Lorentz Invariance of the Number of Photons in a Rectangular Cavity

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1 Problem

In sec. 8 of his 1905 paper introducing special relativity [1], Einstein noted: *It is remarkable that the energy and the frequency of a light complex (Lichtkomplex) vary with the state of motion of the observer in accordance with the same law.* In that paper he did not interpret the ratio of energy to frequency as the number of photons in the "light complex", but this notion was likely on his mind. We are led to the appealing conclusion that the number of photons associated with a "light complex" is Lorentz invariant.

Einstein's argument about a "light complex" was for a finite volume in which the fields were approximated as a monochromatic plane wave (reasonably well realized by a laser beam). Details of the fields near the boundary of the volume were not considered. A variant of Einstein's argument was given by Zeldovich [2], but again its application to a well-defined volume that contained the fields was not pursued. Avron *et al.* [3] considered a cavity filled with monochromatic waves in its rest frame, and noted that one can obtain conflicting results as to the number of photons according to an observer moving with respect to the cavity.[1]

Consider a hollow rectangular cavity of perfectly conducting material with inner dimensions $d_x \times d_y \times d_z$ in its rest frame, where it contains electromagnetic energy U in a single cavity mode of angular frequency ω. The number of photons in this classical electromagnetic field is said to be $N = U/\hbar\omega$, where $\hbar$ is Planck's constant $h/2\pi$.[2,3] Show that in an inertial frame where the cavity has uniform velocity $\mathbf{v}$, an observer reports the same number of photons. You may assume that the motion is perpendicular to a face of the cavity.

2 Solution

2.1 The Cavity Is at Rest

The standing-wave solutions for electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ of angular frequency ω inside a perfectly-conducting, rectangular cavity of extent $0 < x < d_x$, $0 < y < d_y$, $0 < z < d_z$ can be written (in Gaussian units) as,[4]

[1]See also [4, 5].

[2]An amusing polemic against the use of the term "photon" has been given by Lamb [6].

[3]Planck's constant h was introduced by him in 1899 [7] as b in the equation before his eq. (51), for the energy spectrum $U(\nu, T)$ of an electromagnetic oscillator at temperature T as a function of its frequency ν, inspired by Wien's discussion of blackbody radiation [8]. This was more that a year before Planck's identification [9] of $b = h$ as the quantum of action of an oscillator. Already in 1899, Planck recognized the importance of his constant $b = h$, and discussed what are now called the Planck mass and time in sec. 26 of [7].

[4]See, for example, http://kirkmcd.princeton.edu/examples/ph501/ph501lecture14.pdf

$$
\begin{aligned}
E_x &= E_{0x}\, \cos k_x x\, \sin k_y y\, \sin k_z z\, e^{-i\omega t}, \\
E_y &= E_{0y}\, \sin k_x x\, \cos k_y y\, \sin k_z z\, e^{-i\omega t}, \\
E_z &= E_{0z}\, \sin k_x x\, \sin k_y y\, \cos k_z z\, e^{-i\omega t},
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
B_x &= B_{0x}\, \sin k_x x\, \cos k_y y\, \cos k_z z\, e^{-i\omega t}, \\
B_y &= B_{0y}\, \cos k_x x\, \sin k_y y\, \cos k_z z\, e^{-i\omega t}, \\
B_z &= B_{0z}\, \cos k_x x\, \cos k_y y\, \sin k_z z\, e^{-i\omega t},
\end{aligned} \tag{2}
$$

where the wave vector $\mathbf{k}$ is,[5,6]

$$
\mathbf{k} = (k_x, k_y, k_z) = \left(\frac{l\pi}{d_x}, \frac{m\pi}{d_y}, \frac{n\pi}{d_z} \right), \qquad \text{and} \qquad k = \frac{\omega}{c}, \tag{3}
$$

for any set of integers $\{l, m, n\}$, and c is the speed of light in vacuum (and inside the cavity). Further, Faraday's law,

$$
\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{implies that} \qquad i\mathbf{B}_0 = \hat{\mathbf{k}} \times \mathbf{E}_0, \tag{4}
$$

so,

$$
\mathbf{B}_0 \cdot \mathbf{E}_0 = 0, \quad |B_0| = |E_0|, \quad \text{and} \quad i\mathbf{B}_0 = \frac{(k_y E_{0z} - k_z E_{0y}, k_z E_{0x} - k_x E_{0z}, k_x E_{0y} - k_y E_{0x})}{k}. \tag{5}
$$

The free-space Maxwell equations $\boldsymbol{\nabla} \cdot \mathbf{B} = 0 = \boldsymbol{\nabla} \cdot \mathbf{E}$ imply that,

$$
\hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0 = \hat{\mathbf{k}} \cdot \mathbf{B}_0. \tag{6}
$$

For each set of nonzero integers $\{l, m, n\}$ there are two orthogonal polarizations of the vectors $\mathbf{E}_0$ and $\mathbf{B}_0$. As usual, the physical fields are the real parts of eqs. (1)-(2).

