Laser Tweezers

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(November 13, 1998; updated November 6, 2017)

1 Problem

It is well known that a charged particle cannot be held a rest by purely electrostatic fields (Earnshaw's theorem [1, 2]). Give a simple classical explanation of how a neutral atom of polarizability α can be "trapped" at the focus of a laser beam.

a) First, ignore magnetic interactions, and deduce that there is a (time-averaged) trapping force dependent on the electric field of the laser.

b) Atoms have some probability of absorbing photons from the laser beam, thereby being kicked along the direction of the beam. This processes can be modeled classically by supposing that the polarizability of the atom has an imaginary part: $\alpha = \alpha' + i\alpha''$. Deduce the (time-averaged) force on an atom along the direction of propagation of a linearly polarized plane electromagnetic wave in terms of α'', the imaginary (absorptive) part of the polarizability.

c) For an idealized atom with a single natural frequency ω_0, deduce the ratio α'/α'' at the frequency ω for which the real part, α', of the polarizability is a maximum. For this, you may use a classical model of an atom as an electron on a spring of frequency ω_0, subject to a damping force $-\gamma m\dot{\mathbf{x}}$, where $\gamma \ll \omega_0$ is the reciprocal of the lifetime of the 'excited state'.

d) In practice, the trapping force a) must be larger than the longitudinal force b). This requires the laser beam to be tightly focused. Deduce the $f_\#$ of the lens needed for trapping under the conditions of part c).

2 Solution

This concept was proposed by Ashkin in 1977 [3],[1] and first realized by Ashkin *et al.* in 1985 [5]. Aspects of this problem have been discussed in the Journal from a semiclassical view [6]. Here, a completely classical approach displays the essential results quickly.

Undergraduate laboratory experiments with laser tweezers have been described in [7].

a) The important hint is that the atom is polarizable, so it takes on an induced dipole moment $\mathbf{p} = \alpha\mathbf{E}$ where E is the electric field strength, and α is the atomic polarizability. We suppose this simple relation holds for a wave field as well as for a static field.

[1] It was anticipated to some extent in [4].

We now want the force $\mathbf{F}$ on the dipole. This is obtained from the form $(\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E}$ which holds even when $\nabla \times \mathbf{E} \neq 0$,

$$\mathbf{F} = (\mathbf{p} \cdot \boldsymbol{\nabla})\mathbf{E} = \alpha(\mathbf{E} \cdot \boldsymbol{\nabla})\mathbf{E} = \frac{\alpha}{2}\boldsymbol{\nabla} E^2. \tag{1}$$

So the polarizable atom can be "trapped" at a point where E^2 takes on a local maximum, since the force (1) is restoring for departures in any direction from that point.

There is no such place in a charge-free region of an electrostatic field, as demonstrated in Appendix A. Thus, Earnshaw's theorem can be extended to include the case of a polarizable atom.

However, E^2 can have a local maximum in a nonelectrostatic field, such as at the focus of a laser beam.

For oscillatory fields, it is appropriate to restrict the discussion to the time-averaged behavior of the atom. The time-average of the trapping force (1) is written as,

$$\langle \mathbf{F} \rangle_{\rm trapping} = \frac{\alpha}{2}\boldsymbol{\nabla} \left\langle E^2 \right\rangle. \tag{2}$$

b) The induced, oscillating dipole is equivalent to an oscillating charge with velocity along the direction of $\mathbf{E}$. Then the $\mathbf{v} \times \mathbf{B}$ force is in the direction of $\mathbf{E} \times \mathbf{B}$, *i.e.*, along the direction of propagation of the wave. This is consistent with the force being due to absorption of photons from the wave.

Since it is stated that the polarizability has an imaginary part, we describe the plane wave using complex notation. The wave is polarized in, say, the x direction: $\mathbf{E}(\mathbf{x}, t) = E_0 \hat{\mathbf{x}} e^{i(kz - \omega t)}$. It suffices to consider the dipole as being at the origin, where,

$$\mathbf{E} = E_0 \hat{\mathbf{x}} e^{-i\omega t}, \qquad \mathbf{B} = E_0 \hat{\mathbf{y}} e^{-i\omega t}, \qquad \text{and} \qquad \mathbf{p} = \alpha E_0 \hat{\mathbf{x}} e^{-i\omega t} = e\mathbf{x}(t). \tag{3}$$

