A Capacitor Paradox

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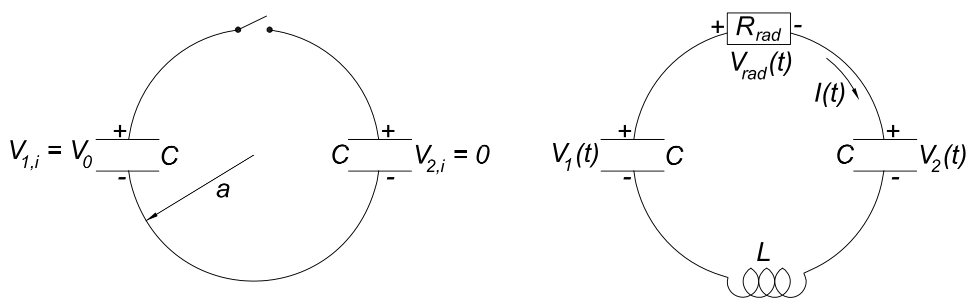
1 Problem

Two capacitors of equal capacitance C are connected in parallel by wires of negligible resistance and a switch, as shown in the lefthand figure below. Initially the switch is open, one capacitor is charged to voltage V_0, and charge $Q_0 = CV_0$, while the other is uncharged. At time $t = 0$ the switch is closed. If there were no damping (dissipative) mechanism, the circuit would then oscillate forever, at a frequency dependent on the self inductance $L \approx \mu_0 \ln a/b$ of the loop of radius a of wire of radius $b \ll a$ and the total capacitance $C_{\rm tot} \approx C/2$, namely $f = \omega/2\pi \approx 1/2\pi\sqrt{LC_{\rm tot}} \approx 1/2\pi\sqrt{(\mu_0 C/2)\ln(a/b)} \approx 200$ Hz$/\sqrt{C \ln(a/b)}$ for C in farards. However, even in a circuit with zero Ohmic resistance, damping occurs due to the radiation of the oscillating charges, and eventually a static charge distribution results, with charge $Q_0/2$ and voltage $V_0/2$, on each capacitor.



The "paradox" is that the final stored energy is $U_{\rm f} = 2(CV_{\rm f}^2/2) = CV_0^2/4 = U_{\rm i}/2$, where $U_{\rm i} = CV_0^2/2$ is the initial stored energy.[1] Hence, half the initial energy is "missing" in the final state.

Where is the "missing" energy?[2]

2 Solution

This problem is (in the view of this author) meant to illustrate the limitations of "ordinary" circuit analysis,[3] and has been discussed many times, including [1]-[40]. A substantial fraction of these papers argue that "ordinary" circuit analysis suffices for a practical understanding of the two-capacitor problem, remarking that if the circuit contains a large enough

[1]If the two capacitances were unequal, more than half of the initial energy would go "missing". Better energy efficiency while charging a capacitor can be obtained using nonlinear circuit elements, as in sec. 9.1 of http://www.ti.com/lit/ds/symlink/lm2664.pdf.

[2]This problem can also be posed for a single capacitor that is initially charged with $\pm Q$ on its plates, and then discharged by "shorting" its terminals with a wire. This can be dangerous, so "don't try this at home". That is, a spark generally occurs during the discharge, which is a clue that the physics here can be intricate. The experiment discussed in sec. 2.3 below is for the single-capacitor version of the "paradox".

[3]Another example that illustrates the limitations of "ordinary" circuit analysis is [41].

Ohmic resistance, the associated Joule heating accounts for essentially all of the "missing" energy.[4,5]

Recall that in Poynting's view [42], the energy the energy that is transferred from one capacitor to the other passes through the intervening space, not down the connecting wires. In the present example, some of the energy in the electrostatic field of the initially charged capacitor "escapes" from the circuit in the form of electromagnetic radiation.[6] Hence, we should also examine the possibility that radiation carries away a significant fraction of the "missing" energy.[7]