The electromagnetic energy stored in the cavity is,

$$
U = \frac{1}{2} \int \frac{|E|^2 + |B|^2}{8\pi}\, d\mathrm{Vol} = \frac{d_x d_y d_z\, |E_0|^2}{64\pi}, \tag{7}
$$

independent of the polarization of the fields, and of the mode indices $\{l, m, n\}$. The classical number of photons in the cavity (at rest) is therefore,

$$
N = \frac{U}{\hbar\omega} = \frac{d_x d_y d_z\, |E_0|^2}{64\pi\hbar\omega}. \tag{8}
$$

[5] The fields (1)-(2) can be decomposed into eight plane waves with the eight wave vectors $(\pm l\pi/d_x, \pm m\pi/d_y, \pm n\pi/d_z)$, that each propagate with phase velocity c. See sec. 2.9 of [10].

[6] Nontrivial cavity modes exist only with two or three of $\{l, m, n\}$ nonzero, so there is no cavity mode in which the wave vector $\mathbf{k}$ is parallel to a wall of the cavity. Hence, the arguments of Einstein [1] and of Avron *et al.* [3] are not strictly applicable to the present example.

For completeness, we compute the total electromagnetic field momentum $\mathbf{P}$ in the rest frame of the cavity, as the volume integral of the field momentum density $\mathbf{p} = \mathbf{E} \times \mathbf{B}/4\pi c$,

$$\mathbf{P} = \int \mathbf{p}\, d\mathrm{Vol} = \int \frac{\mathbf{E} \times \mathbf{B}}{4\pi c}\, d\mathrm{Vol}. \tag{9}$$

In particular,

$$\begin{aligned} P_x &= \int \frac{Re(E_y)Re(B_z) - Re(E_z)Re(B_y)}{4\pi c}\, d\mathrm{Vol} \qquad (10) \\ &= \int \frac{d\mathrm{Vol}}{4\pi c}[Re(E_{0y}\, e^{-i\omega t}) \sin k_x x\, \cos k_y y\, \sin k_z z Re(B_{0z}\, e^{-i\omega t}) \cos k_x x\, \cos k_y y\, \sin k_z z \\ &\qquad -Re(E_{0z}\, e^{-i\omega t}) \sin k_x x\, \sin k_y y\, \cos k_z z Re(B_{0y}\, e^{-i\omega t}) \cos k_x x\, \sin k_y y\, \cos k_z z]. \end{aligned}$$

Both terms of eq. (10) involve the factor,

$$\int_0^{d_x} \sin k_x x \cos k_x x\, dx = \frac{1}{2}\int_0^{d_x} \sin 2k_x x\, dx = 0, \tag{11}$$

and hence $P_x = 0$. Similarly, $P_y = 0 = P_z$, and the total electromagnetic field momentum is zero, $\mathbf{P} = 0$, at all times in the rest frame of the cavity (while the local field momentum density $\mathbf{p}$ is nonzero and oscillatory).

2.2 The Cavity Has Velocity $\mathbf{v} = v\,\hat{\mathbf{x}}'$.

We now consider the cavity as observed in the $'$ frame, in which it has velocity $\mathbf{v} = v\,\hat{\mathbf{x}}' = v\,\hat{\mathbf{x}}$.

The fields inside the cavity are no longer perceived as standing waves, but rather as the sum of waves of two different frequencies. To prepare for this, we re-express the standing wave (1)-(2) in the rest frame of the cavity as a sum of waves traveling in the $+x$ and $-x$ directions: $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_-$, $\mathbf{B} = \mathbf{B}_+ + \mathbf{B}_-$, where,

$$\begin{aligned} \mathbf{E}_+ &= (E_{0x} \sin k_y y\, \sin k_z z, -iE_{0y}\, \cos k_y y\, \sin k_z z, -iE_{0z}\, \sin k_y y\, \cos k_z z)\frac{e^{i(k_x x - \omega t)}}{2}, && (12) \\ \mathbf{E}_- &= (E_{0x} \sin k_y y\, \sin k_z z, iE_{0y}\, \cos k_y y\, \sin k_z z, iE_{0z}\, \sin k_y y\, \cos k_z z)\frac{e^{-i(k_x x + \omega t)}}{2}, && (13) \\ \mathbf{B}_+ &= (-iB_{0x}\, \cos k_y y\, \cos k_z z, B_{0y}\, \sin k_y y\, \cos k_z z, B_{0z}\, \cos k_y y\, \sin k_z z)\frac{e^{i(k_x x - \omega t)}}{2}, && (14) \\ \mathbf{B}_- &= (iB_{0x}\, \cos k_y y\, \cos k_z z, B_{0y}\, \sin k_y y\, \cos k_z z, B_{0z}\, \cos k_y y\, \sin k_z z)\frac{e^{-i(k_x x + \omega t)}}{2}. && (15) \end{aligned}$$

We also note that the total energies of the left- and right-moving waves are equal, and each equal to $U/2$,

$$\begin{aligned} U &= \int \frac{|\mathbf{E}_+ + \mathbf{E}_-|^2 + |\mathbf{B}_+ + \mathbf{B}_-|^2}{8\pi}\, d\mathrm{Vol} = \int \frac{|\mathbf{E}_+|^2 + |\mathbf{B}_+|^2}{8\pi}\, d\mathrm{Vol} + \int \frac{|\mathbf{E}_+|^2 + |\mathbf{B}_+|^2}{8\pi}\, d\mathrm{Vol} \\ &= U_+ + U_-, \qquad (16) \end{aligned}$$

since the cross terms in the first integral involve $\int_0^{d_x} \cos(2k_x x)\, dx = 0$.

We now need expressions for the fields (12)-(15) in the $'$ frame, where $x' = \gamma(x - vt)$, $y' = y$, $z' = z$, $t' = \gamma(t + vx/c^2)$, and $\gamma = 1/\sqrt{1 - v^2/c^2}$.