The oscillating dipole is equivalent to a charge e at distance $\mathbf{x}(t)$ from the nucleus. The velocity of the charge is, of course, $\dot{\mathbf{x}}$. Then, the Lorentz force on the moving charge is (in Gaussian units),

$$\mathbf{F} = e\frac{\dot{\mathbf{x}}}{c} \times \mathbf{B} = \frac{\dot{\mathbf{p}}}{c} \times \mathbf{B} \tag{4}$$

More precisely, since we are using complex notation, the time-average detrapping force is,

$$\langle \mathbf{F} \rangle_{\rm detrapping} = \frac{1}{2} Re \left(\frac{\dot{\mathbf{p}}}{c} \times \mathbf{B}^\star \right) = \frac{1}{2} Re \left(-i\frac{\omega}{c}\alpha E_0^2 \right) \hat{\mathbf{z}} = \frac{k}{2}\alpha'' E_0^2 \hat{\mathbf{z}}, \tag{5}$$

where $k = 2\pi/\lambda$ is the wave number.

c) To model the polarizability, it suffices to consider the response of the model atom to the electric field. The equation of motion is then,

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{eE_0}{m} e^{-i\omega t}. \tag{6}$$

The trial solution $x = x_0 e^{-i\omega t}$ leads to,

$$x_0 = \left(\frac{eE_0}{m}\right) \frac{\omega_0^2 - \omega^2 + i\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \tag{7}$$

Since the magnitude of the dipole moment is $p_0 = ex_0 = \alpha E_0$, the polarizability $\alpha = \alpha' + i\alpha''$ is,

$$\alpha' = \left(\frac{e^2}{m}\right) \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \qquad \alpha'' = \left(\frac{e^2}{m}\right) \frac{\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}. \tag{8}$$

Since $\alpha'' > 0$, the force (5) is in the $+z$ direction, as expected for photon absorption from a wave that moves in the $+z$ direction.

To find the frequency at which α' is maximum, we take the derivative,

$$0 = \frac{-2\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} - \frac{(\omega_0^2 - \omega^2)[-4\omega(\omega_0^2 - \omega^2) + 2\omega\gamma^2]}{[(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^2}. \tag{9}$$

Thus, $\gamma\omega_0 = \omega_0^2 - \omega^2$ at the maximum, and α' is maximal for ω slightly less than ω_0,

$$\omega_{\rm max} \approx \omega_0 - \gamma/2. \tag{10}$$

Approximating $\gamma\omega_{\rm max}$ by $\gamma\omega_0$ since $\gamma \ll \omega_0$, we have,

$$\alpha'(\omega_{\rm max}) = \frac{e^2}{m} \frac{1}{2\gamma\omega_0} = \alpha''(\omega_{\rm max}). \tag{11}$$

The desired ratio is $\alpha'/\alpha'' = 1$.

d) Comparing the results of a) and b) for the trapping and detrapping forces, for trapping to occur we must have,

$$\frac{\alpha'}{2} \frac{\partial \langle E^2 \rangle}{\partial z} > \frac{\pi}{\lambda} \alpha'' E_0^2 = \frac{2\pi}{\lambda} \alpha'' \langle E^2 \rangle. \tag{12}$$

From optics [8], we know that the characteristic length for changes in the electric field along the axis near the focus is the Rayleigh range,

$$z_0 \approx \pi f_\#^2 \lambda, \tag{13}$$

where the $f_\#$ is the usual f/D ratio of the focusing lens. That is, we can approximate,

$$\frac{\partial \langle E^2 \rangle}{\partial z} \approx \frac{\langle E^2 \rangle}{z_0} \approx \frac{\langle E^2 \rangle}{\pi f_\#^2 \lambda}. \tag{14}$$

The maximum trapping force occurs about one Rayleigh range away from the focus. Using (14) in (12), we find the requirement,

$$f_\# < \frac{1}{2\pi} \sqrt{\frac{\alpha'}{\alpha''}}. \tag{15}$$

From part c), if we run the trap at the frequency $\omega = \omega_0 - \gamma/2$ where the trapping term is maximal, the detrapping term is large: $\alpha'' = \alpha'$. In this case, we need,

$$f_\# < \frac{1}{2\pi} = 0.16, \tag{16}$$

which is a very strong focus! For $\omega < \omega_0 - \gamma/2$, the ratio α'/α'' grows rapidly, and a softer focus can be used. But the trapping is not so strong, so other detrapping effects become important.

A Appendix: Maxima of $\mathbf{E}^2$

We show that an electrostatic field $\mathbf{E}$ (or magnetostatic field $\mathbf{B}$) cannot have a local maximum of $E^2 = |\mathbf{E}|^2$ at a charge-free point.