2.1 Ordinary Circuit Analysis of the Two-Capacitor Problem

If the quantity labeled $R_{\rm rad}$ in the circuit diagram on p. 1 were an ordinary resistor of value R, then the circuit equation would be (for $t \geq 0$),

$$-V_1 + V_2 + L\dot{I} + IR = 0, \qquad \frac{Q}{C} + \frac{L\ddot{Q}}{2} + \frac{R\dot{Q}}{2} = 0, \tag{1}$$

where $L \approx \mu_0 \ln(a/b)$ is the self inductance of the circuit, $Q \equiv Q_2 - Q_1$ and we note that $I = \dot{Q}_2 = -\dot{Q}_1 = \dot{Q}/2$. The initial conditions are $Q(0) = -Q_1(0) = -CV_0$ and $\dot{Q}(0) = 0$. Use of a trial solution of the form $e^{i\omega t}$ leads to,

$$\omega = \frac{iR}{2L} \pm \omega_0, \qquad \omega_0 \equiv \sqrt{\frac{2}{LC} - \frac{R^2}{4L^2}}, \tag{2}$$

so the (real) solution that obeys the initial conditions can be written as,

$$Q(t \geq 0) = -CV_0\, e^{-Rt/2L}\left(\cos\omega_0 t + \frac{R}{2L\,\omega_0}\sin\omega_0 t\right), \quad I(t \geq 0) = \frac{\dot{Q}}{2} = \frac{V_0}{L\,\omega_0}\, e^{-Rt/2L}\sin\omega_0 t. \tag{3}$$

Then, $Q_{\rm f} = 0$, $Q_{1,\rm f} = CV_{1,\rm f} = Q_{2,\rm f} = CV_{2,\rm f}$, and hence $V_{1,\rm f} = V_{2,\rm f}$. The energy dissipated by the resistor R is,

$$\Delta U \;=\; \int_0^\infty I^2 R\, dt = \frac{V_0^2 R}{L^2\,\omega_0^2}\int_0^\infty e^{-Rt/L}\sin^2\omega_0 t\, dt = \frac{V_0^2 R}{L^2\,\omega_0^2}\frac{2\omega_0^2}{(R/L)(R^2/L^2 + 4\omega_0^2)}$$

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[4]The YouTube video https://www.youtube.com/watch?v=cmverrUVOQA adds a motor + mechanical load to the circuit, so that the "missing" energy can be "seen" as the mechanical work done after the switch is closed, thereby avoiding the need to consider radiation or even Joule heating, which concepts the video author finds too abstract. Yet, this author gets it right that a dissipative mechanism is required for the circuit to end up with only half of the initial stored field energy.

[5]Two papers [25, 39] supposed that the switch is capable of dissipating energy, via an effective resistance (not modeled in the papers). No mention was made of radiation, or of a spark or related plasma physics.

[6]As noted in [13], when a capacitor is discharged near a radio, the latter detects a burst of noise at any frequency, associated with the initial "switching" transients that last a few nsec. See also the caption of Fig. 3 of [7]. For additional commentary on this phenomenon, see [43].

[7]Some authors [12, 17, 18, 20, 21, 30, 31, 32, 34] have argued that the two-capacitor problem is analogous to the "two-tank problem," in which water is transfered from one tank to another via a connecting pipe (although this "plumbing analogy" was objected to already in [13]). If the water were frictionless, the eventual "missing" potential energy (*i.e.*, gravitational-field energy) would be radiated away by gravitational waves. Since this is a very weak process, the frictionless water would oscillate from one tank to the other for a very long time, before eventually coming to equilibrium with each tank half full. In practice, the friction (viscosity) of water is large enough that there would be no observable oscillation of the water (*i.e.*, overdamped "oscillation").

$$= \frac{CV_0^2}{4} = \frac{U_{\rm i}}{2}, \qquad \text{and so} \qquad U_{\rm f} = \frac{CV_0^2}{4} = CV_{1,\rm f}^2, \qquad V_{1,\rm f} = \frac{V_0}{2} = V_{2,\rm f}, \tag{4}$$

using Dwight 861.10 [44].[8] Thus, if the voltage drop associated with the dissipative mechanism has the form IR for a constant R, the dissipated energy equals the "missing" energy $U_{\rm i}/2 = CV_0^2/4$. It does not, however, follow that this demonstrates R to be purely an Ohmic resistance.

Indeed, for low Ohmic resistance, the current in the circuit would perform a damped oscillation with nominal angular frequency $\omega_0 \approx \sqrt{2/LC}$, and the associated electric and magnetic dipole radiation would have power well described by $P_{\rm rm}(t) = I^2(t) R_{\rm rad}$ where $R_{\rm rad}$ is a constant with dimensions of electrical resistance.

