Power in a Variable Capacitor

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1 Problem

Discuss the various forms of power in a parallel-plate capacitor with plates of constant area A but variable separation $d(t) = d_0 - \dot{d}_0 t$, where d_0 and $\dot{d}_0$ are positive constants, when connected to a battery for voltage difference V (for time $t < d_0/\dot{d}_0$).

This problem appears as no. 3.5 in [1].

2 Solution

The variable capacitance is (in Gaussian units), and assuming the medium between the plates to have unit relative permeability) approximately $C(t) = A/d(t)$. The battery delivers power,

$$P_{\text{battery}} = VI = V\frac{dQ}{dt} = V^2\frac{dC}{dt} = \frac{CV^2\dot{d}_0}{d}\,. \tag{1}$$

The electrical energy stored in the capacitor is $U_C = CV^2/2$, so the power of the changing electrical energy is only,

$$P_C = \frac{dU_C}{dt} = \frac{V^2}{2}\frac{dC}{dt} = \frac{P_{\text{battery}}}{2}\,. \tag{2}$$

The other half of the power delivered by the battery is transferred to the external agent that keeps the capacitor plates apart with variable distance $d(t)$. We recall that the electrical force on each plate has magnitude $F = QE/2 = QV/2d$ (and not QE), so the mechanical power absorbed by the external agent is,

$$P_{\text{mech}} = -Fv = F\dot{d}_0 = \frac{QV\dot{d}_0}{2d} = \frac{CV^2\dot{d}_0}{2d} = \frac{P_{\text{battery}}}{2}\,. \tag{3}$$

The electric dipole moment of the capacitor is $p = Qd = CVd = AV$, which is constant, so the circuit emits no dipole radiation.[1] *The magnetic dipole moment of the circuit of total area A' is $\mu = IA'/c = AA'V\dot{d}_0/d^2$, so the power of the magnetic dipole radiation is $P_{M2} = 2\ddot{\mu}^2/3c^3 = 24A^2A'^2V^2\dot{d}_0^6/c^5d^8 \propto 1/c^5$, where c is the speed of light in vacuum. Hence, negligible power is radiated by this time-dependent circuit (until the plates get so close together that a spark discharge occurs).*[2]

[1] This holds in the quasistatic approximation that $Q = CV$. In a more transient analysis, if the plates begin to move towards one another, this causes changes in the electric field that propagate with the speed of light. When this "radiation" reaches the battery (after ≈ 1 nsec) it alters $\int \mathbf{E}\cdot d\mathbf{l}$ across the battery, which stimulates a chemical reaction that drives some charge out of the battery. In the quasistatic state after such transients have died down, there is no electric dipole radiation.

[2] For a capacitor problem in which radiation cannot be ignored, see [2].

2.1 Effect of Self Inductance

The circuit of battery plus capacitor has a small self inductance L that changes only slightly with the separation of the capacitor plates. The current $I = AV\dot{d}_0/d^2$ that flows as the separation of the plates decreases with constant speed $\dot{d}_0$ induces a (back) $\mathcal{EMF} = L\dot{I} \approx 2LAV\dot{d}_0^2/d^3$, which slightly reduces the voltage drop across the capacitor, which slightly reduces the current I. Thus, Lenz' law applies to this example, but does not significantly change the details of the increased charge on the capacitor plates as their separation decreases.

References

[1] J.A. Cronin, D.F. Greenberg and V.I. Telegdi, *University of Chicago Graduate Problems in Physics* (U. Chicago Press, 1967), http://kirkmcd.princeton.edu/examples/EM/cronin_67.pdf

[2] K.T. McDonald, *A Capacitor Paradox* (July 10, 2002), http://kirkmcd.princeton.edu/examples/twocaps.pdf