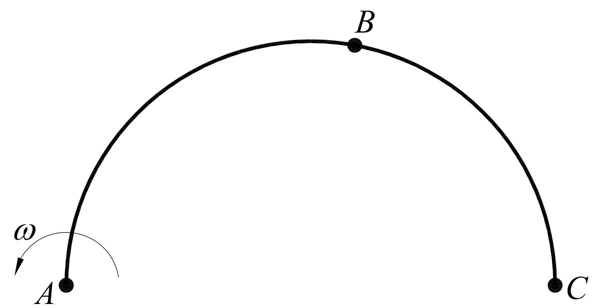
1 Problem

The Cambridge "Mathematical" Tripos are well known for posing problems that appear to be quite complex but can be solved relatively simply if one is sufficiently aware of various somewhat arcane lore. A case of this type is Ex. 3, p. 291 of [1].[1]

A rigid semicircular wire rotates in its plane about its fixed end A with constant angular velocity ω, as sketched below. What is the torque couple (bending moment) at a point B on the wire? For what point B is the wire most likely to bend/break for large ω?



This example is a variant of the famous falling-chimney problem [2].

2 Solution

If we imagine that the wire could bend at point B, then segment BC would lag behind segment AB, which implies the that (vector) torque couple $\boldsymbol{\tau}$ at point B has the opposite sense (clockwise in the figure) to the (counterclockwise) angular velocity $\boldsymbol{\omega}$.

For constant angular velocity ω, the torque on the system is zero (in any frame).

While it suffices to consider an accelerated frame with origin at point B and nonrotating axes, here we consider the (rotating) body frame of the wire, with origin at point B, and perform a torque analysis for the segment BC.[2]

In the accelerating and rotating frame, with origin at point B and angular velocity $\boldsymbol{\omega}$, we must consider the four "fictitious" forces: $-m_i\,\mathbf{a}_B + m_i\mathbf{r}_i \times \dot{\boldsymbol{\omega}} + 2m_i\mathbf{v}_i \times \boldsymbol{\omega} + m_i\,\boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega})$, that act on each mass element m_i (at position $\mathbf{r}_i$, with velocity $\mathbf{v}_i$ with respect to point B), where $\mathbf{a}_B$ is the acceleration of point B in the (inertial) lab frame. See, for example, eq. (39.7) of [3] and pp. 168-172 of [4]. Of these four, the second and third (the Coriolis) force are zero (as $\dot{\boldsymbol{\omega}} = 0 = \mathbf{v}_i$ in the present example), while the fourth (centrifugal) force is along $\mathbf{r}_i$ and so produces no torque about point B.

[1]See also p. 97 of the Solution section of [1].

[2]We could also do a torque analysis for the segment AB, which would involve the as-yet-unknown force on the pivot point A. Since there is no external force on the "free" endpoint C, it is simpler to consider segment BC.

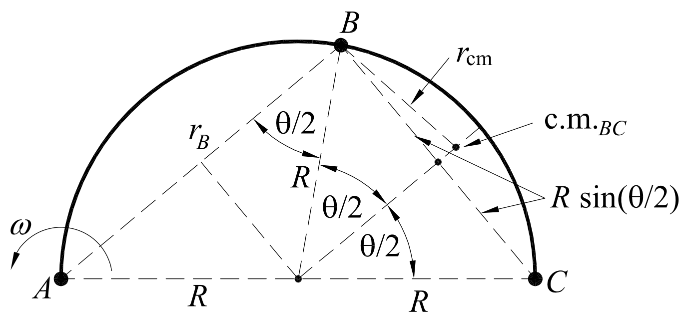
Then, the torque equation in the accelerated, rotating frame is[3]

$$0 = \boldsymbol{\tau} + \sum \mathbf{r}_i \times (-m_i\,\mathbf{a}_B), \tag{1}$$

$$\boldsymbol{\tau} = \sum \mathbf{r}_i \times m_i\,\mathbf{a}_B = m_{BC}\,\mathbf{r}_{\rm cm} \times \mathbf{a}_B = m_{BC}\,\omega^2\,\mathbf{r}_B \times \mathbf{r}_{\rm cm}, \tag{2}$$

noting that $\mathbf{a}_B = -\omega^2\,\mathbf{r}_B$ is the acceleration of point B in the lab frame, $\mathbf{r}_B$ is the position vector from point A to point B, m_{BC} is the mass of segment BC, and $\mathbf{r}_{\rm cm}$ is the position vector from point B to the center of mass of segment BC.

We introduce θ as the angle subtended by segment BC with respect to the center of the semicircular wire (of radius R), as sketched below.[4]



Then, the mass of segment BC is $m_{BC} = m\theta/\pi$, where m is the mass of the semicircular wire. The length of $\mathbf{r}_B$ is $r_B = 2R\cos(\theta/2)$. Since the center of mass of segment BC lies on the radius vector that makes angle $\theta/2$ to the diameter of the semicircular wire, the vector cross product $\mathbf{r}_B \times \mathbf{r}_{\rm cm}$ has magnitude $r_B R\sin(\theta/2) = 2R^2\cos(\theta/2)\sin(\theta/2) = R^2\sin\theta$.

Altogether, eq. (2) implies that the magnitude of the torque couple at point B is

$$\tau = m\omega^2 R^2 \frac{\theta}{\pi}\sin\theta, \tag{3}$$

for θ in radians. This is maximal for $\tan\theta = -\theta$, at $\theta \approx 2.028$ rad $= 116°$ (for $0 < \theta < \pi$, thanks to Wolfram Alpha, https://www.wolframalpha.com/input?i=tan%28x%29+%3D+-x), which is the angle for point B where the rotating wire is most likely to bend/break when the angular velocity ω is large.

2.1 Comment

We succeeded in using accelerated, rotating axes in the torque analysis for the special case that the axes are the body axes, considering the "fictitious" torques asscociated with the four types of "fictitious" torques in such frames of reference. However, it seems that for any other rotating axes there must be additional "fictitious" torques, such that the torque analysis reduces, in effect, to use of nonrotating axes.

This reinforces the well known advice not to use rotating axes in torque analyses.

[3] The torque equation is the same in the accelerated, but nonrotating frame with point B as its origin. The torque couple $\boldsymbol{\tau}$ is the same in any frame.

[4] Note that triangle ABC is a right triangle.

References

[1] S.L. Loney, *Dynamics of Rigid Bodies* (Kindle Edition, 1926, 2018),
http://kirkmcd.princeton.edu/examples/mechanics/loney_26.pdf

[2] K.T. McDonald, *Falling Chimney* (Oct. 1, 1980),
http://kirkmcd.princeton.edu/examples/chimeny.pdf

[3] L.D. Landau and E.M. Lifshitz, *Mechanics*, 3$^{\rm rd}$ ed. (Pergamon, 1976),
http://kirkmcd.princeton.edu/examples/mechanics/landau_mechanics.pdf

[4] K.T. McDonald, *Accelerated Coordinate Systems* (1980),
http://kirkmcd.princeton.edu/examples/ph205l16.pdf