2.2 Model Calculation of Magnetic Dipole Radiation

We assume that the wires form a circle of radius a and we neglect charge accumulation in the wires compared to that on the capacitor plates. In this approximation the current in the wires is spatially uniform, and the total electric dipole moment of the system (with symmetrically arrayed capacitors) is constant. Then, electric dipole radiation does not exist, and magnetic dipole radiation dominates.

The "radiation resistance" of this circuit causes a voltage drop $V_{\rm rad}$ within the circuit that can be identified as,

$$V_{\rm rad}(t) = \frac{P_{\rm rad}(t)}{I(t)} = I(t)\frac{P_{\rm rad}(t)}{I^2(t)} \equiv I(t) R_{\rm rad}\,, \tag{5}$$

where $P_{\rm rad}$ is the radiated power, $I(t)$ is the current in the wire, and the radiation resistance is $R_{\rm rad} = P/I^2$. The latter is constant in the further approximation that the damping time is large compared to the period of oscillation of the current, *i.e.*, $\ddot I \approx -\omega_0^2 I \approx 2I/LC$.

To estimate the radiated power we note that the magnetic moment m of the circuit is (in Gaussian units),

$$m(t) = \frac{\pi a^2 I(t)}{c}\,, \tag{6}$$

where c is the speed of light. According to the Larmor formula [45], the radiated power is,

$$P_{\rm rad} = \frac{2\ddot m^2}{3c^3} = \frac{2\pi^2 a^4 \ddot I^2}{3c^5} \approx \frac{2\pi^2 a^4 \omega_0^4 I^2}{3c^5}\,. \tag{7}$$

The radiation resistance is,

$$R_{\rm rad} = \frac{P_{\rm rad}}{I^2} \approx \frac{2\pi^2}{3c}\left(\frac{a\omega_0}{c}\right)^4 = \frac{2^5\pi^6}{3c}\left(\frac{a}{\lambda}\right)^4 \approx 3\times 10^5 \left(\frac{a}{\lambda}\right)^4\ \Omega, \tag{8}$$

noting that $\omega_0 = 2\pi c/\lambda$, and $1/c$ in Gaussian units equals 30 Ω.

While this radiation resistance appears large at first glance, in practice a/λ (the ratio of the size of the circuit compared to the wavelength of the radiation) will be quite small, and the circuit would oscillate a very long time before the "missing" energy $CV_0^2/4$ would be radiated away.

[8] In [1], the self inductance was ignored, so the resulting circuit equation $\dot Q = -2Q/RC$ has the solution $Q(t \ge 0) = -CV_0\, e^{-2t/RC}$, with $I(t \ge 0) = \dot Q/2 = (V_0/R)\, e^{-2t/RC}$. Then, the total energy dissipated by the resistor R is, $\Delta U = \int_0^\infty I^2 R\, dt = (V_0^2/R)\int_0^\infty e^{-4t/RC}\, dt = CV_0^2/4 = U_{\rm i}/2$.

2.3 An Experiment (Jan. 11, 2018)

Hence, it is useful to consider experimental data in the literature related to the two-capacitor problem, such as Fig. 3 of [7], shown on left below, where the current trace has 10 μs per horizontal (time) division. This experiment was on the "short circuit" discharge of a single capacitor with $C = 11.5\ \mu$F, where the observed frequency of the damped oscillations was $\omega/2\pi = f = 41$ kHz ($\lambda = 7.3 \times 10^5$ cm), and the damping time was observed to be approximately two periods, $\tau \approx 2/f$. Considering the equivalent circuit to be a series R-L-C circuit, the current $I(t > 0) = K\, e^{-\alpha t}$, obeys (for complex constants K and α),

