Motion of a Cylinder Tied to a Slope by a String

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1 Problem

Discuss the motion of a cylinder that slides without friction down a slope to the right, but with a massless string that had been wound counterclockwise around it, with one end to a point on the slope.[1]

What happens after the string completely unwinds, assuming that the string is inelastic and remains attached to the cylinder?

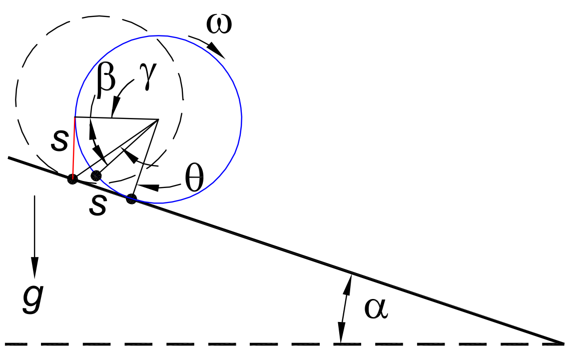
This problem was suggested by Johann Otto.

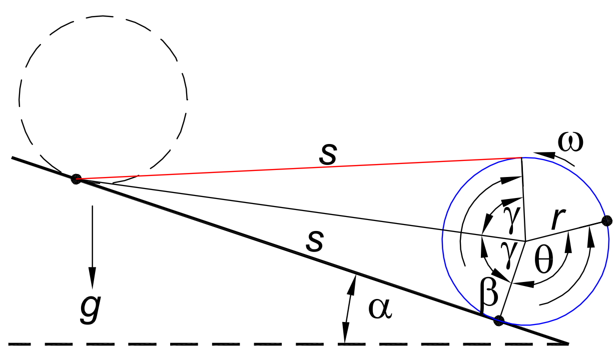
2 Solution

2.1 Motion Down the Slope

We suppose that the axis of the cylinder remains horizontal throughout its motion down the slope, which makes angle α to the horizontal.

Coordinate s is the distance along the slope between the line of contact with the cylinder and the point where the constraining string is tied. Then, the length of string that has unrolled off the cylinder is also s, as sketched in the figure below.





A second coordinate is needed to characterize the motion, which we take to be the angle θ between the perpendicular to slope through the axis of the cylinder and the line on the cylinder that was in contact with the slope initially, when the entire string was coiled on the cylinder. We define θ to be positive when the initial line of contact is rotating in a counterclockwise sense, as in the right figure above.

The possibly surprising feature is that angle θ is negative during the early motion of the cylinder and becomes positive only for distances s greater than $2.331\,r$, where r is the radius of the cylinder.

[1] If the string had been wound clockwise, the motion of the cylinder would simply be rolling without slipping with clockwise angular velocity down the slope.

Referring to the figure, we introduce angle $\beta = s/r$ between the present position of the initial line of contact of the cylinder with the slope and the present line where the string comes free of the cylinder. We also introduce angle γ, with $\tan\gamma = s/r$, such that angle θ is related by,

$$\theta = \beta - 2\gamma = \frac{s}{r} - 2\tan^{-1}\frac{s}{r}\,. \tag{1}$$

For small s, $\theta \approx -s/r$, while for large s, $\theta \approx s/r$. Angle θ is zero when $s = 0$, and also when $s/r = \tan(s/2r)$, *i.e.*, when $s/r = 2.331$.

For a cylinder of mass m and momentum of inertia $I = kmr^2$, its kinetic energy is related by,

$$\mathrm{KE} = \frac{m\dot{s}^2}{2} + \frac{I\dot{\theta}^2}{2} = \frac{m\dot{s}^2}{2}\left[1 + k\left(\frac{s^2 - r^2}{s^2 + r^2}\right)^2\right], \tag{2}$$

noting that,

$$\dot{\theta} = \frac{\dot{s}}{r} - 2\frac{r\dot{s}}{r^2 + s^2} = \frac{\dot{s}}{r}\left(1 - 2\frac{r^2}{s^2 + r^2}\right) = \frac{\dot{s}}{r}\frac{s^2 - r^2}{s^2 + r^2}\,, \tag{3}$$

which latter equation shows that the angular velocity $\omega = \dot{\theta}$ is positive (counterclockwise) only for $s > r$; the angular velocity is negative (clockwise) for $s < r$.

For motion without friction, the kinetic energy equals the change $mgs\sin\alpha$ in potential energy after the cylinder has slid/rolled distance s down the slope. Hence, the velocity of the cylinder is given by,

$$\dot{s} = \sqrt{\frac{2gs\sin\alpha}{1 + k\left(\frac{s^2 - r^2}{s^2 + r^2}\right)^2}}\,, \tag{4}$$

which could be integrated numerically to find $s(t)$ (after which θ would follow from eq. (1).)

Another example in which a reversal of the direction of the angular velocity of a cylinder can occur is when one cylinder rolls without slipping on another cylinder which latter rolls without slipping on a horizontal plane, http://kirkmcd.princeton.edu/examples/2cylinders.pdf

2.2 Motion after the String Has Completely Unwound

We suppose the string has length $l \gg r$, and is inelastic.

Just before the string is completely unwound, the maximum linear velocity of the cylinder is,

$$v_{\max} \approx \dot{s}(l) \approx \sqrt{\frac{2gl\sin\alpha}{1 + k}}, \tag{5}$$

and the maximum (counterclockwise) angular velocity is,

$$\omega_{\max} \approx \frac{v_{\max}}{r}\,. \tag{6}$$

At the moment when the string becomes completely unwound, it exerts a large impulsive force on the cylinder. For $l \gg r$ the string is essentially parallel to the slope at this moment, so the impulsive force exerts no torque about the center of mass of the cylinder, and the angular momentum of the cylinder is conserved during the impulse.

Shortly after this impulse, the unwound portion of the string lies along the slope, and the cylinder (whose sense of rotation is still counterclockwise) rolls back up the slope, rewinding the string.

During this rewinding, the angular velocity $\dot{\theta}$ is related to the linear velocity $\dot{s}$ (which is now negative) by $\dot{\theta} = -\dot{s}/r$. Hence, conservation of angular momentum during the impulse impels that the linear velocity of the cylinder simply reverses at the moment when the string becomes completely unwound. Thereafter, the cylinder rolls back up the slope, essentially to its initial position (like a "yo-yo"[2]) in the approximation of no friction (and no energy loss during the impulse at the moment when the string becomes completely unwound). During this upslope motion, the string becomes wound in the sense we would call counterclockwise if the cylinder were at rest while the string was wound around it.

After the cylinder comes to rest near its initial position, it rolls down the slope again, but this time with clockwise (negative) angular velocity, so when the string becomes completely unwound for a second time, the subsequent motion up the slope is with clockwise angular velocity (which results in the string being wound ins the sense we call counterclockwise)

When the cylinder comes to rest for a second time near its initial position, the string now wound in the same sense as at the very beginning of the problem. Thereafter, the motion repeats as from the very beginning. That is, while the motion is cyclic, its period is two up-down cycles, not one.

[2]See also http://kirkmcd.princeton.edu/examples/yoyo.pdf