Decomposition of Electromagnetic Fields into Electromagnetic Plane Waves

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1 Problem

Give a Fourier analysis of electromagnetic fields $\mathbf{E}(\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t)$ in the half spaces $z \gtrless 0$ (which are uniform, isotropic, nonconducting media with known free currents $\mathbf{J}(\mathbf{x},t)$) in terms of plane-wave solutions to Maxwell's equations. This analysis can be interpreted as a representation of these fields in terms of "classical photons." In particular, consider the fields of a charge q that moves with uniform velocity $\mathbf{v} = v\,\hat{\mathbf{z}}$ in a medium with index of refraction $n(\omega)$, including the limit that $v = 0$ as well as the case that $v > c/n$ (Čerenkov radiation [1]), where c is the speed of light in vacuum.

In this classical problem, consider only plane waves that satisfy Maxwell's equations as the basis for the Fourier synthesis of the electromagnetic fields (and potentials).[1]

To analyze as large a class of time-dependent fields as possible, consider plane waves for which the wave vector $\mathbf{k}$ is complex, such that the "plane waves" for which $Im(k) \neq 0$ are physically significant only close to their source charge-current distributions. Waves with such limited spatial extent are often called evanescent.

[1] For a static electric field, such as that of a point charge q at the origin, $\mathbf{E} = q\,\hat{\mathbf{r}}/r^2$ (in Gaussian units), relevant plane waves of the form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ have zero frequency. Then, a Fourier analysis of the scalar potential $V = q/r$ has the Fourier expansion,

$$V(\mathbf{x}) = \frac{q}{r} = \int V_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3\mathbf{k}, \tag{1}$$

where the Fourier coefficient $\tilde{V}$ is given by,

$$\begin{aligned} V_{\mathbf{k}} &= \int V(\mathbf{x})\, e^{-i\mathbf{k}\cdot\mathbf{x}} \frac{d^3\mathbf{x}}{(2\pi)^3} = \frac{q}{(2\pi)^2}\int_0^\infty r\,dr \int_{-1}^{1} e^{-ikr\cos\theta}\, d\cos\theta \\ &= -\frac{q}{2\pi^2 k}\int_0^\infty \sin kr\, dr = -\frac{q}{2\pi^2 k}\cos kr\big|_0^\infty = \frac{q}{2\pi^2 k^2}\,, \end{aligned} \tag{2}$$

on averaging to zero the rapid oscillations of the term $\cos kr$ for large r.

Then,

$$\mathbf{E} = \frac{q}{r^2}\,\hat{\mathbf{r}} = -\boldsymbol{\nabla} V = -\boldsymbol{\nabla}\int \frac{q}{2\pi^2 k^2}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3\mathbf{k} = -\frac{iq}{2\pi^2}\int \frac{\mathbf{k}}{k^2}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3\mathbf{k} = -\frac{iq}{2\pi^2}\int \hat{\mathbf{k}}\,\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{k}\, d^3\mathbf{k}. \tag{3}$$

This result suggests that the static Coulomb fields can be regarded as consisting of longitudinal "plane waves", $\hat{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}}$, of zero frequency. This description is appealing from a quantum view, in which we identify the "wave" $\hat{\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}}$ with a longitudinal virtual photon (of mass m given by $m^2 = -(\hbar k/c)^2$). However, this "wave" does not satisfy the free-space Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{E} = 0$, and so is somewhat unsatisfactory from a classical perspective.

We return to this case in sec. 2.7.2.

2 Solution

This problem is a variant on the analysis of Huygens and Fresnel of a scalar field (with time dependence $e^{-i\omega t}$) in terms of spherical waves generated by sources on a surface, typically planar. Here, we seek a description of the vector electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ in terms of plane waves rather than spherical waves.