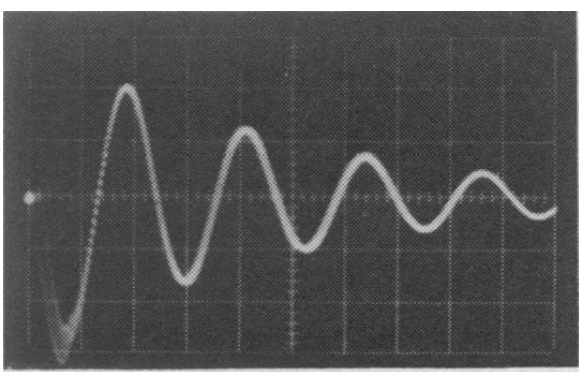
$$\ddot{I}L + IR + \frac{Q}{C} = 0, \qquad \alpha = \frac{R}{2L} \pm i\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \approx \frac{R}{2L} \pm \frac{i}{\sqrt{LC}} \equiv \beta \pm i\omega, \tag{9}$$

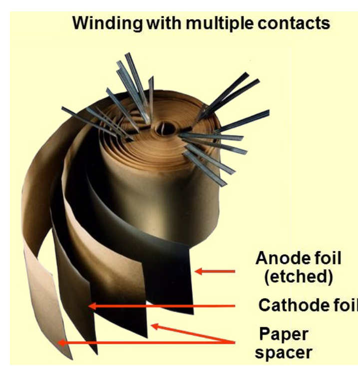
where the approximation holds for small resistance R, as in this example. The discharge begins at time $t = 0$, when $I_0 = 0$ and $Q_0 = CV_0$, and the waveforms are, using Dwight 575.1 [44],[9]

$$I(t > 0) = A\, e^{-\beta t} \sin\omega t = -\frac{CV_0(\beta^2 + \omega^2)}{\omega}\, e^{-\beta t} \sin\omega t \approx -CV_0\, \omega\, e^{-Rt/2L} \sin\omega t, \tag{10}$$

$$Q(t > 0) = Q_0 + \int_0^t A\, e^{-\beta t'} \sin\omega t'\, dt' = CV_0\, e^{-Rt/2L} \left(\cos\omega t + \frac{R}{2\omega L} \sin\omega t\right). \tag{11}$$

Then, the observed frequency implies that the self inductance of the circuit was $L = 1/4\pi^2 f^2 C = 1.3\ \mu$H (consistent with the circuit being a loop of 2-cm radius made of 24-gauge wire), and the observed damping time implies that the effective resistance was $R = fL = 0.05\ \Omega$.





The wires in the circuit were stated to be "very short," such that it is implausible that the Ohmic resistance of the circuit was 0.05 Ω (for example, the resistance of 2000 feet of 24-gauge wire is 0.05 Ω). However, the *conventional capacitor contained a "rolled up" sandwich of foil and dielectric*, for which the equivalent series resistance of the thin foil was very plausibly close to the observed 50 mΩ.[10] In contrast, the radiation resistance (8) is only $1.7 \times 10^{-17}\ \Omega$ for $a = 2$ cm and $\lambda = 7.3 \times 10^5$ cm.

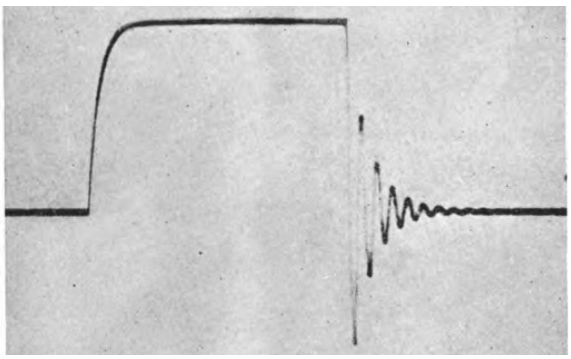
[9]Note that the damping time constant is $2L/R$ and not RC. That is, most textbook discussions of the discharge of a capacitor are naïve in ignoring the self inductance, and claiming that $Q(t > 0) = CV_0\, e^{-t/RC}$.

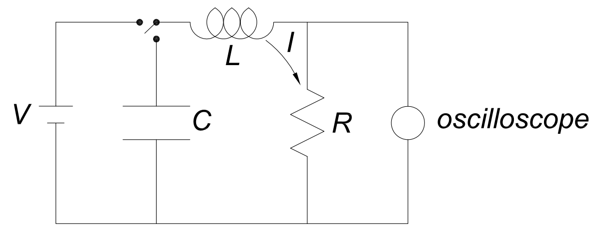
[10]https://en.wikipedia.org/wiki/Aluminum_electrolytic_capacitor

This supports the view in many of the discussions of the two-capacitor problem [1]-[38] that the Ohmic resistance of the circuit dissipates the vast majority of the "missing" energy (unless, of course, the electrical circuit is used to drive a nonelectrical load that dissipates the energy, as in footnote 4 above).

