Electric Field of a Uniform Charge Density

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1 Problem

Discuss the electric field associated with a uniform volume charge density ϱ, when this is surrounded by a nonuniform surface charge density σ. In particular, show that a sphere with a uniform volume charge density can have its interior electric field normal to an axis of the sphere, given an appropriate surface charge density.

2 Solution

The electric field $\mathbf{E}$ is a vector, but a uniform charge distribution is not associated with any direction, other than directions related to the boundary of the charge distribution. This can be expressed formally by first noting that the electric field $\mathbf{E}$ associated with a volume charge density ϱ that is surrounded by a surface charge density σ is (in Gaussian units)

$$\mathbf{E}(\mathbf{r}) = \int \varrho(\mathbf{r}') \frac{\hat{\mathbf{R}}}{R^2} \, d\text{Vol}' + \oint \sigma(\mathbf{r}') \frac{\hat{\mathbf{R}}}{R^2} \, d\text{Area}', \tag{1}$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Then, since $\hat{\mathbf{R}}/R^2 = \boldsymbol{\nabla}'(1/R)$, the electric field associated with a uniform charge density ϱ can be expressed as

$$\mathbf{E}(\mathbf{r}) = \varrho \oint \frac{\hat{\mathbf{n}}}{R} \, d\text{Area}' + \oint \sigma(\mathbf{r}') \frac{\hat{\mathbf{R}}}{R^2} \, d\text{Area}', \tag{2}$$

where $\hat{\mathbf{n}}$ is the unit vector normally outward from the bounding surface.

For cases of zero surface charge density and simple geometry, the electric field of a uniform charge density ϱ has well-known forms:

$$\mathbf{E} = 4\pi\varrho \, x \, \hat{\mathbf{x}} \tag{3}$$

for an infinite slab of charge whose midplane is $x = 0$,

$$\mathbf{E} = 2\pi\varrho \, \rho \, \hat{\boldsymbol{\rho}} \tag{4}$$

for an infinite cylinder of charge with axis $\rho = 0$ in coordinates (ρ, ϕ, z), and

$$\mathbf{E} = \frac{4\pi\varrho}{3} \, \hat{\mathbf{r}} \tag{5}$$

for a sphere of charge centered on the origin in coordinates (r, θ, ϕ). Of course, all three forms (3)-(5) satisfy the time-independent Maxwell equations $\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\varrho$ and $\boldsymbol{\nabla} \times \mathbf{E} = 0$.

Suppose we wish the electric field inside a sphere of radius a to have the form (4). For this, the field (5) due to the uniform charge density ϱ inside the sphere is modified by the effect of a surface charge distribution σ according to eq. (2). However, we will analyze this effect by considering the scalar potential V. Inside the sphere, the relation $\mathbf{E} = -\boldsymbol{\nabla} V$ implies that

$$V(r < a) = V_0 - \pi\varrho\,\rho^2 = V_0 - \pi\varrho\,r^2\sin^2\theta = V_0 + \pi\varrho\,r^2(\cos^2\theta - 1), \tag{6}$$

where V_0 is the potential on the z-axis inside the sphere. Outside the sphere, where there is no charge, the scalar potential can be expanded in a series of Legendre polynomials P_i according to

$$V(r > a) = \sum_i \frac{A_i P_i(\cos\theta)}{r^{i+1}}\,. \tag{7}$$

Continuity of the potential across the surface $r = a$ requires that

$$\sum_i \frac{A_i P_i(\cos\theta)}{a^{i+1}} = V_0 + \pi\varrho\,a^2(\cos^2\theta - 1) = \left(V_0 - \frac{2\pi\varrho\,a^2}{3}\right)P_0 + \frac{2\pi\varrho\,a^2}{3}P_2. \tag{8}$$

Hence, the exterior potential is

$$V(r > a) = V_0\frac{a}{r}P_0 + \frac{2\pi\varrho\,a}{3}\left(\frac{a^4}{r^3}P_2 - \frac{a^2}{r}P_0\right). \tag{9}$$

The surface charge distribution $\sigma(\theta)$ is given by

$$\begin{aligned}\sigma &= \frac{1}{4\pi}\left[E_r(r = a^+) - E_r(r = a^-)\right] = \frac{1}{4\pi}\left[-\frac{\partial V(r = a^+)}{\partial r} - 2\pi\varrho\,a\sin^2\theta\right] \\ &= \frac{1}{4\pi}\left[\frac{V_0}{a}P_0 + 2\pi\varrho\,a\left(P_2 - \frac{P_0}{3}\right) + \frac{4\pi\varrho\,a}{3}(P_2 - P_0)\right] \\ &= \frac{V_0}{4\pi a}P_0 + \frac{\varrho\,a}{2}\left(\frac{5}{3}P_2 - P_0\right) = \frac{V_0}{4\pi a} + \frac{\varrho\,a}{4}\left(5\cos^2\theta - \frac{11}{3}\right). \end{aligned} \tag{10}$$

The total surface charge is

$$Q_{\rm surface} = 2\pi a^2\int_{-1}^{1}\sigma(\theta)\,d\cos\theta = V_0\,a - 2\pi\varrho\,a^3 = V_0\,a - \frac{3}{2}Q_{\rm interior}\,. \tag{11}$$

A solution for which the interior electric field has the cylindrical form (4) exists for any value of V_0, since this only contributes to the uniform component of the surface charge distribution, which does not affect the interior electric field. In particular, there exists a solution ($V_0 = 2\pi\varrho\,a^2/3$) for which the total charge in/on the sphere is zero.

In principle, a charged (or neutral), conducting, rotating sphere takes on a uniform charge density in its interior, and the associated interior electric field is normal to the axis of rotation [1]. However, this effect is too small to be observed in practice, being second-order in $\omega a/c$, where ω is the angular velocity and c is the speed of light.

References

[1] K.T. McDonald, *Charged, Conducting, Rotating Sphere* (July 22, 2009),
http://physics.princeton.edu/~mcdonald/examples/chargedsphere.pdf