The solution builds on the spirit of Weyl's representation [2] of a scalar spherical wave in terms of scalar plane waves,

$$\frac{e^{ikr}}{r} = \frac{i}{2\pi} \int\int \frac{e^{i(k_x x + k_y y + k_z |z|)}}{k_z}\, dk_x\, dk_y = \frac{i}{2\pi} \int\int \frac{e^{i\mathbf{k}^{\pm}\cdot\mathbf{x}}}{k_z}\, dk_x\, dk_y, \tag{4}$$

where,

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} = \begin{cases} \sqrt{k^2 - k_x^2 - k_y^2} & \text{if } k_x^2 + k_y^2 \le k^2, \\ i\sqrt{k_x^2 + k_y^2 - k^2} & \text{if } k_x^2 + k_y^2 > k^2, \end{cases} \tag{5}$$

and,[2]

$$\mathbf{k}^{\pm} = \begin{cases} (k_x, k_y, k_z) & \text{if } z \ge 0, \\ (k_x, k_y, -k_z) & \text{if } z < 0. \end{cases} \tag{6}$$

The plane waves are homogeneous when $k_x^2 + k_y^2 \le k^2$, but they are inhomogeneous (evanescent, and significant only close to the plane $z = 0$) otherwise. The plane-wave decomposition (4) is not spherically symmetric, which is a reminder that all plane waves (and especially evanescent plane waves = "classical virtual photons") are convenient mathematical fictions, rather than entities with crisp physical reality.

The decomposition of three-dimensional scalar waves with a velocity c into homogenous plane waves of the same velocity was perhaps first considered by Stoney [4], and elaborated upon by Whittaker [5]. Inclusion of inhomogeneous waves in the decomposition of scalar waves was first considered by Weyl [2]. The representation of electromagnetic fields in terms of plane-wave solutions to Maxwell's equations was perhaps first considered by Clemmow [6] and [7], which latter work this note follows.[3]

2.1 Temporal Fourier Analysis

As usual, a (possibly complex) scalar function $f(\mathbf{x}, t)$ has a temporal Fourier representation,

$$f(\mathbf{x}, t) = \int_{-\infty}^{\infty} f_\omega(\mathbf{x}, \omega)\, e^{-i\omega t}\, d\omega, \tag{7}$$

where,

$$f_\omega(\mathbf{x}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\mathbf{x}, t)\, e^{i\omega t}\, dt. \tag{8}$$

[2]The notation $\mathbf{k}^{\pm}$ follows [3].

[3]If one's attention is restricted to the fields far from the source, a somewhat simpler analysis obtains, as discussed in [8] (which seems unaware of the full solution of [6, 7]).

It is customary to consider only positive angular frequencies, so by noting that $f_\omega(\mathbf{x}, -\omega) = f_\omega^\star(\mathbf{x}, \omega)$ we can also write,

$$f(\mathbf{x}, t) = 2Re \int_0^\infty f_\omega(\mathbf{x}, \omega)\, e^{-i\omega t}\, d\omega. \tag{9}$$

Throughout the rest of this note we assume that such a temporal Fourier analysis can be made for all relevant scalar functions.

2.2 Plane Electromagnetic Waves

We consider uniform, isotropic, nonconducting media with (frequency-dependent) relative permittivity ϵ and relative permeability μ. Maxwell's equations for the Fourier components $\mathbf{E}_\omega$ and $\mathbf{B}_\omega$ of the electric and magnetic fields in such media are,

$$\boldsymbol{\nabla}\cdot\mathbf{E}_\omega = \frac{4\pi\rho_\omega}{\epsilon}, \qquad \boldsymbol{\nabla}\times\mathbf{E}_\omega = ik_0\mathbf{B}_\omega, \qquad \boldsymbol{\nabla}\cdot\mathbf{B}_\omega = 0, \qquad \boldsymbol{\nabla}\times\mathbf{B}_\omega = \frac{4\pi\mu}{c}\mathbf{J}_\omega - in^2k_0\mathbf{E}_\omega, \tag{10}$$

where ρ_ω and $\mathbf{J}_\omega$ are the Fourier components of the free charge and current densities,

$$k_0 = \frac{\omega}{c}\,, \tag{11}$$

and $n = \sqrt{\epsilon\mu}$ is the index of refraction of the medium.

In regions with no free charge or current all six scalar components of $\mathbf{E}_\omega$ and $\mathbf{B}_\omega$ obey the Helmholtz equation,

$$\nabla^2 f_\omega + k^2 f_\omega = 0, \tag{12}$$

where,

$$k = \sqrt{\epsilon\mu}k_0 = nk_0 = \frac{n\omega}{c}\,. \tag{13}$$

In general, ϵ and μ are frequency dependent, which implies that they are complex functions. However, at frequencies not close to the natural frequencies of the medium the imaginary parts of ϵ and μ are very small in many (transparent) media of interest, so we approximate the index n and the wave number k as being purely real throughout the rest of this note.