2.3.1 Another Experiment (July 1, 2022)

Another experiment involving a capacitor in a series L-R-C circuit was reported in Fig. 81, p. 226 of [46]. The figure caption was somewhat ambiguous, but I believe the circuit studied was that shown on the right below.





At time $t = 0$, a switch connected a battery of voltage V to an inductance L and resistance R in series. The voltage drop IR across the resistor was observed with an oscilloscope (of large input impedance). The current waveform corresponding to this position of the switch was,

$$I(t > 0) = \frac{V}{R}\left(1 - e^{Rt/L}\right), \tag{12}$$

as reviewed in eq. (414), p. 219 of [46], and seen on the left side of the photo (oscillogram) above.

At time t_1 large enough that the current I was essentially V/R, the switch was thrown so as to disconnect the battery and connect L and R to a capacitor C that was initially uncharged. The circuit equation for $t > t_1$ again follows from eq. (9), whose solution can now be written as,

$$I(t > t_1) = \frac{V}{R} e^{-R(t-t_1)/2L} \cos\omega(t - t_1), \qquad \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}, \tag{13}$$

such that $I(t_1) = V/R$. The waveform for $t > t_1$ is that seen in the right side of the photo above, and is oscillatory (rather than a simple exponential as would be inferred if the inductance were neglected).

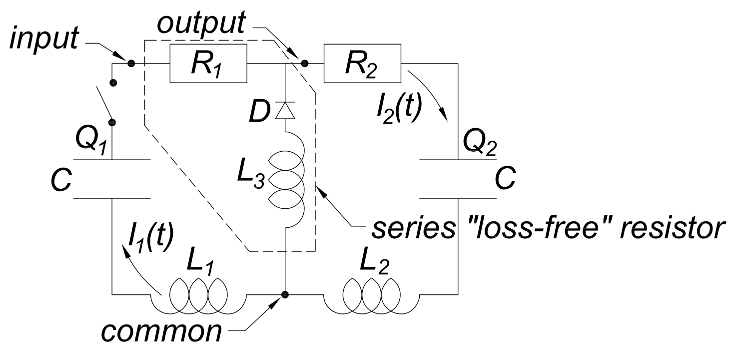
A Appendix: Loss-Free Resistor (June 30, 2022)

This Appendix was inspired by e-discussions with Ivo Barbi and Sigmond Singer

Circuits have been developed that are called "loss-free resistors". The original version [47, 48] is a 4-terminal device that appears to the two input terminals as a resistance R, while transferring power I^2R to a load connected to the two output terminal, where I is the input current. Another version is a 3-terminal device [49], with input, output and common terminals, where this "series loss-free resistor" appears to be between the input and output terminals. Both of these devices are not totally loss free, and the "series loss-free resistor" of [49] operates only for currents of one sign.

We now consider the two-capacitor circuit with a series "loss-free" resistor in the configuration of [49], which involves two circuit loops.[11]

Nominally, R_1 is "loss free", although it includes a small dissipative resistance, while R_2 is a small dissipative resistance.[12] Each loop has self inductance, some of which is shared between the two loops, which we indicate by the three inductances L_j in the figure below.[13] A diode D next to L_3 keeps the current $I_1 - I_2$ through this inductor always negative.



Initially, $Q_1 = CV_0$, $Q_2 = 0$, and the switch is open, to be closed at time $t = 0$. The steady-state voltage and charge on the two capacitors are equal.

In the approximation that the system is actually "lossless", the final (positive) charge on each capacitor is $Q_1(0)/\sqrt{2}$.[14] The total final positive charge is greater than the initial positive charge, which is possible in the two-loop circuit with the series "loss-free" resistor, whose internal circuitry can generate additional charge separation.[15]

After the switch is closed at time $t = 0$, the circuit relations are, so long as $I_2 - I_1 > 0$,

$$I_1 = -\dot{Q}_1, \qquad I_2 = \dot{Q}_2, \tag{14}$$

$$L_1\dot{I}_1 + L_3(\dot{I}_1 - \dot{I}_2) + I_1 R_1 - \frac{Q_1}{C} = 0, \qquad (L_1 + L_3)\ddot{I}_1 - L_3\ddot{I}_2 + R_1\dot{I}_1 + \frac{I_1}{C} = 0, \tag{15}$$

$$L_2\dot{I}_2 + L_3(\dot{I}_2 - \dot{I}_1) + I_2 R_2 + \frac{Q_2}{C} = 0, \qquad (L_2 + L_3)\ddot{I}_2 - L_3\ddot{I}_1 + R_2\dot{I}_2 + \frac{I_2}{C} = 0. \tag{16}$$

Supposing the currents I_j have the time dependences as (the real part of) ...

