Oscillating Fluid in a U-Tube

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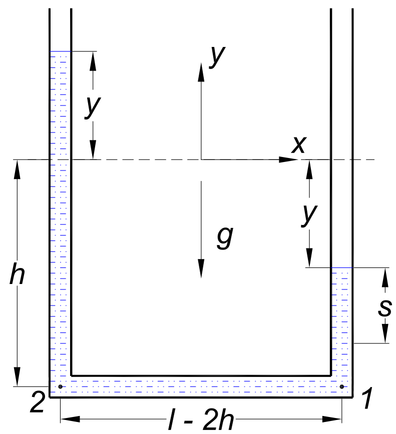
1 Problem

Discuss the motion of a "frictionless" fluid in a vertical U-tube that is constrained to be at rest. The fluid levels in the two arms of the tube are initially unequal, and vacuum exists in the tube above these levels. Discuss also the pressure inside the fluid. Does Bernoulli's equation apply here?



2 Solution

Taking the total mass of the fluid to be m and the length of the fluid in the tube to be l, the equation of motion for the fluid is,[1]

$$m\ddot{y} = -\frac{2mg}{l} y \left(= -ky \qquad \text{with} \qquad k = \frac{2mg}{l} \right), \tag{1}$$

when the level of the fluid in, say, the left arm of the U-tube is at height y above the equilibrium value $y = 0$. Thus, the fluid level oscillates vertically according to,

$$y(t) = y_0 \cos \omega t, \qquad \text{where} \qquad \omega = \sqrt{\frac{2g}{l}} \left(= \sqrt{\frac{k}{m}} \right). \tag{2}$$

The center of mass of the fluid, relative to the center of the U-tube ($x = 0$), is,

$$x_{\rm cm}(t) = \frac{(h-y)(l/2-h) - (h+y)(l/2-h)}{l} = -y(t)\left(1 - \frac{2h}{l}\right) = -y_0 \left(1 - \frac{2h}{l}\right) \cos \omega t, \tag{3}$$

[1] The equation of motion (1) can be confirmed via the Lagrangian $\mathcal{L} = \frac{1}{2} m\dot{y}^2 - \frac{mg}{l} y^2 = \frac{1}{2} m\dot{y}^2 - \frac{1}{2}\frac{2mg}{l} y^2 = \frac{1}{2} m\dot{y}^2 - \frac{1}{2} k y^2$.

where h is the height of the equilibrium level of the fluid above the bottom segment (here assumed to be straight), and we restrict out attention to the case that $y_0 < h$. If the tube were free to move in the x-direction, it would oscillate horizontally so as to keep the center of mass fixed at $x = 0$.[2]

The internal pressure P can used to describe the motion of fluid elements according to,

$$
\begin{aligned}
[P(s) - P(s+ds)]A + \rho g A\, ds &= \rho A\, ds\, \ddot{s} = \rho A\, ds\, \ddot{y} = -\frac{2\rho g A y\, ds}{l} \qquad (0 < s < h - y), && (4)\\
\frac{dP}{ds} &= \rho g \left(1 + \frac{2y}{l}\right) \qquad (0 < s < h - y), && (5)\\
[P(s) - P(s+ds)]\, A &= \rho A\, ds\, \ddot{s} = -\frac{2\rho g A y\, ds}{l} \qquad (h - y < s < l - h - y), && (6)\\
\frac{dP}{ds} &= \rho g \frac{2y}{l} \qquad (h - y < s < l - h - y), && (7)\\
[P(s) - P(s+ds)]\, A - \rho g A\, ds &= \rho A\, ds\, \ddot{s} = -\frac{2\rho g A y\, ds}{l} \qquad (l - h - y < s < l), && (8)\\
\frac{dP}{ds} &= -\rho g \left(1 - \frac{2y}{l}\right) \qquad (l - h - y < s < l), && (9)
\end{aligned}
$$

where s is the path length along the fluid starting from the upper surface on the right, A is the cross-sectional area of the tube, and $\rho = m/Al$ is the mass density of the fluid. Integrating eqs. (5), (7) and (9), we find,

$$
\begin{aligned}
P(s) &= \rho g s \left(1 + \frac{2y}{l}\right) \qquad (0 < s < h - y), && (10)\\
P(s) &= \rho g s \frac{2y}{l} + \rho g (h - y) \qquad (h - y < s < l - h - y), && (11)\\
P(s) &= \rho g (l - s) \left(1 - \frac{2y}{l}\right) \qquad (l - h - y < s < l), && (12)
\end{aligned}
$$

noting that the pressures at the corners of the tube are,

$$
P_1 = \rho g (h - y) \left(1 + \frac{2y}{l}\right), \qquad P_2 = \rho g (h + y) \left(1 - \frac{2y}{l}\right). \tag{13}
$$

The pressure at the left fluid surface is $P(l) = 0$ as expected, while the pressure on the left side at the height of the surface on the right side is,

$$
P(l - 2y) = 2\rho g y \left(1 - \frac{2y}{l}\right), \tag{14}
$$

rather than $2\rho g y$ as would hold at depth $2y$ in a static column of the fluid.

Since the fluid motion is not steady, we do not expect Bernoulli's equation to apply (even though energy is conserved).[3,4] For example, the pressure (11) in the bottom, horizontal segment varies with position s although the velocity and height are independent of s.

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[2]Additional aspects of this theme are pursued in [1].

[3]An elegant derivation of Bernoulli's equation is given in sec. 5 of [2], starting from Euler's equation (sec. 2). This derivation clarifies that the flow must be steady for Bernoulli's equation to hold.

[4]For another example in which care is required in the use of Bernoulli's equation, see [3].

References

[1] K.T. McDonald, *"Hidden" Momentum in an Oscillating Tube of Water* (June 24, 2012), http://kirkmcd.princeton.edu/examples/utube.pdf

[2] L.D. Landau and E.M. Lifshitz, *Fluid Mechanics*, 2nd ed. (Butterworth-Heinemann, 1998), http://kirkmcd.princeton.edu/examples/fluids/landau_fluids_59.pdf

[3] K.T. McDonald, *Is Bernoulli's Equation Relativistically Invariant?* (Sept. 13, 2009), http://kirkmcd.princeton.edu/examples/bernoulli.pdf