Plane-wave solutions to the Helmholtz equation (12) have the form,

$$f_\omega(\mathbf{x}) = f_0\, e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{14}$$

where the (possibly complex) wave vector $\mathbf{k} = Re\mathbf{k} + iIm\mathbf{k}$ obeys,

$$k^2 = \mathbf{k}^2 = k_x^2 + k_y^2 + k_z^2. \tag{15}$$

The plane waves are homogeneous when $Im\mathbf{k}$ is zero, and inhomogeneous when $Im\mathbf{k}$ is nonzero.

In general, any of the components (k_x, k_y, k_z) of the wave vector might be complex. We consider the case that two components are real and one, k_z say, is complex. In the approximation that k is real, eq. (15) tells us that k_z is real when $k_x^2 + k_y^2 \le k^2$, and purely imaginary when $k_x^2 + k_y^2 > k^2$. When k_z is imaginary the plane waves are evanescent, dying out in the z-coordinate, and propagating in x and y.

We can introduce the (possibly complex) unit vector $\hat{\mathbf{k}} = \mathbf{k}/k$, such that $\hat{\mathbf{k}}^2 = 1$. Using eq. (14) for components of the plane wave $\mathbf{E}_{\omega,\mathbf{k}}$, $\mathbf{B}_{\omega,\mathbf{k}}$ we see that Maxwell's equations (10) (in regions with no free charge or current) imply that,

$$\hat{\mathbf{k}}\cdot\mathbf{E}_{\omega,\mathbf{k}} = 0, \qquad \hat{\mathbf{k}}\times\mathbf{E}_{\omega,\mathbf{k}} = \frac{k_0}{k}\mathbf{B}_{\omega,\mathbf{k}} = \frac{\mathbf{B}_{\omega,\mathbf{k}}}{n}, \qquad \hat{\mathbf{k}}\cdot\mathbf{B}_{\omega,\mathbf{k}} = 0, \qquad \hat{\mathbf{k}}\times\mathbf{B}_{\omega,\mathbf{k}} = -n\mathbf{E}_{\omega,\mathbf{k}}. \quad (16)$$

Thus, $\hat{\mathbf{k}}$, $\mathbf{E}_{\omega,\mathbf{k}}$ and $\mathbf{B}_{\omega,\mathbf{k}}$ are mutually orthogonal, and $|\mathbf{B}_\omega| = n\,|\mathbf{E}_\omega|$.

The time-average density of electromagnetic energy stored in a plane wave is,

$$\langle u_{\omega,\mathbf{k}}\rangle = \frac{\epsilon\,|\mathbf{E}_{\omega,\mathbf{k}}|^2}{16\pi} + \frac{|\mathbf{B}_{\omega,\mathbf{k}}|^2}{16\pi\mu} = \frac{\epsilon\,|\mathbf{E}_{\omega,\mathbf{k}}|^2}{8\pi}\,. \quad (17)$$

The transport of energy by a plane wave is described by the Poynting vector, whose time average is,

$$\langle\mathbf{S}_{\omega,\mathbf{k}}\rangle = \frac{c}{8\pi}Re(\mathbf{E}_{\omega,\mathbf{k}}\times\mathbf{H}^\star_{\omega,\mathbf{k}}) = \frac{cn}{8\pi\mu}Re[\mathbf{E}_{\omega,\mathbf{k}}\times(\hat{\mathbf{k}}^\star\times\mathbf{E}^\star_{\omega,\mathbf{k}})] = \frac{c\,\epsilon}{8\pi n}\,|\mathbf{E}_{\omega,\mathbf{k}}|^2\,Re\hat{\mathbf{k}}^\star = \frac{c}{n}\,\langle u_{\omega,\mathbf{k}}\rangle\,. \quad (18)$$

In the case of evanescent waves, $Re\hat{\mathbf{k}}^\star$ has no component in the z-direction, and their energy flows only perpendicular to the direction in which these waves die out.

