Squeezing Flow
as an Application of the Extended Bernoulli Equation

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1 Problem

What is the force required to squeeze two coaxial disks of radius R together such that their separation is a known function $h(t)$ when the space between the disks is filled with an incompressible, inviscid fluid of density ρ? You may ignore gravity and rotation in this problem.

This problem was suggested by Johann Otto.

2 Solution

The nominal form of Bernoulli's equation is for steady, incompressible, inviscid fluid flow in an inertial frame of reference, relating the fluid pressure P and velocity $\mathbf{u}$ at two points along a streamline via conservation of energy,

$$P_1 + \frac{\rho\, u_1^2}{2} + \rho\, g h_1 = P_2 + \frac{\rho\, u_2^2}{2} + \rho\, g h_2 \qquad \text{(steady Bernoulli)}, \tag{1}$$

where h is the height of a point in a gravitational field with acceleration g. Bernoulli's equation can be extended to the case of nonsteady, compressible, rotational, elasto-viscoplastic flow in a noninertial reference frame by the addition of a "correction" term obtained by an appropriate integration along the streamline,

$$P_1 + \frac{\rho\, u_1^2}{2} + \rho\, g h_1 = P_2 + \frac{\rho\, u_2^2}{2} + \rho\, g h_2 + \int_1^2 \text{“correction”}, \qquad \text{(extended Bernoulli)}, \tag{2}$$

where the (complicated) "correction" term is displayed in eq. (12) of [1].

In the present example of unsteady, but incompressible flow, in an inertial frame where rotation of the fluid is neglected, only a simple "correction" applies,[1]

$$P_1 + \frac{\rho\, u_1^2}{2} + \rho\, g h_1 = P_2 + \frac{\rho\, u_2^2}{2} + \rho\, g h_2 + \int_1^2 \rho\, \frac{\partial \mathbf{u}}{\partial t} \cdot d\mathbf{l}. \tag{3}$$

We make the approximation that fluid velocity is $\mathbf{u}(\mathbf{r}, t) = u(r, t)\,\hat{\mathbf{r}}$, ignoring the small u_z, in a cylindrical coordinate system (r, θ, z) with the z-axis being that of the two disks. This ignores the usual boundary condition that $u = 0$ next to the surfaces of the disks. We also approximate the fluid pressure at the surface $r = R$ as atmospheric pressure P_A.

[1] This relatively simple form of the extended/unsteady Bernoulli equation is deduced from Euler's equation in [2]. See Appendix A of [3] for comments on that paper.

The time rate of change of the mass $M = \pi\rho r^2 h$ of a cylindrical volume of radius r of incompressible fluid between the two disks is related to the mass flow across the cylindrical surface at r by,

$$\frac{dM}{dt} \equiv \dot{M} = \pi\rho r^2 \dot{h} = -2\pi\rho r h\, u(r), \tag{4}$$

such that,

$$u = -r\frac{\dot{h}}{2h}, \qquad \dot{u} = \frac{r}{2}\left(\frac{\dot{h}^2}{h^2} - \frac{\ddot{h}}{h}\right). \tag{5}$$

Of course, $u(r = 0) = 0$.

Using the extended Bernoulli equation (3) for the streamline from point 1 at $(r, \theta, z) = (0, 0, z)$ to point 2 at $(r, 0, z)$, both between the disks, we have,

$$P_0 = P_r + \frac{\rho\, u^2(r)}{2} + \int_0^r \rho\, \frac{\partial u(r')}{\partial t}\, dr' = P_r + \frac{\rho\, r^2 \dot{h}^2}{8h^2} + \frac{\rho\, r^2}{4}\left(\frac{\dot{h}^2}{h^2} - \frac{\ddot{h}}{h}\right) = P_r + \frac{3\rho\, r^2 \dot{h}^2}{8h^2} - \frac{\rho\, r^2 \ddot{h}}{4h}. \tag{6}$$