[11]If a series "loss-free" resistor were used in a simple R-L-C circuit, its output and common terminals would have to be shorted together, such that there would be no "load" to accept the I^2R power. That is, there cannot be a "loss-free" R-L-C circuit with a series "loss-free" resistor.

[12]The dissipative resistances are partly Ohmic and partly due to radiation.

[13]The complex internal circuitry of the series "loss-free" resistor is not shown in the figure. See [49].

[14]Of course, a capacitor supports equal and opposite charges on its two plates, such that the total charge associated with a capacitor is zero. The total charge in the two-capacitor circuit is zero at all times.

We follow the usual convention in describing the positive charge on one of the capacitor plates as "the" charge of the capacitor.

[15]Recall that for the classic two-capacitor problem in a single-loop circuit, $Q_1(\infty) + Q_2(\infty) = Q_1(0)$, and half the initial energy has been "lost" in the final configuration.

$I_j(t \geq 0) = K_j\, e^{\alpha t}$ for complex constants α and K_j, the loop equations take the forms,[16]

$$\alpha^2(L_1 + L_3)K_1 - \alpha^2 L_3 K_2 + \alpha R_1 A_1 + \frac{K_1}{C} = 0, \tag{17}$$

$$\alpha^2(L_2 + L_3)K_2 - \alpha^2 L_3 K_1 + \alpha R_2 A_2 + \frac{K_2}{C} = 0. \tag{18}$$

We have two simultaneous equations in the two unknowns K_1 and K_2 (in terms of the unknown parameter α),

$$\left[\alpha^2(L_1 + L_3) + \alpha R_1 + \frac{1}{C}\right] K_1 - \alpha^2 L_3 K_2 = 0, \tag{19}$$

$$-\alpha^2 L_3 K_1 + \left[\alpha^2(L_2 + L_3) + \alpha R_2 + \frac{1}{C}\right] K_2 = 0. \tag{20}$$

For a solution to exist, the determinant of the 2×2 coefficient matrix must vanish,

$$\begin{aligned} 0 &= \left[\alpha^2(L_1 + L_3) + \alpha R_1 + \frac{1}{C}\right]\left[\alpha^2(L_2 + L_3) + \alpha R_2 + \frac{1}{C}\right] - \alpha^4 L_3^2 \\ &= \alpha^4(L_1 L_2 + L_1 L_3 + L_2 L_3) + \alpha^3[R_1(L_2 + L_3) + R_2(L_1 + L_3)] \\ &\qquad + \alpha^2\left[R_1 R_2 + \frac{L_1 + L_2 + 2L_3}{C}\right] + \alpha\frac{R_1 + R_2}{C} + \frac{1}{C^2}\,. \end{aligned} \tag{21}$$

This is a quartic equation for α, for which an analytic solution exists, but which is very cumbersome. Instead, we content ourselves with a numerical example, based on parameters of a proposed experiment (Ivo Barbi, private communication) in which the series "loss-free" resistor has $R_1 = 150\ \Omega$, and $C = 1$ mF. We take $R_2 = 0.05\ \Omega$, as in the experiment described in sec. 2.3 above. As noted in sec. 1 above, the self inductances are of order $\mu_0 \ln(a/b)$, and we take $L_1 = L_2 = 2L_3 = 5\ \mu$H. With these parameters, the quartic equation is (in SI units),

$$5 \times 10^{-11}\alpha^4 + 0.075\,\alpha^3 + 7.5\,\alpha^2 + 1.5 \times 10^5\alpha + 1 \times 10^6 = 0. \tag{22}$$