For each direction of the unit wave vector $\hat{\mathbf{k}}$ there are two independent plane-wave solutions, commonly characterized by their polarization. We can, however, omit further discussion of polarization and proceed to relate the plane waves to the source charges and currents.[4]

2.3 Four-Dimensional Fourier Analysis

We can augment the temporal Fourier analysis that determined the fields $\mathbf{E}_\omega$, $\mathbf{B}_\omega$ and $\mathbf{J}_\omega$ with 3-dimensional spatial Fourier analyses of the form,

$$\mathbf{B}_\omega(\mathbf{x}) = \int\!\!\int\!\!\int \mathbf{B}_{\omega,\mathbf{k}}\,e^{i\mathbf{k}\cdot\mathbf{x}}\,d^3\mathbf{k}, \qquad \text{where} \qquad \mathbf{B}_{\omega,\mathbf{k}} = \frac{1}{(2\pi)^3}\int\!\!\int\!\!\int \mathbf{B}_\omega(\mathbf{x})\,e^{-i\mathbf{k}\cdot\mathbf{x}}\,d^3\mathbf{x}, \quad (19)$$

and the wave vector $\mathbf{k}$ is purely real. The Fourier components $\mathbf{E}_{\omega,\mathbf{k}}$, $\mathbf{B}_{\omega,\mathbf{k}}$ and $\mathbf{J}_{\omega,\mathbf{k}}$ are related to one another by the transforms of Maxwell's equations (10),

$$\mathbf{k}\cdot\mathbf{E}_{\omega,\mathbf{k}} = -4\pi i\rho_{\omega,\mathbf{k}}, \quad \mathbf{k}\times\mathbf{E}_{\omega,\mathbf{k}} = k_0\mathbf{B}_{\omega,\mathbf{k}}, \quad \mathbf{k}\cdot\mathbf{B}_{\omega,\mathbf{k}} = 0, \quad \mathbf{k}\times\mathbf{B}_{\omega,\mathbf{k}} = -\frac{4\pi i\mu}{c}\mathbf{J}_{\omega,\mathbf{k}} - n^2k_0\mathbf{E}_{\omega,\mathbf{k}}. \quad (20)$$

[4]For evanescent waves the condition that $\mathbf{k}^2$ be real implies that the vectors $Re\mathbf{k}$ and $Im\mathbf{k}$ are orthogonal. These two directions, together with the direction of their cross product, can be taken as the basis for a definition of the wave polarization. For example, taking $Re\mathbf{k}$ along the x-axis and $Im\mathbf{k}$ along the z-axis we can write $\mathbf{k} = k\,\hat{\mathbf{k}} = k\,(\cosh a, 0, i\sinh a)$ for any real number a, whence $\mathbf{k}^2 = k^2$.

One polarization (called E-polarization by Clemmow [7]) has $\mathbf{E}_\omega$ perpendicular to both $Re\mathbf{k}$ and $Im\mathbf{k}$ and $\mathbf{B}_\omega$ in their plane, $\mathbf{E}_{\omega,E} = (0, E_E, 0)\,e^{i\mathbf{k}\cdot\mathbf{x}}$. The other polarization (H-polarization) has $\mathbf{B}_{\omega,H} = (0, nE_H, 0)\,e^{i\mathbf{k}\cdot\mathbf{x}}$. The relations $\mathbf{B}_\omega = \hat{\mathbf{k}}\times n\mathbf{E}_\omega$ and $\mathbf{E}_\omega = \hat{\mathbf{k}}\times\mathbf{B}_\omega/n$ imply that $\mathbf{B}_{\omega,E} = (inE_E\sinh a, 0, nE_E\cosh a)\,e^{i\mathbf{k}\cdot\mathbf{x}}$ and $\mathbf{E}_{\omega,H} = (iE_H\sinh a, 0, E_H\cosh a)\,e^{i\mathbf{k}\cdot\mathbf{x}}$. For both polarizations, $Re(\mathbf{E}_\omega\times\mathbf{B}^\star_\omega) = nE^2\cosh a\,\hat{\mathbf{x}} = n\,|\mathbf{E}_\omega|^2\,Re\hat{\mathbf{k}}^\star$, which provides a more detailed justification of eq. (18).