The phase of a plane wave is a Lorentz invariant, so we can write,

$$\mathbf{E}' = \mathbf{E}'_+ + \mathbf{E}'_- = \mathbf{E}'_+(y,z)\, e^{i(k'_{x+}x' - \omega'_+ t')} + \mathbf{E}'_-(y,z)\, e^{i(k'_{x-}x' + \omega'_- t')}, \tag{17}$$

$$\mathbf{B}' = \mathbf{B}'_+ + \mathbf{B}'_- = \mathbf{B}'_+(y,z)\, e^{i(k'_{x+}x' - \omega'_+ t')} + \mathbf{B}'_-(y,z)\, e^{i(k'_{x-}x' + \omega'_- t')}. \tag{18}$$

The Lorentz transformations of the wave 4-vectors $(\omega_\pm = kc, k_{x\pm}c = \pm k_x c, k_y c, k_z c)$ are,

$$\omega'_\pm = k'_\pm c = \gamma(\omega \pm k_x v) = \gamma\omega\left(1 \pm \frac{k_x}{k}\frac{v}{c}\right), \quad k'_{x\pm} = \gamma(\pm k_x + \omega v/c^2), \quad k'_y = k_y, \quad k'_z = k_z. \tag{19}$$

Waves that move in the $+x'$ (right) direction have the higher frequency ω'_+.

The transformations of the electromagnetic fields are,

$$E'_x = E_x, \qquad E'_y = \gamma\left(E_y + \frac{v}{c}B_z\right), \qquad E'_z = \gamma\left(E_z - \frac{v}{c}B_y\right), \tag{20}$$

and,

$$B'_x = B_x, \qquad B'_y = \gamma\left(B_y - \frac{v}{c}E_z\right), \qquad B'_z = \gamma\left(B_z + \frac{v}{c}E_y\right), \tag{21}$$

To calculate the electromagnetic energy inside the moving cavity at time t' we need,

$$\begin{aligned} & |E'|^2 + |B'|^2 \\ = \; & |E_x|^2 + |B_x|^2 + \gamma^2\left[\left(1 + \frac{v^2}{c^2}\right)\left(|E_y|^2 + |E_z|^2 + |B_y|^2 + |B_z|^2\right) + 4\frac{v}{c}Re(E_y B_z^\star - E_z B_y^\star)\right]. \end{aligned} \tag{22}$$

The volume of the moving cavity is $d_x d_y d_z/\gamma$, due to the Lorentz contraction. Also, the cross terms due to fields with different frequencies will vanish in the calculation the time-average energy stored in the moving cavity, so we can write,[7]

$$\langle U' \rangle = \langle U'_+ \rangle + \langle U'_- \rangle, \tag{23}$$

where,

$$\begin{aligned} \langle U'_\pm \rangle &= \frac{1}{2}\int \frac{|E_\pm|^2 + |B_\pm|^2}{8\pi}\, d\mathrm{Vol}' = \frac{d_x d_y d_z}{256\pi\gamma}\Bigg\{|E_{0x}|^2 + |B_{0x}|^2 \\ &\quad + \gamma^2\left[\left(1 + \frac{v^2}{c^2}\right)\left(|E_{0y}|^2 + |E_{0z}|^2 + |B_{0y}|^2 + |B_{0z}|^2\right) + 4\frac{v}{c}Re(E_{0y\pm}B^\star_{0z\pm} - E_{0z\pm}B^\star_{0y\pm})\right]\Bigg\} \\ &= \frac{\gamma d_x d_y d_z}{256\pi}\Bigg\{\left(1 - \frac{v^2}{c^2}\right)\left(|E_{0x}|^2 + |B_{0x}|^2\right) + \left(1 + \frac{v^2}{c^2}\right)\left(|E_{0y}|^2 + |E_{0z}|^2 + |B_{0y}|^2 + |B_{0z}|^2\right) \\ &\quad \mp 4\frac{v}{c}Re(iE_{0y}B^\star_{0z} - iE_{0z}B^\star_{0y})\Bigg\} \\ &= \frac{\gamma d_x d_y d_z}{256\pi}\Bigg\{2|E_0|^2 - \frac{v^2}{c^2}\left(|E_{0x}|^2 + |B_{0x}|^2 - |E_{0y}|^2 - |E_{0z}|^2 - |B_{0y}|^2 - |B_{0z}|^2\right) \\ &\quad \mp 4\frac{v}{c}Re(iE_{0y}B^\star_{0z} - iE_{0z}B^\star_{0y})\Bigg\}. \end{aligned} \tag{24}$$

[7](Nov. 29, 2020) The electromagnetic field energy U' is not constant in time, but oscillates, as discussed at the end of sec. 3 of [13]. Conservation of energy is presumably satisfied by the existence of an oscillatory energy associated with the stresses in the cavity walls. See also sec. 2.3 below.

