The Velocity Factor of an Insulated Two-Wire Transmission Line

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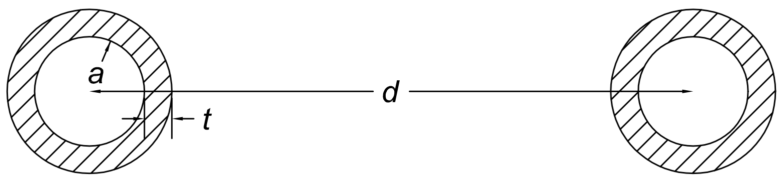
1 Problem

Estimate the velocity factor $F = v/c$ and the impedance Z of a two-wire transmission line made of cylindrical conductors of radius a whose centers are separated by distance d, when each wire is insulated by a layer of (relative) dielectric constant ϵ of thickness t, as shown in the figure below.



The thickness t is of the same order as radius a, but $a + t \ll d$. The space outside the insulated wires has unit (relative) dielectric constant. All media in this problem have unit (relative) magnetic permeability.

2 Solution

2.1 Velocity Factor

The propagation speed v of waves on a transmission line is,[1]

$$v = \frac{1}{\sqrt{LC}}\,, \tag{1}$$

where L and C are the inductance and capacitance per unit length of the two-line system. If there is no insulation on the wires, the propagation speed is the speed of light c.[2] Thus,

$$c = \frac{1}{\sqrt{L_0 C_0}}\,, \tag{2}$$

where L_0 and C_0 are the inductance and capacitance per unit length of the two-line system without insulation on the wires. Since the inductance is unaffected by the presence of insulation (assumed to have unit magnetic permeability), the velocity factor of the insulated-wire transmission line can be written as,

$$F = \frac{v}{c} = \sqrt{\frac{C_0}{C}}\,. \tag{3}$$

[1] See, for example, sec. 2(a) of http://kirkmcd.princeton.edu/examples/impedance_matching.pdf
[2] See, for example, pp. 16-17 of http://kirkmcd.princeton.edu/examples/ph501lecture13.pdf

The approximation employed here is that for $d \gg a + t$ the potential difference between the two wires can be calculated as twice the potential between radii a and d when only a single wire is present.

In this approximation the electric field E outside an uninsulated wire is (in Gaussian units),

$$E = \frac{2Q}{r}, \tag{4}$$

where Q is the electric charge per unit length on the wire. Then, the potential difference between radii a and d is $2Q \ln(d/a)$, so the capacitance C_0 of an uninsulated two-wire line is,

$$C_0 = \frac{Q}{V} \approx \frac{1}{4 \ln \frac{d}{a}}. \tag{5}$$

To convert to SI units, replace the factor 1/4 by $\pi\epsilon_0 = 27.82$ pF/m. An "exact" expression for C_0 is,[3]

$$C_0 = \frac{1}{4 \ln \frac{d+\sqrt{d^2-4a^2}}{2a}} \approx \frac{1}{4 \ln \left(\frac{d}{a} - \frac{a}{d}\right)} \approx \frac{1}{4 \ln \frac{d}{a}} \left(1 + \frac{a^2}{d^2 \ln d/a}\right), \tag{6}$$

which shows that the approximation (5) is rather good when $a \ll d$.

Similarly, the electric displacement D outside an insulated wire is,

$$D = \frac{2Q}{r}, \tag{7}$$

and hence the electric field is,

$$E(a < r < a + t) = \frac{2Q}{\epsilon r}, \tag{8}$$

$$E(r > a + t) = \frac{2Q}{r}, \tag{9}$$

where the layer of dielectric constant ϵ extends from radius a to $a+t$. The potential difference between radii a and d outside a single such wire is,

$$\frac{2Q}{\epsilon} \ln \frac{a+t}{a} + 2Q \ln \frac{d}{a+t}. \tag{10}$$

From this we estimate the capacitance C to be

$$C \approx \frac{1}{\frac{4}{\epsilon} \ln \frac{a+t}{a} + 4 \ln \frac{d}{a+t}}, \tag{11}$$

and therefore the velocity factor (3) is,

$$F \approx \sqrt{\frac{\frac{1}{\epsilon} \ln \frac{a+t}{a} + \ln \frac{d}{a+t}}{\ln \frac{d}{a}}}. \tag{12}$$

An Excel spreadsheet that implements eq. (12) is available at
http://kirkmcd.princeton.edu/examples/velocity_factor.xls

[3]See, for example, prob. 11 of http://physics.princeton.edu/~mcdonald/examples/ph501set3.pdf