Combining these, we have,

$$\mathbf{B}_{\omega,\mathbf{k}} = -\frac{4\pi i\mu}{c}\frac{\mathbf{k}\times\mathbf{J}_{\omega,\mathbf{k}}}{n^2k_0^2-\mathbf{k}^2} = \frac{4\pi i\mu}{c}\frac{\mathbf{k}\times\mathbf{J}_{\omega,\mathbf{k}}}{\mathbf{k}^2-k^2} = \frac{4\pi i\mu}{c}\frac{\mathbf{k}\times\mathbf{J}_{\omega,\mathbf{k}}}{k_z^2-(k^2-k_x^2-k_y^2)}\,, \tag{21}$$

and hence,

$$\mathbf{B}_\omega(\mathbf{x}) = \frac{4\pi i\mu}{c}\int\int\int \frac{\mathbf{k}\times\mathbf{J}_{\omega,\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, d^3\mathbf{k}}{\left(k_z-\sqrt{k^2-k_x^2-k_y^2}\right)\left(k_z+\sqrt{k^2-k_x^2-k_y^2}\right)}\,. \tag{22}$$

The expansion (22) expresses the magnetic field in terms of mathematical plane waves, but are these electromagnetic plane waves? As discussed in sec. 2.2, electromagnetic planes waves must satisfy the conditions $k^2 = \mathbf{k}^2$, eq. (15), and $\mathbf{k}\cdot\mathbf{B}_{\omega,\mathbf{k}} = 0$, eq. (16), in a medium of index n. The second, but not the first of these conditions is met by the expansion (21)-(22).

Note that the denominator of the integrand of eq. (22) vanishes when the first condition holds. If we consider, say, the integration over k_z to be a contour integration with the contour completed at infinity, then Cauchy's integral theorem has the effect of enforcing the condition (15). When $k_x^2+k_y^2 > k^2$, $k_z = i\sqrt{k_x^2+k_y^2-k^2}$ is pure imaginary and the plane waves are evanescent. Physically, these waves should die out, rather than grow, with $|z|$, which is insured when $z > 0$ by completing the contour on a semicircle at infinity for positive $\mathrm{Im}(k_z)$ and deforming the contour along the $Re(k_z)$ axis to enclose the pole at $k_z = \sqrt{k^2-k_x^2-k_y^2}$ (when this is real) but not that at the negative of this value (and completing the contour on a semicircle at infinity for negative $\mathrm{Im}(k_z)$ if $z < 0$). Using Cauchy's integral theorem we now obtain,[5]

$$\mathbf{B}_\omega(\mathbf{x}) = -\frac{4\pi^2\mu}{c}\int\int\frac{\mathbf{k}^\pm\times\mathbf{J}_{\omega,\mathbf{k}^\pm}}{k_z}\,e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\,dk_x\,dk_y, \tag{23}$$

where k_z and $\mathbf{k}^\pm$ are defined in eqs. (5)-(6) and the $\pm$ sign holds for $z \gtrless 0$. Then, using the last of eq. (16), $\mathbf{E}_{\omega,\mathbf{k}} = -\mathbf{k}\times\mathbf{B}_{\omega,\mathbf{k}}/nk$, to determine the electric field of the plane wave we find,

$$\mathbf{E}_\omega(\mathbf{x}) = \frac{4\pi^2\mu}{cnk}\int\int\frac{\mathbf{k}^\pm\times(\mathbf{k}^\pm\times\mathbf{J}_{\omega,\mathbf{k}^\pm})}{k_z}\,e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\,dk_x\,dk_y. \tag{24}$$

The expansions (23)-(24) are the electromagnetic equivalent of the more familiar Fresnel diffraction integral[6] for scalar fields, and have the advantage of being "exact" throughout all space. Qualitatively, these expansions indicate that the fields in space are a kind of Fourier transform of the source distribution, which is often restricted to an aperture in the plane $z = 0$.

2.4 Surface Currents Only on the Plane $z = 0$

We can confirm the general result (23) by a different argument for the case that the currents are confined to the plane $z = 0$. Then, the magnetic field obeys the following symmetry with

[5] This argument follows sec. 3.4 of [7], where the discussion is artificially restricted to currents on the plane $z = 0$.