The demonstration makes use of the mean-value theorem [9], that the average value of the electrostatic field in a charge-free sphere is equal to the value of the field at the center of the sphere.

If E^2 has a local maximum at some point P in a charge-free region, then there is a nonzero r such that $E^2 < E^2(P)$ for all points (other than P) within a sphere of radius r about P. Consequently, $E < E(P)$ in that sphere.

Let $\hat{\mathbf{z}}$ point along $\mathbf{E}(P)$. Then, the mean-value theorem can be written as,

$$\int E_z \, d\text{Vol} = \frac{4\pi r^3}{3} E(P), \tag{17}$$

for the sphere about P. In general, $E_z \le E$, and by assumption $E < E(P)$ for all points other than P within the sphere, so,

$$\int E_z \, d\text{Vol} \le \int E \, d\text{Vol} < \int E(P) \, d\text{Vol} = \frac{4\pi r^3}{3} E(P), \tag{18}$$

which contradicts eq. (17). Hence, E^2 cannot be locally maximal at P.

However, E^2 can take on a local minimum. This has been shown by explicit examples in Ref. [10], which also provides an alternative demonstration that E^2 cannot have a local maximum. A brief discussion of this issue in Ref. [11] shows that $\nabla^2 E^2 \ge 0$, which does not exclude the possibility that at a maximum both the first and second derivatives of E^2 vanish. However, as noted in refs. [11] and [12], the condition that $\nabla^2 E^2 \ge 0$ is sufficient to exclude the possibility of trapping.

B Appendix: The Ponderomotive Force and the Abraham Force *(Nov. 2017)*

We model the electric dipole $\mathbf{p}$ as consisting of a pair of electric charges $\pm q$ at positions $\mathbf{x} \pm \mathbf{d}/2$, where $\mathbf{p} = q\mathbf{d}$, and $\mathbf{x}$ is the position of the electric center of the dipole. Then, the force on the dipole due to external fields $\mathbf{E}$ and $\mathbf{B}$ is the sum of the Lorentz forces on the charges $\pm q$,

$$\begin{aligned}\mathbf{F} &= q[\mathbf{E}(\mathbf{x}+\mathbf{d}/2)-\mathbf{E}(\mathbf{x}-\mathbf{d}/2)]+q[\dot{\mathbf{x}}+\dot{\mathbf{d}}/2]\times\mathbf{B}(\mathbf{x}+\mathbf{d}/2)-q[\dot{\mathbf{x}}-\dot{\mathbf{d}}/2]\times\mathbf{B}(\mathbf{x}-\mathbf{d}/2)\\ &\to (\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}+\frac{d\mathbf{p}}{dt}\times\mathbf{B}+\mathbf{v}\times(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{B}, \qquad (19)\end{aligned}$$

in the limit of small $\mathbf{d}$, where $\mathbf{v}=\dot{\mathbf{x}}$ is the velocity of the center of the dipole. For a dipole at rest in only a static electric field, we recover the familiar form $\mathbf{F}=(\mathbf{p}\cdot\boldsymbol{\nabla})\mathbf{E}$.

In the rest of this Appendix we ignore the velocity $\mathbf{v}$ of the dipole, in which case $\partial/\partial t = d/dt$ for derivatives at the position of the dipole.

When $\mathbf{p}=\alpha\mathbf{E}$, eq. (19) becomes,

$$\mathbf{F}=\alpha(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E}+\alpha\frac{d\mathbf{E}}{dt}\times\mathbf{B}=\frac{\alpha}{2}\boldsymbol{\nabla}E^2+\alpha\frac{d}{dt}(\mathbf{E}\times\mathbf{B})=\frac{\alpha}{2}\boldsymbol{\nabla}E^2+\frac{d}{dt}(\mathbf{p}\times\mathbf{B}), \qquad (20)$$

noting that $\boldsymbol{\nabla}E^2=\boldsymbol{\nabla}(\mathbf{E}\cdot\mathbf{E})=2(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E}+2\mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})=2(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E}-2\mathbf{E}\times d\mathbf{B}/dt$.

The term $\alpha\boldsymbol{\nabla}E^2/2$ is called the ponderomotive force, and can lead to "trapping" at a minium of E^2, as at the focus of a laser beam. The term $(d/dt)(\mathbf{p}\times\mathbf{B})$ is called the Abraham force.