Using eq. (5) and noting that $\mathbf{k} \cdot \mathbf{E}_0^\star = 0$, we have,

$$
\begin{aligned}
& |E_{0x}|^2 + |B_{0x}|^2 - |E_{0y}|^2 - |E_{0z}|^2 - |B_{0y}|^2 - |B_{0z}|^2 \\
= \;& 2\,|E_0|^2 - 2\,|E_{0y}|^2 - 2\,|E_{0z}|^2 - 2\,|B_{0y}|^2 - 2\,|B_{0z}|^2 \\
= \;& 2\,|E_{0x}|^2 \\
& - \frac{2}{k^2}\left\{k_z^2\,|E_{0x}|^2 + k_x^2\,|E_{0z}|^2 - 2k_xk_zRe(E_{0x}E_{0z}^\star)\right\} \\
& - \frac{2}{k^2}\left\{k_y^2\,|E_{0x}|^2 + k_x^2\,|E_{0y}|^2 - 2k_xk_yRe(E_{0x}E_{0y}^\star)\right\} \\
= \;& 2\,|E_{0x}|^2 - \frac{2}{k^2}\left\{k^2\,|E_{0x}|^2 + k_x^2\,|E_0|^2 - 2k_xRe[E_{0x}(k_xE_{0x}^\star + k_yE_{0y}^\star + k_zE_{0z}^\star)]\right\} \\
= \;& -\frac{2k_x^2}{k^2}\,|E_0|^2 .
\end{aligned} \tag{25}
$$

and similarly,

$$
iE_{0z}B_{0y}^\star - iE_{0y}B_{0z}^\star = -\frac{k_x}{k}\,|E_0|^2 . \tag{26}
$$

Thus, recalling eq. (7),

$$
\langle U'_\pm \rangle = \frac{\gamma d_x d_y d_z}{128\pi}\left(1 \pm \frac{k_x}{k}\frac{v}{c}\right)^2 |E_0|^2 = \gamma\left(1 \pm \frac{k_x}{k}\frac{v}{c}\right)^2 \frac{U}{2} . \tag{27}
$$

The total field energy U', eq. (23), inside the moving cavity is independent of time,

$$
\langle U' \rangle = U' = \gamma\left(1 + \frac{k_x^2}{k^2}\frac{v^2}{c^2}\right) U. \tag{28}
$$

The numbers of classical photons in the moving cavity are, recalling eqs. (8) and (19),

$$
N'_\pm \;=\; \frac{\langle U'_\pm \rangle}{\hbar\omega_\pm} = \gamma\left(1 \pm \frac{k_x}{k}\frac{v}{c}\right)^2 \frac{U}{2\hbar\gamma\omega(1 \pm k_xv/kc)} = \frac{N}{2}\left(1 \pm \frac{k_x}{k}\frac{v}{c}\right) . \tag{29}
$$

While there are more photons of higher frequency (ω_+, right-moving) than of lower frequency (ω_-, left-moving) in the moving cavity, the total number $N' = N'_+ + N'_- = N$ of photons observed at a time t' is the same as in a frame where the cavity is at rest (at a time t).[8,9]

[8]For observations in the $'$ frame at time $t' = 0$, the corresponding times of observation in the rest frame are $t = 0$ for the left wall at $x = 0$ and $t = -vd_x/c^2$ for the right wall at $x = d_x$. The $'$ observer counts some photons near the right wall as being right-moving at time $t' = 0$ ($t < 0$), while the rest frame observer will count these as left-moving at the later time $t = 0$ after they have reflected off the right wall. That is, the $'$ observer counts more right-moving (higher-energy) than left-moving photons.

[9](Nov. 22, 2020) Thanks to William Celmaster for spotting several typos in a previous version of this section.

2.3 Is the Field Energy Part of the Rest Energy of the System?

A side issue of possible interest is the question of whether the field energy U can/should be considered as contributing to the "rest" energy of the system, such that,

$$U_0 = Mc^2 + U, \tag{30}$$

in the rest frame of the cavity, whose rest mass without fields inside is M.

If so, then in the frame where the cavity has velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$, the energy of the system should be,

$$U' = \gamma U_0 = \gamma(Mc^2 + U). \tag{31}$$

However, in this frame the cavity itself has energy,

$$U'_{\rm cavity} = \gamma Mc^2, \tag{32}$$

and the fields have energy,

$$U' = \gamma U \left(1 + \frac{k_x^2}{k^2}\frac{v^2}{c^2}\right), \tag{33}$$

so that the actual energy of the system is,

$$U' = \gamma \left[Mc^2 + U\left(1 + \frac{k_x^2}{k^2}\frac{v^2}{c^2}\right)\right], \tag{34}$$

rather than expression (31).

Missing from the above analysis is the energy associated with the stress on the walls of the cavity due to the "radiation pressure" of the fields inside. If these stresses, and their Lorentz transformations, were included in the analysis, a meaningful rest energy could be defined for the system, and the total energy-momentum of the system would constitute a 4-vector.[10] See [13] for a model of stresses in the walls of a box that contains bouncing, massive particles. For the simpler case of a capacitor with a dielectric block between its plates, see [12].

[10](Dec. 8, 2020) Rohrlich [11] has proposed that we define an energy-momentum 4-vector $P_{\rm Rohrlich,\mu}$ for electromagnetic fields that have zero field momentum in some inertial frame * (as holds for the present example), taking this 4-vector to be simply $(U^\star, 0)$ in the * frame, where $U^\star = \int (E^{\star 2} + B^{\star 2})\, d{\rm Vol}^\star/8\pi$ is the total electromagnetic field energy, and the total electromagnetic field momentum $\mathbf{P}^\star = \int \mathbf{E}^\star \times \mathbf{B}^\star\, d{\rm Vol}^\star/4\pi c$ is zero.