Because the terms higher than linear are small (and $R_2 \ll R_1$), there is a solution with no oscillation and simple exponential damping with time constant $-1/\alpha \approx R_1 C = 0.15$ s; however, this requires the charge on both capacitors to decrease with time, and so is not the solution we seek. Other solutions (found by several online quartic-equation solvers) include,

$$\alpha \approx -50 \pm 1500\,i \equiv -\beta + i\,\omega_0. \tag{23}$$

corresponding to rapidly damped oscillations,

$$I_j(t \geq 0) = A_j\, e^{-\beta t} \sin \omega_0 t, \tag{24}$$

with real, positive constants A_j and about 10 oscillations per damping time constant.[17]

[16] As will be seen below, this assumption is not consistent with $I_2 - I_1 > 0$ at all times.

[17] This behavior is similar to that observed in the discharge of a single capacitor, as discussed in sec. 2.3 above. There is also an unphysical solution with $\alpha \approx -10^7$.

Use of eq. (24) for all $t > 0$ would imply that the charges Q_j on the two capacitors obey, recalling eq. (14),

$$Q_1(t \geq 0) = Q_1(0) - \int_0^t I_1(t')\,dt' = Q_1(0) - A_1 \int_0^t e^{-\beta t'} \sin\omega_0 t'\,dt'$$
$$= Q_1(0) - \frac{A_1\,\omega_0}{\omega_0^2 + \beta^2} + \frac{A_1\,\omega_0\,e^{-\beta t}}{\omega_0^2 + \beta^2}\left(\cos\omega_0 t - \frac{\beta}{\omega_0}\sin\omega_0 t\right), \tag{25}$$
$$Q_2(t \geq 0) = Q_2(0) + \int_0^t I_1(t')\,dt' = A_2 \int_0^t e^{-\beta t'} \sin\omega_0 t'\,dt'$$
$$= \frac{A_2\,\omega_0}{\omega_0^2 + \beta^2} - \frac{A_2\,\omega_0\,e^{-\beta t}}{\omega_0^2 + \beta^2}\left(\cos\omega_0 t - \frac{\beta}{\omega_0}\sin\omega_0 t\right), \tag{26}$$

using Dwight 575.1 [44]. The final charges as $t \to \infty$ are,

$$Q_1(\infty) = Q_1(0) - \frac{A_1\,\omega_0}{\omega_0^2 + \beta^2} \approx Q_1(0) - \frac{A_1}{\omega_0} \qquad Q_2(\infty) = \frac{A_2\,\omega_0}{\omega_0^2 + \beta^2} \approx \frac{A_2}{\omega_0}\,. \tag{27}$$

As noted earlier, the design of the series "loss-free" resistor of [49] is such that its input and output terminals are connected by effective resistance R_1, which means that the final (steady-state) voltages on the two capacitors are the same. For two capacitors of the same capacitance, this implies that $Q_1(\infty) = Q_2(\infty)$, and hence that,

$$A_2 = -A_1 + \frac{Q_1(0)\,(\omega_0^2 + \beta^2)}{\omega_0} \approx -A_1 + Q_1(0)\,\omega_0. \tag{28}$$

Also, as noted earlier, the above analysis is valid only if $I_2 - I_1 > 0$, to be consistent with the presence of the diode in the circuit. However, in our model, $I_2 - I_1 = (A_2 - A_1)\,e^{-\beta t}\sin\omega_0 t$, which is negative half of the time for any values of A_1 and A_2. Hence, the above analysis cannot be valid for all times, and it is perhaps a matter for experiment to determine what actually happens in the two-capacitor circuit with a series "loss-free" resistor.[18]

B Appendix: A Two-Inductor Circuit (July 4, 2022)

This Appendix is based on an e-print by Ivo Barbi [50].

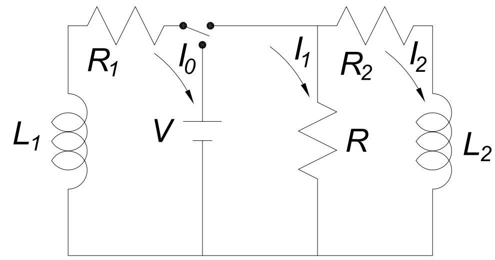
A circuit somewhat analogous (dual) to that of the two capacitors on p. 1 above, but with two inductors rather than two capacitors, is sketched below.