2.2 Impedance

The characteristic impedance Z of a transmission line is related to its inductance and capacitance per unit length according to,

$$Z = \sqrt{\frac{L}{C}} . \tag{13}$$

The inductance L per unit length of a transmission line that is surrounded by media of unit magnetic permeability can be related to the capacitance C_0 for this case by eq. (2). Thus,

$$Z = \frac{1}{c\sqrt{C_0 C}} . \tag{14}$$

In particular, the impedance Z_0 of an uninsulated two-wire transmission line is,

$$Z_0 = \frac{1}{cC_0} = \frac{4}{c} \ln \frac{d + \sqrt{d^2 - 4a^2}}{2a} \approx \frac{4}{c} \ln \frac{d}{a} = 120 \ln \frac{d}{a}\, \Omega, \tag{15}$$

recalling that $1/c = 30\,\Omega$. Then, using eq. (11), the impedance of an insulated two-wire transmission line is,

$$Z \approx 120 \sqrt{\ln \frac{d}{a} \left(\frac{1}{\epsilon} \ln \frac{a+t}{a} + \ln \frac{d}{a+t} \right)}\, \Omega. \tag{16}$$

Furthermore, eliminating C in eq. (14) by use of expression (3) for the velocity factor, we find,

$$Z = \frac{F}{cC_0} = F Z_0. \tag{17}$$

2.3 Estimate of the Capacitance by an Energy Method

An alternative computation of the capacitance per unit length C can be based on the relation for stored electrostatic energy U per unit length when charge $\pm Q$ per unit length is placed on the two wires,

$$\begin{aligned} U &= \frac{Q^2}{2C} = \int \frac{ED}{8\pi}\, d\text{Area} = \int \frac{D^2}{8\pi\epsilon}\, d\text{Area} = \int \frac{D^2}{8\pi}\, d\text{Area} + \int \left(\frac{1}{\epsilon} - 1 \right) \frac{D^2}{8\pi}\, d\text{Area} \\ &\approx \frac{Q^2}{2C_0} + \frac{2}{8\pi} \left(\frac{1}{\epsilon} - 1 \right) \int_a^{a+t} \frac{4Q^2}{r^2}\, 2\pi r\, dr \approx \frac{Q^2}{2} \left[4 \ln \frac{d}{a} + 4 \left(\frac{1}{\epsilon} - 1 \right) \ln \frac{a+t}{a} \right] \\ &= \frac{Q^2}{2} \left[\frac{4}{\epsilon} \ln \frac{a+t}{a} + 4 \ln \frac{d}{a+t} \right] . \end{aligned} \tag{18}$$

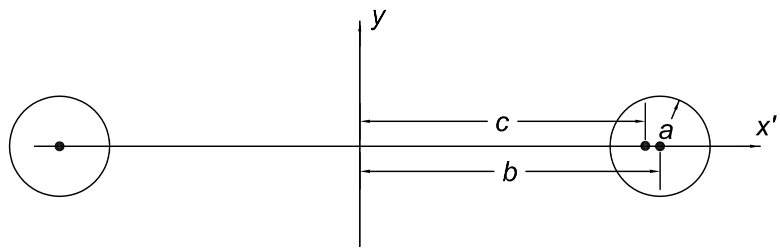
This yields the same estimate for C as eq. (11).

We can also use the energy method to find the next correction to eq. (11) for the capacitance. For this, we note that the electric displacement $\mathbf{D}$ when a dielectric is present is the same as the electric field $\mathbf{E}_1$ if the dielectric constant were unity. For the latter case, the corresponding electric potential V_1 can be determined from an appropriate complex logarithmic function,[4]

$$V_1 = -Q \ln \frac{(x' - c)^2 + y^2}{(x' + c)^2 + y^2}\,, \tag{19}$$

in a coordinate system (x', y) whose origin is midway between the two conductors, as sketched below. The conductors are centered at $x' = \pm b = \pm d/2$, and the parameter c is, given by

$$c = \sqrt{b^2 - a^2} \approx b - \frac{a^2}{2b} = b - \frac{a^2}{d}\,. \tag{20}$$