Then, in the $'$ frame, which has velocity $-\mathbf{v}$ with respect to the * frame, $P_{\rm Rohrlich,\mu} = (U'_{\rm Rohrlich}, \mathbf{P}'_{\rm Rohrlich}) = U^\star u_\mu/c = \gamma U^\star(1, \mathbf{v}/c)$, where $u_\mu = \gamma(c, \mathbf{v})$ is the velocity 4-vector of the * frame relative to the $'$ frame, $U'_{\rm Rohrlich}$ and $\mathbf{P}'_{\rm Rohrlich}$ are given by eqs. (3.23-24) of [11] (noting that $d^3\sigma' = \gamma\, d{\rm Vol}'$), and are not the volume integrals of the field energy and momentum densities in the $'$ frame; their physical significance is unclear. See also footnote 4, p. 5, of [12].

Rohrlich's prescription could be applied to any scalar quantity $Q^\star$ measured in some rest frame, to define a 4-vector $Q_\mu = Q^\star u_\mu$. This had been applied to $Q^\star$ being the temperature $T^\star$, and to the inverse temperature $1/T^\star$, in the rest frame of some thermodynamic system, prior to Rohrlich's usage, as reviewed in [14].

2.4 Blackbody Radiation (Dec. 6, 2020)

If the rectangular cavity is heated, but otherwise not excited, the electromagnetic cavity radiation approximates the (continuous) Planck spectrum [15]-[17].[11] Blackbody radiation in a rectangular cavity can be thought of as a sum of the electromagnetic cavity modes of the same cavity but with perfectly conducting walls.[12,13] The electromagnetic field energy, in the rest frame of the cavity, is the sum over the energies of the excited cavity modes with various wave vectors $\mathbf{k}$,

$$U = \sum_{\mathbf{k}} U_{\mathbf{k}}, \tag{35}$$

and according to eq. (28), the (time average) field energy of a mode with wave vector $\mathbf{k}$ in the rest frame of the cavity is,

$$U' = \gamma \sum_{\mathbf{k}} \left(1 + \frac{k_x^2}{k^2} \frac{v^2}{c^2} \right) U_{\mathbf{k}}, \tag{36}$$

in a frame where the (rectangular) cavity has velocity $v\,\hat{\mathbf{x}}$. In the rest frame of the cavity, blackbody radiation is isotropic, and independent of the shape of the cavity, which implies

[11] The radiation that Planck studied was obtained inside metallic cavities, lined with an optically thick, black, metal-oxide layer [18]-[20]. Blackbody radiation is absorbed and re-emitted by the cavity wall with a $\cos\theta$ (Lambert) angular distribution, in contrast to the specular reflection of electromagnetic waves by a perfect conductor. Also, the waves reflected by a perfect conductor have the same frequency as the incident wave, in the rest frame of the reflector, while an ideal black body emits waves of all frequencies upon absorption of one with a single frequency.

[12] This argument was introduced by Rayleigh in 1900 [21], but using modes of sound waves in a cavity as an analogy. The famous Rayleigh-Jeans law followed in 1905 [22], in which Jean's contribution was to point out a factor of 8 error by Rayleigh [23, 24]. The counting of electromagnetic modes in a cavity with perfectly conducting walls was discussed by Jeans in [25].

[13] *Jan. 5, 2021.* Further development of the quantum theory of blackbody radiation was slow. In 1907, Planck intuited (p. 567 of [26]) that there should exist nonzero cavity radiation (*latente Energie* = zero-point energy of the electromagnetic oscillators) even at zero temperature, and contemplated the gravitational interaction with this zero-point energy/mass (p. 568 of [26]). (For a black body to emit a photon of energy higher than that of one absorbed, the body must contain internal energy. In particular, a black body must have nonzero energy even at zero temperature.) He only gave a more detailed discussion of this notion in 1911-12 [27, 28] (see also [29, 30]). In 1913, Einstein [31] built on his earlier work [32] to note that if one accepts the notion of zero-point energy, then one can infer the Planck spectrum for blackbody radiation without any assumption as to quanta of energy (*ohne die Annahme von Quanten*). Einstein, like Planck, associated zero-point energy with oscillators, but not with the electromagnetic field, and Einstein needed to double Planck's zero-point energy to explain the blackbody spectrum (which, in effect, associates Planck's zero-point energy with both the field and the oscillators).

In the 1960's, Einstein's (largely forgotten) argument was rediscovered [33, 34], now with emphasis on the zero-point energy of the fields. This has been enshrined in so-called stochastic electrodynamics *(see also [35]-[37]), a supposedly "classical" theory involving random fluctuations of the electromagnetic field. At present, the essence of quantum theory is often regarded as the existence of entangled states, and intrinsic randomness of Nature (both of which bothered Einstein greatly), rather than of quantization (which exists in some form in many "classical" systems). So, it seems misguided that a theory based on intrinsic randomness be characterized as "classical" by its proponents.*

that,[14]

$$\langle k_x^2 \rangle = \frac{k^2}{3} \,. \tag{37}$$

Hence,[15]

$$U' = \gamma \sum_{\mathbf{k}} \left(1 + \frac{v^2}{3c^2}\right) U_{\mathbf{k}} = \gamma \left(1 + \frac{v^2}{3c^2}\right) U. \tag{38}$$

The blackbody radiation is associated with a temperature $T^\star$ in its rest frame. While it is tempting to use the transformation (38) as a basis for defining a temperature T' for the radiation according to observers in a $'$ frame in which the cavity is moving, the procedure for this is ambiguous. Rather, it seems best to consider that the temperature is only well defined in the rest frame of the cavity [47].