B.1 Time-Harmonic Fields

If the external fields have the form $\mathbf{E}_0\,e^{-i\omega t}$ and $\mathbf{B}_0\,e^{-i\omega t}$, then the time-average force is,

$$\langle\mathbf{F}\rangle=\frac{Re(\alpha)}{4}\boldsymbol{\nabla}\left|E_0\right|^2+\frac{1}{2}\frac{d}{dt}Re(\alpha\mathbf{E}_0\times\mathbf{B}_0^\star)=\frac{Re(\alpha)}{4}\boldsymbol{\nabla}\left|E_0\right|^2. \qquad (21)$$

That is, the time-average Abraham force vanishes in the case of time-harmonic electromagnetic fields.

However, this formal result is not consistent with the fact that if the dipole absorbs energy and momentum from the external fields, there is a force on the dipole. And, this force is nonzero for plane waves where $\boldsymbol{\nabla}\left|E_0\right|^2=0$.

For a plane wave (in vacuum), such as $\mathbf{E}(\mathbf{x},t)=E_0\,e^{i(kz-\omega t)}\,\hat{\mathbf{x}}$, $\mathbf{B}(\mathbf{x},t)=E_0\,e^{i(kz-\omega t)}\,\hat{\mathbf{y}}$ with wavelength $\lambda=2\pi/k$) large compared to the size of the electric dipole, it suffices to consider the dipole as being at the origin, where,

$$\mathbf{E}=E_0\,e^{-i\omega t}\,\hat{\mathbf{x}},\qquad \mathbf{B}=\frac{E_0}{c}\,e^{-i\omega t}\,\hat{\mathbf{y}},\qquad \text{and}\qquad \mathbf{p}=\alpha E_0\,e^{-i\omega t}\,\hat{\mathbf{x}}=e\mathbf{x}(t), \qquad (22)$$

where we now suppose the oscillating dipole is equivalent to a charge e at distance $\mathbf{x}(t)$ from the nucleus (at the origin). The velocity of the moving charge is, of course, $\dot{\mathbf{x}}=\dot{\mathbf{p}}/e$.

The wave field does work on the dipole at time-average rate,

$$\left\langle\frac{dU}{dt}\right\rangle=\frac{1}{2}Re(e\mathbf{E}\cdot\dot{\mathbf{x}}^\star)=\frac{1}{2}Re(\mathbf{E}\cdot\dot{\mathbf{p}}^\star)=\frac{1}{2}Re(\mathbf{E}\cdot(i\omega\alpha^\star)\mathbf{E}^\star)=\frac{\omega E_0^2}{2}Im(\alpha). \qquad (23)$$

Since the momentum density in the wave equals its energy density times $\hat{\mathbf{z}}/c$, we infer that the momentum absorbed by the dipole, and the force on it, is given by,

$$\langle \mathbf{F}_{\mathrm{absorb}} \rangle = \left\langle \frac{dU}{dt} \right\rangle \frac{\hat{\mathbf{z}}}{c} = \frac{\omega E_0^2}{2c} Im(\alpha)\,\hat{\mathbf{z}} = \frac{k E_0^2}{2} Im(\alpha)\,\hat{\mathbf{z}}. \tag{24}$$

That is, we should expect the force on the dipole to include the force of eq. (24), which was somehow missed in the derivation of eq. (21).

On the other hand, if we return to the derivation of eq. (19), note that the $\mathbf{v} \times \mathbf{B}$ force on the moving charge is,

$$\mathbf{F}_{\mathbf{v}\times\mathbf{B}} = e\dot{\mathbf{x}} \times \mathbf{B} = \dot{\mathbf{p}} \times \mathbf{B}, \tag{25}$$

whose time average is,

$$\langle \mathbf{F}_{\mathbf{v}\times\mathbf{B}} \rangle = \frac{1}{2} Re\,(\dot{\mathbf{p}} \times \mathbf{B}^\star) = \frac{1}{2} Re\left(-i\omega\alpha \frac{E_0^2}{c}\right)\hat{\mathbf{z}} = \frac{k E_0^2}{2} Im(\alpha)\,\hat{\mathbf{z}} = \langle \mathbf{F}_{\mathrm{absorb}} \rangle \tag{26}$$

Thus, it appears that in the transformation of eq. (19) to eq. (20) via the vector-calculus identity, $(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E} = \mathbf{E} \times d\mathbf{B}/dt + \boldsymbol{\nabla} E^2/2$, a more physical result is obtained if we simply replace $(\mathbf{E}\cdot\boldsymbol{\nabla})\mathbf{E}$ by $\boldsymbol{\nabla} E^2/2$, at least in the case of time-harmonic fields.[2]