For time $t < 0$ the switch connects the battery to inductor L_1, such that the steady-state current through that inductor (for $t < 0$) is $I_0 = V/R_1$. Hence, magnetic energy $U_0 = L_1 I_0^2/2$ exists in the circuit at $t = 0$.

[18] In the approximation that the circuit is loss free, $Q_1(\infty) = Q_0/\sqrt{2} = Q_2(\infty)$, so from eq. (27) we would have,

$$A_2 = \frac{Q_0\,\omega_0(\sqrt{2} - 1)}{\sqrt{2}}\,, \qquad A_2 = \frac{Q_0\,\omega_0}{\sqrt{2}}\,, \tag{29}$$

and $A_2 > A_1$. Hence, the above analysis should be a good approximation to the actual performance for the first half cycle of the oscillations, *i.e.*, for $t < \pi/\omega_0 \approx 2$ ms.



At time $t = 0$, the switch is moved so as to disconnect the battery from L_1, and send the current I_0 into the part of the circuit on the right of the figure. Eventually the currents I_1 and I_2 drop to zero, and all the initial magnetic energy U_0 is dissipated in the resistors of the circuit. However, if R_1 and R_2 are small compared to R, there exists an interesting intermediate state of the circuit in which the current $I_1 - I_2$ is negligible, and a nearly steady current $I_1 \approx I_2$ flows through the two inductors, such that the magnetic energy of the system is still nonzero.

An analytic solution can be given if we neglect the small resistances R_1 and R_2. Then, the circuit equations are (for $t > 0$),

$$L_1 \dot{I}_1 + R(I_1 - I_2) = 0, \qquad L_2 \dot{I}_2 + R(I_2 - I_1) = 0, \tag{30}$$

with initial conditions that $I_1(0) = I_0$ and $I_2(0) = 0$. Supposing the currents to have the forms $I_1(t > 0) = I_0(A + B\,e^{-\alpha t})/(A + B)$ and $I_2(t > 0) = I_0 A(1 - e^{-\alpha t})/(A + B)$, the solutions are readily verified to be,

$$I_1(t > 0) = I_0 \frac{L_1 + L_2\,e^{-R(L_1+L_2)t/L_1L_2}}{L_1 + L_2}, \qquad I_2(t > 0) = \frac{I_0 L_1}{L_1 + L_2}\left(1 - e^{-R(L_1+L_2)t/L_1L_2}\right). \tag{31}$$

At large times (in this approximation), the "final" currents are steady and equal, $I_{1f} = I_{2f} = I_0 L_1/(L_1 + L_2)$, and the "final" magnetic energy is (independent of R),[19]

$$U_f = \frac{L_1 I_{1f}^2 + L_2 I_{2f}^2}{2} = \frac{L_1^2 I_0^2}{2(L_1 + L_2)} = \frac{L_1}{L_1 + L_2} U_0. \tag{32}$$

It is interesting to consider the magnetic fluxes in the inductors:

$$\Phi_1(t > 0) = L_1 I_1 = \frac{L_1 I_0 \left(L_1 + L_2\,e^{-R(L_1+L_2)t/L_1L_2}\right)}{L_1 + L_2}, \tag{33}$$

$$\Phi_2(t > 0) = L_2 I_2 = \frac{L_1 L_2 I_0 \left(1 - e^{-R(L_1+L_2)t/L_1L_2}\right)}{L_1 + L_2}, \tag{34}$$

$$\Phi(t > 0) = \Phi_1(t > 0) + \Phi_2(t > 0) = L_1 I_0 = \Phi_0. \tag{35}$$

The total magnetic flux through the two inductors is the same as the initial magnetic flux $\Phi_0 = L_1 I_0$ at any time $t > 0$. The fluxes Φ_1 and Φ_2 are not equal except for large times and when $L_1 = L_2$.

Of course, this situation is not strictly true, and the "final" currents $I_{1f} = I_{2f} = I_0 L_1/(L_1 + L_2)$, and energy U_f of eq. (32), eventually go to zero, as the small resistances

[19]For equal inductances, $L_1 = L_2$, the "final" stored energy is $U_f = U_0/2$, which parallels the result for the two-capacitor circuit with equal capacitors.

R_1 and R_2 were neglected in the preceding analysis. As mentioned previously, the "final" currents and energy are eventually dissipated in those resistors.[20]

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