A Appendix: Are the Cavity Photons Localized?

The cavity photons are "localized" in the sense of being "trapped" inside the cavity. A cavity photon of energy $E = \hbar\omega$ has no momentum, $\mathbf{P} = 0$, and so can be said to have effective mass $m = \sqrt{E^2 - c^2 P^2}/c^2 = \hbar\omega/c^2$. The cavity photons are "virtual photons", with nonzero mass.[16]

However, people also speak of "Anderson localization" [48] of quantum waves/particles in disordered media, when ordinary diffusion is absent, due to the localizing effect of multiple scattering. Anderson localization of light was perhaps first discussed in [49], and is considered to have been observed (see, for example, [50], but apparently some reports of this must be regarded with skepticism (see, for example, [51]).

References

[1] A. Einstein, *Zur Elektrodynamik bewegten Körper*, Ann. Phys. **17**, 891 (1905),
http://kirkmcd.princeton.edu/examples/EM/einstein_ap_17_891_05.pdf
http://kirkmcd.princeton.edu/examples/EM/einstein_ap_17_891_05_english.pdf

[2] Y.B. Zeldovich, *Number of Quanta as an Invariant of the Classical Electromagnetic Field*, Sov. Phys. Doklady **10**, 771 (1996),
http://kirkmcd.princeton.edu/examples/EM/zeldovich_spd_10_771_66.pdf

[14] This behavior is not self evident from consideration of the electromagnetic modes of a general cavity, but it clearly holds for a cubical (or spherical) cavity. Indeed, the idealization of the Planck blackbody spectrum holds only for cavities large compared to wavelengths of interest (see, for example, [38]).

[15] The transformation (38) was deduced in [39, 40] by other methods. The angular dependence of the transformation of blackbody radiation was discussed in [41]–[46].

[16] (Nov. 29, 2020) While the classical field inside that cavity can be regarded as the superposition of eight plane waves that move with the speed of light (see footnote 3 above), perhaps suggesting the superposition of eight massless photons, the quantum field could consist of only a single (massive) photon.

[3] J.E. Avron *et al.*, *Is the number of photons a classical invariant?* Eur. J. Phys. **20**, 153 (1999), http://kirkmcd.princeton.edu/examples/EM/avron_ejp_20_153_99.pdf

[4] J.-J. Labarthe, *Einstein's answer to 'Is the number of photons a classical invariant?'* Eur. J. Phys. **20**, L37 (1999), http://kirkmcd.princeton.edu/examples/EM/labarthe_ejp_20_L37_99.pdf

[5] D. Redžić and J. Strnad, *Einstein's Light Complex*, Fizika A **13**, 113 (2004), http://kirkmcd.princeton.edu/examples/GR/redzic_fizika_a13_113_04.pdf

[6] W.E. Lamb, Jr, *Anti-photon*, Appl. Phys. B **60**, 77 (1995), http://kirkmcd.princeton.edu/examples/QM/lamb_apb_60_77_95.pdf

[7] M. Planck, *Über irreversible Strahlungsvorgänge*, Sitzb. König. Preuss. Akad. Wiss. **26**, 440 (1899), http://kirkmcd.princeton.edu/examples/statmech/planck_skpaw_26_440_99.pdf

[8] W. Wien *On the Division of Energy in the Emission-Spectrum of a Black Body*, Phil. Mag. **43**, 214 (1897), http://kirkmcd.princeton.edu/examples/statmech/wien_pm_43_214_97.pdf

[9] M. Planck, *Zur Theorie des Gesetzes der Energieverteilung im Normalspectrum*, Verh. Deutsch. Phys. Gesell. **2**, 237 (1900),
http://kirkmcd.princeton.edu/examples/statmech/planck_vdpg_2_237_00.pdf
http://kirkmcd.princeton.edu/examples/statmech/planck_vdpg_2_202_237_00_english.pdf
Ueber die Elememtarquanta der Materie und der Electricität, Ann. d. Phys. **4**, 553 (1901), http://kirkmcd.princeton.edu/examples/statmech/planck_ap_4_553_01.pdf
http://kirkmcd.princeton.edu/examples/statmech/planck_ap_4_553_01_english.pdf

[10] K.T. McDonald, *Decomposition of Electromagnetic Fields into Electromagnetic Plane Waves* (July 11, 2010), http://kirkmcd.princeton.edu/examples/virtual.pdf

[11] F. Rohrlich, *Electromagnetic Momentum, Energy and Mass*, Am. J. Phys. **38**, 1310 (1970), http://kirkmcd.princeton.edu/examples/EM/rohrlich_ajp_38_1310_70.pdf

[12] K.T. McDonald, *Stress and Momentum in a Capacitor That Moves with Constant Velocity* (Apr. 21, 1984), http://kirkmcd.princeton.edu/examples/cap_stress.pdf

[13] W. Celmaster, *Regarding the energy Lorentz transformation of classical particles and waves bouncing in a box* (Jan. 10, 2021), http://kirkmcd.princeton.edu/examples/balls_in_cavity_210110.pdf

[14] K.T. McDonald, *Temperature and Special Relativity* (Dec. 2, 2020), http://kirkmcd.princeton.edu/examples/temperature_rel.pdf