An issue with the approach that led to eq. (19) is the use of the "external" electromagnetic fields, when in fact the interaction of these fields with the dipole modifies them, and makes the time-average derivative $\langle (d/dt)(\mathbf{E} \times \mathbf{B}) \rangle$ nonzero. A practical question is how to use only the "external" fields, yet arrive at a result that accommodates their interaction with the dipole. Equations (19)-(21) do not do this well, while it appears (to this author) that the form,

$$\mathbf{F} = \frac{\alpha}{2} \boldsymbol{\nabla} E^2 + \dot{\mathbf{p}} \times \mathbf{B}. \tag{27}$$

is a better approximation.

B.2 Has the Abraham Force Been Observed in Experiment?

In a broad sense, the Abraham force is any force on nonconducting matter other than the ponderomotive force $\boldsymbol{\nabla}(\alpha E^2/2)$, which latter vanishes for uniform electric fields. It is therefore related to a "radiation pressure."

B.2.1 Radiation Pressure

Apparently, Kepler considered the pointing of comets' tails away from the Sun as evidence for radiation pressure of light [13].[3] After his unification of electricity, magnetism and light [15], Maxwell argued (sec. 792 of [16]) that the radiation pressure P of light is equal to its energy density u,

$$P = u = \frac{D^2}{4\pi} = \frac{H^2}{4\pi} \tag{28}$$

[2] The latter substitution was used, perhaps naïvely, in sec. 2b) above.

[3] Several inconclusive attempts to confirm Kepler's conjecture are reviewed in [14].

for an electromagnetic wave with fields **D** and **H** in vacuum. The first convincing experimental evidence for the radiation pressure of light was given by Lebedev in 1901 [17], and confirmed by Nichols and Hull [18].[4]

B.2.2 The Abraham Force

Observation of the Abraham force has been reported in an experiment in which an electromagnetic field oscillated at 0.3 Hz in the presence of a static magnetic field [22, 23, 24, 25].

A recent report [26] claims to have observed an Abraham force of a laser beam on a liquid-core optical fiber, although this effect seems to this author to be more of a radiation pressure.

Acknowledgement

The author wishes to thank Paul Chaikin and Will Happer for discussions of this problem, and the Editor of the American Journal of Physics for pointing out refs. [10, 11, 12] which discuss the "well known to those who know well" theorem [12] of Appendix A.

References

[1] S. Earnshaw, *On the Nature of the Molecular Forces which regulate the Constitution of the Luminiferous Ether*, Trans. Camb. Phil. Soc. **7**, 97 (1839), particularly secs. 11-15. http://kirkmcd.princeton.edu/examples/EM/earnshaw_tcps_7_97_39.pdf

[2] W.R. Smythe, *Static and Dynamic Electricity*, 3rd ed. (McGraw-Hill), 1968), sec. 1.11, http://kirkmcd.princeton.edu/examples/EM/smythe_50.pdf

[3] A. Ashkin, *Trappings of Atoms by Resonance Radiation Pressure*, Phys. Rev. Lett. **40**, 729 (1978), http://kirkmcd.princeton.edu/examples/optics/ashkin_prl_40_729_78.pdf

[4] L.T. Wood, *Electromagnetic Acceleration of Neutral Molecules*, Am. J. Phys. **42**, 1020 (1974), http://kirkmcd.princeton.edu/examples/EM/wood_ajp_42_1020_74.pdf

[5] A. Ashkin *et al.*, *Observation of a single-beam gradient force optical trap for dielectric particles*, Opt. Lett. **11**, 288 (1986), http://kirkmcd.princeton.edu/examples/optics/ashkin_ol_11_288_86.pdf

[6] Y. Shimizu and H. Sasada, *Mechanical force in laser cooling and trapping*, Am. J. Phys. **66**, 960 (1998), http://kirkmcd.princeton.edu/examples/optics/shimizu_ajp_66_960_98.pdf

[7] S.P. Smith *et al.*, *Inexpensive optical tweezers for undergraduate laboratories*, Am. J. Phys. **67**, 26 (1999), http://kirkmcd.princeton.edu/examples/optics/smith_ajp_67_26_99.pdf

[4] The famous Crookes radiometer [19] does not demonstrate electromagnetic radiation pressure, as argued by Schuster [20]. See also [21].