[6] See, for example, http://www.fourieroptics.org.uk/fresnelDiff.html

respect to this plane,[7]

$$B_x(x, y, -z) = -B_x(x, y, z), \quad B_y(x, y, -z) = -B_y(x, y, z), \quad B_z(x, y, -z) = B_z(x, y, z). \tag{25}$$

From consideration of small loops with areas normal to the x and y axes and which surround a surface-current-density element $\mathbf{K}(x, y, 0, t)$, the fourth Maxwell equation and eq. (25) imply that,

$$B_x(x, y, 0^+, t) = \frac{2\pi\mu}{c} K_y(x, y, 0, t), \qquad \text{and} \qquad B_y(x, y, 0^+, t) = -\frac{2\pi\mu}{c} K_x(x, y, 0, t). \tag{26}$$

Of course, the temporal Fourier components obey the same form,

$$B_{\omega,x}(x, y, 0^+) = \frac{2\pi\mu}{c} K_{\omega,y}(x, y, 0), \qquad \text{and} \qquad B_{\omega,y}(x, y, 0^+) = -\frac{2\pi\mu}{c} K_{\omega,x}(x, y, 0). \tag{27}$$

The surface current density $\mathbf{K}_\omega$ can be further analyzed in a two-dimensional Fourier transform,

$$\mathbf{K}_\omega(x, y, 0) = \int\int \mathbf{K}_{\omega,\mathbf{k}} e^{i(k_x x + k_y y)}\, dk_x\, dk_y, \tag{28}$$

where,

$$\mathbf{K}_{\omega,\mathbf{k}} = \frac{1}{(2\pi)^2} \int\int \mathbf{K}_\omega(x, y, 0) e^{-i(k_x x + k_y y)}\, dx\, dy. \tag{29}$$

The expansion of $\mathbf{B}_\omega$ into homogeneous and inhomogeneous electromagnetic plane waves (which latter die out with increasing $|z|$) has the general form,

$$\mathbf{B}_\omega(\mathbf{x}) = \int\int \mathbf{B}_{\omega,\mathbf{k}^\pm}\, e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\, dk_x\, dk_y, \tag{30}$$

so that,

$$\mathbf{B}_\omega(x, y, 0^+) = \int\int \mathbf{B}_{\omega,\mathbf{k}}\, e^{i(k_x x + k_y y)}\, dk_x\, dk_y. \tag{31}$$

Then, equations (27) and (28) tell us that,

$$B_{\omega,\mathbf{k}^+,x} = \frac{2\pi\mu}{c} K_{\omega,\mathbf{k},y}, \qquad B_{\omega,\mathbf{k}^+,y} = -\frac{2\pi\mu}{c} K_{\omega,\mathbf{k},x}, \tag{32}$$

and the condition (16) that $\mathbf{k}^+ \cdot \mathbf{B}_{\omega,\mathbf{k}^+} = 0$ leads to,

$$B_{\omega,\mathbf{k}^+,z} = -\frac{2\pi\mu}{c} \frac{k_x K_{\omega,\mathbf{k},y} - k_y K_{\omega,\mathbf{k},x}}{k_z}\,. \tag{33}$$

Equations (32)-(33) can be combined into the form,

$$\mathbf{B}_{\omega,\mathbf{k}^+} = -\frac{2\pi\mu}{c} \frac{\mathbf{k}^+ \times \mathbf{K}_{\omega,\mathbf{k}}}{k_z} = -\frac{2\pi\mu}{c} \frac{\mathbf{k}^+ \times \mathbf{K}_{\omega,\mathbf{k}^+}}{k_z} \qquad (z > 0), \tag{34}$$

[7] See, for example, [9].

where we note that $\mathbf{K}_{\omega,\mathbf{k}} = \mathbf{K}_{\omega,\mathbf{k}^+} = \mathbf{K}_{\omega,\mathbf{k}^-}$ since the Fourier component $\mathbf{K}_{\omega,\mathbf{k}}$ does not depend on k_z.