For use in energy expression (18), we wish to estimate the integral,

$$2\left(\frac{1}{\epsilon} - 1\right)\int_a^{a+t} dr \int_0^{2\pi} r\, d\phi\, \frac{D^2}{8\pi} = 2\left(\frac{1}{\epsilon} - 1\right)\int_a^{a+t} dr \int_0^{2\pi} r\, d\phi\, \frac{E_1^2}{8\pi}\,, \tag{21}$$

in a coordinate system centered on the righthand wire. So, we make the change of variables $x' = x + b$, leading to,

$$V_1 = -Q \ln \frac{(x + b - c)^2 + y^2}{(x + b + c)^2 + y^2}\,. \tag{22}$$

The electric field $\mathbf{E}_1$ then has components,

$$\begin{aligned}
E_{1,x} &= -\frac{\partial V_1}{\partial x} = Q\frac{2(x + b - c)}{(x + b - c)^2 + y^2} - Q\frac{2(x + b + c)}{(x + b + c)^2 + y^2} \\
&\approx 2Q\left(\frac{x + a^2/d}{r^2 + 2a^2x/d} - \frac{x + d}{d^2 + 2dx}\right) \approx 2Q\left(\frac{x + a^2/d}{r^2 + 2a^2x/d} - \frac{d - x}{d^2}\right), \qquad (23) \\
E_{1,y} &= -\frac{\partial V_1}{\partial y} = Q\frac{2y}{(x + b - c)^2 + y^2} - Q\frac{2y}{(x + b + c)^2 + y^2} \\
&\approx 2Q\left(\frac{y}{r^2 + 2a^2x/d} - \frac{y}{d^2}\right), \qquad (24)
\end{aligned}$$

where $r^2 = x^2 + y^2$. Then,

$$E_1^2 \approx 4Q^2\left(\frac{x^2 + 2a^2x/d + y^2}{(r^2 + 2a^2x/d)^2} - 2\frac{(x + a^2/d)(d - x) + y^2}{d^2(r^2 + 2a^2x/d)}\right)$$

[4]See, for example, pp. 14-16 of
http://kirkmcd.princeton.edu/examples/ph501lecture6.pdf

$$\approx \frac{4Q^2}{r^2 + 2a^2x/d}\left(1 - \frac{2x}{d} - \frac{2a^2}{d^2} + 2\frac{x^2 - y^2}{d^2}\right)$$

$$\approx \frac{4Q^2}{r^2}\left(1 - \frac{2x}{d} - \frac{2a^2x}{r^2d} - \frac{2a^2}{d^2} + \frac{4a^2x^2}{r^2d^2} + 2\frac{x^2 - y^2}{d^2}\right). \qquad (25)$$

The integral over ϕ in eq. (21) leads to the cancelation of all terms in eq. (25) except the first, using $x = r\cos\phi$ and $y = r\sin\phi$. Hence, we find no corrections to the capacitance (11) at either order a/d or a^2/d^2.

That we find no corrections at these orders is surprising in view of numerical computations for the case $a = t = d/2$ in which the insulated wires touch,[5] *indicating a capacitance about twice that predicted by eq. (11).*

2.4 Estimate of the Capacitance Supposing the Insulation Follows Equipotentials

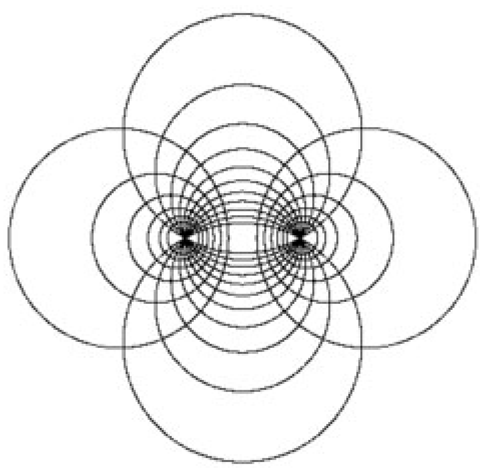
The equipotentials of the potential (19) are circles of the form,

$$\left(x - c\coth\frac{V}{2Q}\right)^2 + y^2 = c^2\text{csch}^2\frac{V}{2Q}, \qquad (26)$$

and the corresponding fields lines are also circles, characterized by a parameter W according to,

$$x^2 + \left(y + c\cot\frac{W}{2Q}\right)^2 = c^2\csc^2\frac{W}{2Q}, \qquad (27)$$

as sketched in the figure below.



If the region inside an equipotential were filled with a dielectric of permittivity ϵ, then the above forms describe the displacement field $\mathbf{D}$ rather than the electric field $\mathbf{E}$. In general, one cannot write $\mathbf{D} = -\boldsymbol{\nabla}V$, because $\boldsymbol{\nabla}\times\mathbf{D}$ is nonzero at the interface between regions of

[5] See, for example, J.C. Clements, C.R. Paul and A.T. Adams, *Computation of the Capacitance Matrix for Systems of Dielectric-Coated Cylinders*, IEEE Trans. Elec. Compat. **17**, 238 (1975), http://kirkmcd.princeton.edu/examples/EM/clements_ieeetec_17_238_75.pdf

differing permittivity. However, if $\mathbf{D}$ (and $\mathbf{E}$) are everywhere perpendicular to such interfaces, then $\boldsymbol{\nabla} \times \mathbf{D} = 0$ everywhere, and the displacement field can be deduced from a scalar potential.