[15] M. Planck, *Ueber eine Verbesserung der Wien'schen Spectralgleichung*, Verh. Deutsch. Phys. Gesell. **2**, 202 (1900), http://kirkmcd.princeton.edu/examples/statmech/planck_vdpg_2_202_00.pdf

[16] M. Planck, *Zur Theorie des Gesetzes der Energieverteilung im Normalspectrum*, Verh. Deutsch. Phys. Gesell. **2**, 237 (1900), http://kirkmcd.princeton.edu/examples/statmech/planck_vdpg_2_237_00.pdf

[17] M. Planck, *Ueber das Gesetz der Energieverteilung im Normalspectrum*, Ann. d. Phys. **4**, 553 (1901), http://kirkmcd.princeton.edu/examples/statmech/planck_ap_4_553_01.pdf http://kirkmcd.princeton.edu/examples/statmech/planck_ap_4_553_01_english.pdf

[18] H. Rubens and F. Kurlbaum, *Über die Emission langewelliger Wärmestrahlen durch den schwarzen Körper bei verschiedenen Temperaturen*, Sitzb. König. Preuss. Akad. Wiss. Berlin, 929 (1900), http://kirkmcd.princeton.edu/examples/statmech/rubens_skpawb_929_00.pdf

[19] H. Rubens and F. Kurlbaum, *Anwendung derMethode der Reststralen zur Prüfung des Strahlungsgesetzes*, Ann. d. Phys. **4**, 64 (1901), http://kirkmcd.princeton.edu/examples/statmech/rubens_ap_4_649_01.pdf

[20] O. Lummer und F. Kurlbaum, *Der elektrisch geglühte "schwarze" Körper*, Ann. d. Phys. **5**, 829 (1901), http://kirkmcd.princeton.edu/examples/statmech/lummer_ap_5_829_01.pdf

[21] Lord Rayleigh, *Remarks upon the Law of Complete Radiation*, Phil. Mag. **49**, 539 (1900), http://kirkmcd.princeton.edu/examples/statmech/rayleigh_pm_49_539_00.pdf

[22] Lord Rayleigh, *The Dynamical Theory of Gases and Radiation*, Nature **72**, 54 (1905), http://kirkmcd.princeton.edu/examples/statmech/rayleigh_nature_72_54_05.pdf

[23] J. Jeans, *On the Partition of Energy between Matter and Æther*, Phil. Mag. **10**, 91 (1905), sec. 10, http://kirkmcd.princeton.edu/examples/statmech/jeans_pm_10_91_05.pdf

[24] Lord Rayleigh, *The Constant of Radiation as Calculated from Molecular Data*, Nature **72**, 243 (1905), http://kirkmcd.princeton.edu/examples/statmech/rayleigh_nature_72_243_05.pdf

[25] J. Jeans, *The Motion of Electrons in Solids. Part II.–Radiation of all Wave-lengths in a perfectly Reflecting Enclosure*, Phil. Mag. **18**, 209 (1909), http://kirkmcd.princeton.edu/examples/statmech/jeans_pm_18_209_09.pdf

[26] M. Planck, *Zur Dynamik bewegter Systeme*, Sitzb. König. Preuss. Akad. Wiss., 542 (1907), http://kirkmcd.princeton.edu/examples/statmech/planck_skpaw_542_07.pdf
Also, Ann. d. Phys. **26**, 1 (1908), http://kirkmcd.princeton.edu/examples/statmech/planck_ap_26_1_08.pdf

[27] M. Planck, *Eine neue Strahlungshypothese*, Verh. Deutsch. Phys. Gesell. Berlin **13**, 138 (1911), http://kirkmcd.princeton.edu/examples/statmech/planck_vdpgb_13_138_11.pdf

[28] M. Planck, *Über die Begründung des Gesetzes der schwarzen Strahlung*, Verh. Deutsch. Phys. Gesell. Berlin **14**, 113 (1912), http://kirkmcd.princeton.edu/examples/statmech/planck_vdpgb_14_113_12.pdf
Also, Ann. d. Phys. **37**, 642 (1912), http://kirkmcd.princeton.edu/examples/statmech/planck_ap_37_642_12.pdf

[29] P.W. Milonni and L.-H. Shih, *Zero-point energy in early quantum theory*, Am. J. Phys. **59**, 684 (1991), http://kirkmcd.princeton.edu/examples/statmech/milonni_ajp_59_684_91.pdf

[30] J. Mehra and H. Rechenberg, *Planck's Half-Quanta: A History of the Concept of Zero-Point Energy*, Found. Phys. **29**, 91 (1999),
http://kirkmcd.princeton.edu/examples/statmech/mehra_fp_29_91_99.pdf

[31] A. Einstein and O. Stern, *Einige Argumemte für die Annahme einer molekularen Agitation beim absolutern Nullpunkt*, Ann. d. Phys. **40**, 551 (1913),
http://kirkmcd.princeton.edu/examples/statmech/einstein_ap_40_551_13.pdf
http://kirkmcd.princeton.edu/examples/statmech/einstein_ap_40_551_13_english.pdf

[32] A. Einstein and L. Hopf, *Statistische Untersuchung der Bewegung Resonators im einem Strahlungsfeld*, Ann. d. Phys. **33**, 1105 (1910),
http://kirkmcd.princeton.edu/examples/statmech/einstein_ap_33_1105_10.pdf
http://kirkmcd.princeton.edu/examples/statmech/einstein_ap_33_1105_10_english.pdf