[8] A.E. Siegman, *Lasers* (University Science Books, Mill Valley, CA, 1986), sec. 17.2,
http://kirkmcd.princeton.edu/examples/EM/siegman_86.pdf

[9] J.D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, 1999), sec. 4.1,
http://kirkmcd.princeton.edu/examples/EM/jackson_ce3_99.pdf

[10] W.H. Wing, *On Neutral Particle Trapping in Quasistatic Electromagnetic Fields*, Prog. Quant. Electr. **8**, 181 (1984),
http://kirkmcd.princeton.edu/examples/EM/wing_pqe_8_181_84.pdf

[11] W. Ketterle and D.E. Pritchard, *Trapping and Focusing Ground State Atoms with Static Fields*, Appl. Phys. B **54**, 403 (1992),
http://kirkmcd.princeton.edu/examples/optics/ketterle_ap_b54_403_92.pdf

[12] M.V. Berry and A.K. Geim, *Of flying frogs and levitrons*, Eur. J. Phys. **18**, 307 (1997),
http://kirkmcd.princeton.edu/examples/mechanics/berry_ejp_18_307_97.pdf

[13] J. Kepler, *De Cometis* (1619),
http://kirkmcd.princeton.edu/examples/astro/kepler_de_cometis.pdf

[14] R.V. Jones, *Pressure or Radiation*, Nature **171**, 1089 (1953),
http://kirkmcd.princeton.edu/examples/EM/jones_nature_171_1089_53.pdf

[15] J.C. Maxwell, *A Dynamical Theory of the Electromagnetic Field*, Phil. Trans. Roy. Soc. London **155**, 459 (1865), http://kirkmcd.princeton.edu/examples/EM/maxwell_ptrsl_155_459_65.pdf

[16] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2, 3rd ed. (Clarendon Press, 1892), secs. 618 and 792, kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_92.pdf

[17] P. Lebedew, *Untersuchen über die Druckkräfte des Lichtes*, Ann. Phys. **6**, 433 (1901),
http://kirkmcd.princeton.edu/examples/EM/lebedev_ap_6_433_01.pdf

[18] E.E. Nichols and G.F. Hull, *The Pressure due to Radiation*, Ap. J. **17**, 315 (1903),
http://kirkmcd.princeton.edu/examples/EM/nichols_apj_17_315_03.pdf
Phys. Rev. **17**, 26,91 (1903),
http://kirkmcd.princeton.edu/examples/EM/nichols_pr_17_26_03.pdf
http://kirkmcd.princeton.edu/examples/EM/nichols_pr_17_91_03.pdf

[19] W. Crookes, *On Repulsion resulting from Radiation*, Phil. Trans. Roy. Soc. London **166**, 325 (1876), http://kirkmcd.princeton.edu/examples/optics/crookes_ptrsl_166_325_76.pdf

[20] A. Schuster, *On the Nature of the Force producing the Motion of a Body exposed to Rays of Heat and Light*, Phil. Trans. Roy. Soc. London **166**, 715 (1876),
http://kirkmcd.princeton.edu/examples/EM/schuster_ptrsl_166_715_76.pdf

[21] A.E. Woodruff, *William Crookes and the Radiometer*, Isis **57**, 188 (1966),
http://kirkmcd.princeton.edu/examples/EM/woodruff_isis_57_188_66.pdf

[22] G.B. Walker and D.G. Lahoz, *Experimental observaion of Abraham force in a dielectric*, Nature **253**, 339 (1975), http://kirkmcd.princeton.edu/examples/EM/walker_nature_253_339_75.pdf

[23] G.B. Walker and D.G. Lahoz, *Measurement of the Abraham Force in a Barium Titanate Specimen*, Can. J. Phys. **53**, 2577 (19775),
http://kirkmcd.princeton.edu/examples/EM/walker_cjp_53_2577_75.pdf

[24] G.B. Walker and G. Walker, *Mechanical forces of electromagnetic origin*, Nature **263**, 401 (1976), http://kirkmcd.princeton.edu/examples/EM/walker_nature_263_401_76.pdf

[25] G.B. Walker and G. Walker, *Mechanical forces in a dielectric due to electromagnetic fields*, Can. J. Phys. **55**, 2121 (1977),
http://kirkmcd.princeton.edu/examples/EM/walker_cjp_55_2121_77.pdf

[26] H. Choi *et al.*, *Optomechanical measurement of the Abraham force in an adiabatic liquid-core optical-fiber waveguide*, Phys. Rev. A **95**, 053817 (2017),
http://kirkmcd.princeton.edu/examples/EM/choi_pra_95_053817_17.pdf