This provides another method of estimating the capacitance of an insultated pair of wires. We suppose that the surface of the insulation is circular, but the center of this circle is displaced from the center of the wire, such that the surface of the insulation is on an equipotential.

The equipotential (26) intercepts the positive x-axis at,

$$x = c \coth \frac{V_x}{2Q} \pm c \,\mathrm{csch} \frac{V_x}{2Q}\,, \qquad i.e., \qquad x = c \tanh \frac{V_x}{4Q} \ \text{ and } \ c \coth \frac{V_x}{4Q}\,, \tag{28}$$

where V_x is the potential at x when $\epsilon = 1$ everywhere. In particular, if $x = b - a$ is on the surface of a wire, the potential of that wire is,

$$V_{\rm wire} = 4Q \tanh^{-1} \frac{b-a}{c} = 2Q \ln \frac{b+c}{a} = \frac{V_0}{2} = \frac{Q}{2C_0}\,, \tag{29}$$

where $V_0 = 2V_{\rm wire}$ is the voltage difference between the wires, so the capacitance of the bare wires is

$$C_0 = \frac{1}{4 \ln \frac{b+c}{a}} = \frac{1}{4 \ln \frac{d+\sqrt{d^2-4a^2}}{2a}}\,, \tag{30}$$

as previously stated in eq. (6).

If instead the wires are surrounded by circular cylinders of insulation of permittivity ϵ that extend from $x = c \tanh V_x/4Q$ to $c \coth V_x/4Q$, then the electric field for $|x| <= c \tanh V_x/4Q$ is the same as for bare wires, but the field for $c \tanh V_x/4Q < |x| < b - a$ is smaller than for bare wires by a factor $1/\epsilon$. Hence, the voltage difference between the wires is now,

$$V = 2\left(V_x + \frac{V_0/2 - V_x}{\epsilon}\right) = \frac{V_0}{\epsilon} + 2V_x \left(1 - \frac{1}{\epsilon}\right) = \frac{Q}{\epsilon C_0} + 4Q \frac{\epsilon - 1}{\epsilon} \ln \frac{c+x}{c-x}\,, \tag{31}$$

and the capacitance is now,

$$C = \frac{\epsilon C_0}{1 + 4(\epsilon - 1) C_0 \ln \frac{c+x}{c-x}}\,. \tag{32}$$

The actual wires have a concentric layer of insulation of thickness t. Setting $x = b - a - t$ in eq. (32) corresponds to increasing the amount of insulation until if fills the equipotential that passes through this point, which overestimates the capacitance,

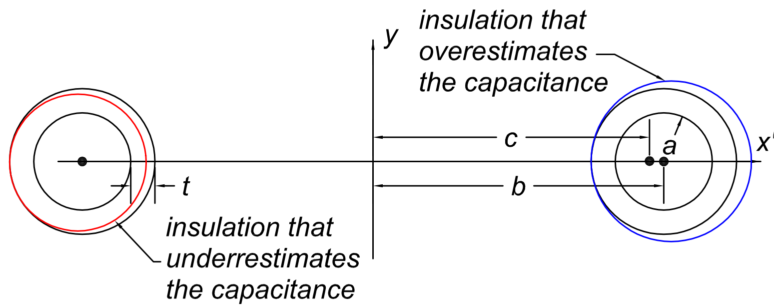
$$C < \frac{\epsilon C_0}{1 + 4(\epsilon - 1) C_0 \ln \frac{c+b-a-t}{c-b+a+t}}\,. \tag{33}$$

On the other hand, we could remove insulation from the wires until it fills the equipotential that passes through point $x = b + a + t$, in which case eq. (32) would underestimate the capacitance. Recalling eq. (28), the surface of this insulation also passes through the point $x = c^2/(b + a + t)$, so we have that,

$$C > \frac{\epsilon C_0}{1 + 4(\epsilon - 1) C_0 \ln \frac{b+a+t+c}{b+a+t-c}}\,. \tag{34}$$

We might then take are our estimate to be a kind of average of eqs. (33) and (34), such as,

$$C \approx \frac{\epsilon C_0}{1 + 2(\epsilon - 1)C_0 \left(\ln \frac{c+b-a-t}{c-b+a+t} + \ln \frac{b+a+t+c}{b+a+t-c}\right)} = \frac{\epsilon C_0}{1 + 2(\epsilon - 1)C_0 \ln \frac{(c+b-a-t)(b+a+t+c)}{(c-b+a+t)(b+a+t-c)}} . \qquad (35)$$



The approximation (35) appears to be significantly better than that of eq. (11) for the case that $b = a + t$.