[33] T.W. Marshall, *Random Electrodynamics*, Proc. Roy. Soc. London A **275**, 475 (1963),
http://kirkmcd.princeton.edu/examples/statmech/marshall_prsla_275_475_63.pdf
Statistical electrodynamics, Proc. Camb. Phil. Soc. **61**, 537 (1965),
http://kirkmcd.princeton.edu/examples/statmech/marshall_pcps_61_537_65.pdf

[34] T.H. Boyer, *Derivation of the Blackbody Radiation Spectrum without Quantum Assumptions*, Phys. Rev. **182**, 1374 (1969),
http://kirkmcd.princeton.edu/examples/statmech/boyer_pr_182_1374_69.pdf
The contrasting roles of Planck's constant in classical and quantum theories, Am. J. Phys. **96**, 280 (2018), http://kirkmcd.princeton.edu/examples/QM/boyer_ajp_86_280_18.pdf
Blackbody radiation in classical physics: A historical perspective, Am. J. Phys. **96**, 495 (2018), http://kirkmcd.princeton.edu/examples/statmech/boyer_ajp_86_495_18.pdf

[35] J.L. Jiménez, L. de la Peña and T.A. Brody, *Zero-pont term in cavity radiation*, Am. J. Phys. **48**, 840 (1980), http://kirkmcd.princeton.edu/examples/statmech/jimenez_ajp_48_840_80.pdf

[36] D.C. Cole, *Reinvestigation of the thermodynamics of blackbody radiation via classical physics*, Phys. Rev. A **45**, 8471 (1992),
http://kirkmcd.princeton.edu/examples/statmech/cole_pra_45_8471_92.pdf

[37] M. Ibison and B. Haisch, *Quantum and classical statistics of the electromagnetic zero-point field*, Phys. Rev. A **54**, 2737 (1996),
http://kirkmcd.princeton.edu/examples/statmech/ibison_pra_54_2737_96.pdf

[38] A.M. García-García *Finite-size corrections to the blackbody radiation laws*, Phys. Rev. A **78**, 023806 (2008), http://kirkmcd.princeton.edu/examples/statmech/garcia_pra_78_023806_08.pdf

[39] J.H. Eberly and A. Kujawski, *Relativistic Statistical Mechanics and Blackbody Radiation*, Phys. Rev. **155**, 10 (1967), eq. (3.5),
http://kirkmcd.princeton.edu/examples/statmech/eberly_pr_155_10_67.pdf

[40] C.K. Yuen, *Lorentz Transformation of Thermodynamic Quantities*, Am. J. Phys. **38**, 246 (1970), eq. (A2), http://kirkmcd.princeton.edu/examples/statmech/yuen_ajp_38_246_70.pdf

[41] C.V. Heer and R.H. Kohl, *Theory for the Measurement of the Earth's Velocity through the 3°K Cosmic Radiation*, Phys. Rev. **174**, 1611 (1968), http://kirkmcd.princeton.edu/examples/statmech/heer_pr_174_1611_68.pdf

[42] P.J.E. Peebles and D.T. Wilkinson, *Comment on the Anisotropy of the Primeval Fireball*, Phys. Rev. **174**, 2168 (1968), http://kirkmcd.princeton.edu/examples/statmech/peebles_pr_174_2168_68.pdf

[43] R.N. Bracewell and E.K. Conklin, *An Observer moving in the 3°K Radiation Field*, Nature **219**, 1343 (1968), http://kirkmcd.princeton.edu/examples/statmech/bracewell_nature_219_1343_68.pdf

[44] G.R. Henry *et al.*, *Distribution of Blackbody Cavity Radiation in a Moving Frame of Reference*, Phys. Rev. **176**, 1451 (1968), http://kirkmcd.princeton.edu/examples/statmech/henry_pr_176_1451_68.pdf

[45] T.K. Nakamura, *Lorentz transform of black-body radiation temperature*, EuroPhys. Lett. **88**, 20004 (2009), http://kirkmcd.princeton.edu/examples/statmech/nakamura_epl_88_20004_09.pdf

[46] G.W. Ford and R.F. O'Connell, *Lorentz transformation of blackbody radiation*, Phys. Rev. E **88**, 044101 (2013), http://kirkmcd.princeton.edu/examples/statmech/ford_pre_88_044101_13.pdf

[47] K.T. McDonald, *Temperature and Special Relativity* (Dec. 2, 2020), http://kirkmcd.princeton.edu/examples/temperature_rel.pdf

[48] P.W. Anderson, *Absence of Diffusion in Certain Random Lattices*, Phys. Rev. **109**, 1492 (1958), http://kirkmcd.princeton.edu/examples/CM/anderson_pr_109_1492_58.pdf

[49] S. John, *Electromagnetic Absorption in a Disordered Medium near a Photon Mobility Edge*, Phys. Rev. Lett. **53**, 2169 (1984), http://kirkmcd.princeton.edu/examples/CM/john_prl_53_2169_84.pdf

[50] M. Segev, Y. Silberberg and D.N. Christodoulides, *Anderson localization of light*, Nat. Phot. **7**, 197 (2013), http://kirkmcd.princeton.edu/examples/CM/segev_np_7_197_13.pdf

[51] T. Sperling *et al.*, *Can 3D light localization be reached in 'white paint'?* New J. Phys. **18**, 013039 (2016), http://kirkmcd.princeton.edu/examples/CM/sperling_njp_18_013039_16.pdf