For $z < 0$, a similar argument for $\mathbf{B}_\omega(x, y, 0^-)$ leads to eqs. (34) with the substitution of $\mathbf{k}^-$ for $\mathbf{k}^+$. Hence,

$$\mathbf{B}_\omega(\mathbf{x}) = -\frac{2\pi\mu}{c}\int\int \frac{\mathbf{k}^\pm \times \mathbf{K}_{\omega,\mathbf{k}^\pm}}{k_z}\, e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\, dk_x\, dk_y, \tag{35}$$

where k_z and $\mathbf{k}^\pm$ are defined in eqs. (5) and (6) and the $\pm$ sign holds for $z \gtrless 0$. This is a special case of the previous result (23), since the volume current density $\mathbf{J}$ corresponding to the surface current density $\mathbf{K}$ on the plane $z = 0$ is,

$$J_x = K_x\, \delta(z), \qquad J_y = K_y\, \delta(z), \qquad J_z = 0, \tag{36}$$

for which the Fourier transforms are,

$$J_{\omega,\mathbf{k}^\pm,x} = \frac{1}{(2\pi)^3}\int\int\int J_{\omega,x}\, \delta(z)\, e^{-i\mathbf{k}^\pm\cdot\mathbf{x}}\, d^3\mathbf{x} = \frac{1}{(2\pi)^3}\int\int K_{\omega,x}\, e^{-i(k_x x + k_y y)}\, dx\, dy = \frac{1}{2\pi} K_{\omega,\mathbf{k}^\pm,x}, \tag{37}$$

etc.

2.5 Oscillating Electric Dipole

As a first example, we consider a small oscillating electric dipole of moment $\mathbf{p} = p_0\, e^{-i\omega t}\, \hat{\mathbf{p}}$, so that $\mathbf{p}_\omega = p_0\, \hat{\mathbf{p}}$. We follow the usual method of Hertz (see, for example, sec. 9.2 of [10]) which shows that, on integration by parts and using the continuity equation $\boldsymbol{\nabla}\cdot\mathbf{J}_\omega = i\omega\rho_\omega$,

$$\int \mathbf{J}_\omega\, d^3\mathbf{x} = -\int \mathbf{x}(\boldsymbol{\nabla}\cdot\mathbf{J}_\omega)\, d^3\mathbf{x} = -i\omega\int \mathbf{x}\rho_\omega\, d^3\mathbf{x} = -i\omega\mathbf{p}_{\boldsymbol{\omega}}. \tag{38}$$

For a "point" dipole at the origin we therefore write,

$$\mathbf{J}_\omega = -i\omega\mathbf{p}_\omega\, \delta^3(\mathbf{x}). \tag{39}$$

Then,

$$\mathbf{J}_{\omega,\mathbf{k}^\pm} = \frac{1}{(2\pi)^3}\int\int\int \mathbf{J}_\omega(\mathbf{x})\, e^{-i\mathbf{k}^\pm\cdot\mathbf{x}}\, d^3\mathbf{x} = \frac{-i\omega\mathbf{p}_\omega}{(2\pi)^3}\,. \tag{40}$$

Taking the dipole to be in vacuum, its magnetic field is given by eq. (23) as,

$$\mathbf{B}_\omega(\mathbf{x}) = \frac{ik}{2\pi}\int\int \frac{\mathbf{k}^\pm \times \mathbf{p}_\omega}{k_z}\, e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\, dk_x\, dk_y = \boldsymbol{\nabla}\times\mathbf{A}_\omega, \tag{41}$$

where $k = k_0 = \omega/c$, $k_z = \sqrt{k^2 - k_x^2 - k_y^2}$ and,

$$\mathbf{A}_\omega(\mathbf{x}) = \frac{k\mathbf{p}_\omega}{2\pi}\int\int \frac{e^{i\mathbf{k}^\pm\cdot\mathbf{x}}}{k_z}\, dk_x\, dk_y = -ik\mathbf{p}_\omega\, \frac{e^{ikr}}{r}\,, \tag{42}$$

recalling Weyl's expansion (4). Thus, we obtain the standard form for the vector potential of an oscillating "point" dipole (or conversely, we can use the standard form of the vector potential together with eq. (42) to provide a "proof" of eq. (4)).