In particular, in the approximation that $P_R = P_A$ (with $P_r = P(r)$ between the disks), we have,

$$P_0 = P_A + \frac{3\rho R^2 \dot{h}^2}{8h^2} - \frac{\rho R^2 \ddot{h}}{4h}, \tag{7}$$

Combining eqs. (6) and (7), we find that,

$$P_r = P_A + \rho\,(R^2 - r^2)\left(\frac{3\dot{h}^2}{8h^2} - \frac{\ddot{h}}{4h}\right). \tag{8}$$

The force, in excess of that of atmospheric pressure, which needs to be applied to one disk to move it toward the other is,

$$F = \int_0^R 2\pi r\,(P_r - P_A)\, dr = \frac{\pi\rho R^4}{2}\left(\frac{3\dot{h}^2}{8h^2} - \frac{\ddot{h}}{4h}\right) = \frac{\pi\rho R^4}{16}\left(\frac{3\dot{h}^2}{h^2} - \frac{2\ddot{h}}{h}\right). \tag{9}$$

Physically, it seems that force F must be positive for disks that move towards one another (*i.e.*, for negative $\dot{h}$). However, the form $h(t > 0) = h_0/(1 + kt)^2$ obeys $2h\ddot{h} = 3\dot{h}^2$, which suggests that even with zero F the disks could move together. That is, the results (5)-(9) are only approximate, and should be used with care.[2]

We have tacitly assumed that the disks remain flat and parallel as they move together. Since the force of the disk on the fluid varies with radius r, the disks must be thick enough to support the internal stresses resulting from the applied force, while remaining flat. Each disk, of mass M, experiences acceleration $-\ddot{h}/2$ (for disks that are squeezed together such that the midplane of the fluid between them is at rest), which requires additional force $F' = -M\ddot{h}/2$ on each disk. This force could be negative.

[2]Viscous flow with velocity $u\,\hat{\mathbf{r}}$ dependent on z as well as r has been considered by Jackson [4], who used a momentum analysis, *i.e.*, the Navier-Stokes equation. Ignoring viscosity and gravity, the momentum equation is $\rho\, D\mathbf{u}/Dt = -\boldsymbol{\nabla} P$, where $D/Dt = \partial/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}$ is the convective derivative. In cylindrical coordinates, with the approximation that $\mathbf{u} = u(r,t)\,\hat{\mathbf{r}}$, this reduces to $\rho\,\partial u/\partial t + \rho u\,\partial u/\partial r = -dP/dr$. Using our eq. (5) in this leads to our eq. (8).

See also [5]. A somewhat related problem of viscous flow between two annular plates is discussed in [6].

2.1 Energy Analysis

As Bernoulli's equation is based on conservation of energy, we could also do an energy analysis instead of invoking the extended Bernoulli equation.

The applied force F does work at rate $-F\dot{h}$, which changes the kinetic energy of the fluid between the disks,

$$\mathrm{KE}_{\mathrm{between}} = \int_0^R 2\pi r h \rho \frac{u^2}{2}\, dr = \frac{\pi \rho R^4}{16} \frac{\dot{h}^2}{h}\,, \tag{10}$$

recalling eq. (5). The kinetic energy generated by force F includes that which leaves the cylindrical surface of radius R with velocity $u(R)$, such that the total time derivative of the kinetic energy between the disks is,

$$\begin{aligned} \frac{d\,\mathrm{KE}}{dt} &= \frac{d\mathrm{KE}_{\mathrm{between}}}{dt} + 2\pi R h\, u(R) \frac{\rho u^2(R)}{2} = \frac{\pi \rho R^4}{16}\left(\frac{2\dot{h}\ddot{h}}{h} - \frac{\dot{h}^3}{h^2}\right) - \frac{\pi \rho R^4}{8}\frac{\dot{h}^3}{h^2} \\ &= \frac{\pi \rho R^4}{16}\left(\frac{2\dot{h}\ddot{h}}{h} - \frac{3\dot{h}^3}{h^2}\right) = -F\dot{h}, \end{aligned} \tag{11}$$

which leads to eq. (9), but not to eqs. (6)-(8).

References

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