The corresponding plane-wave expansion of the electric field follows from eq. (24),

$$\mathbf{E}_\omega(\mathbf{x}) = -\frac{i}{2\pi} \int\int \frac{\mathbf{k}^\pm \times (\mathbf{k}^\pm \times \mathbf{p}_\omega)}{k_z} e^{i\mathbf{k}^\pm \cdot \mathbf{x}} \, dk_x \, dk_y. \tag{43}$$

The expansions (41)-(43) contain both homogeneous and inhomogeneous plane waves, which is satisfactory in that the fields of an oscillating dipole contain energy in the "near zone" that is exchanged with the source rather than radiated away. We can say that this energy corresponds to the inhomogeneous plane waves, whose energy flow is the same in magnitude for $\pm k_x$ and for $\pm k_y$ such that the average flow of energy in the inhomogeneous waves is zero. However, the details of the characterization of the inhomogeneous waves is somewhat unsatisfactory in that these depend on the arbitrary choice of orientation of the plane called $z = 0$. To this author, there is very limited physical reality to the inhomogeneous plane waves identified in the expansions (41)-(43).[8]

2.6 Radiated Power

When considering the power radiated by a time-dependent current distribution, our expansion of the fields in plane waves (in which the plane $z = 0$ plays a special role) leads us to evaluate the power crossing a plane with $z > 0$. Because the orientation of the x-y-z axes is arbitrary, we obtain a general result. In this way we can deduce the total power radiated into any half space, as discussed in sec. 3.1.2 of [7].

However, this does not provide a very detailed picture of the flow of radiated power. Far from all current sources the radiated power flows radially outwards from the centroid of the source, and the plane-wave expansion of the fields, together with appropriate approximations, leads to the usual characterization of the power radiated into a specified solid angle, as discussed in sec. 3.2 of [7]. Of course, in this approximation the spirit of the plane-wave expansion is abandoned in favor of the more usual approach based on spherical waves.

Close to the sources (Fresnel zone in optics, near zone in antenna theory) the plane-wave expansion provides an alternative description to the more usual formulation in terms of the fields $\mathbf{E}_\omega(\mathbf{x})$ and $\mathbf{B}_\omega(\mathbf{x})$. However, because plane waves have, by definition, infinite transverse extent, the physical meaning of such waves in small volumes is ambiguous (to this author). It remains that the standard description of the (time-average) flow of energy through an electromagnetic field is via the Poynting vector,

$$\langle \mathbf{S} \rangle = \frac{c}{8\pi} Re(\mathbf{E} \times \mathbf{H}^\star), \tag{44}$$

and its temporal Fourier components,

$$\langle \mathbf{S}_\omega \rangle = \frac{c}{8\pi} Re(\mathbf{E}_\omega \times \mathbf{H}_\omega^\star). \tag{45}$$

Further decomposition of the fields, by spatial Fourier transforms, into electromagnetic plane waves does not, in general, provide much additional physical insight as to the nature of the flow of energy.

Of possible interest will be the plane-wave expansion of the fields of a charge with uniform velocity.

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[8] Plane-wave representations of arbitrary multipoles are discussed in [11].

2.7 Charge with Uniform Velocity

The first calculation of the fields of a charge with uniform velocity may have been made by Maxwell [12] who understood that the fields are fore-aft symmetric, but whose miscalculation of asymmetric fields via Lorenz' retarded potentials [13, 14] held up acceptance of this powerful tool for many years. The first published calculation of the fields of a charge with $v \ll c$ was made by J.J. Thomson [15], and the first calculation valid for any $v < c$ was given by Heaviside [16]. See also [17]. Shortly thereafter, Heaviside also calculated the fields for motion with $v > c$ [18], anticipating by many years what is now called the Čerenkov effect [1].

The plane-wave expansion of the fields of a uniformly moving charge is treated in sec. 7.2 of [7]. A somewhat earlier discussion was given in [19]. See also [20].

The current density of charge q that moves with velocity v along the x-axis can be written as,

$$\mathbf{J} = qv\,\delta(x - vt)\,\delta(y)\,\delta(z)\,\hat{\mathbf{x}}. \tag{46}$$

Then, its temporal Fourier transform is, recalling eq. (13),

$$\mathbf{J}_\omega = \frac{q\,\hat{\mathbf{x}}}{2\pi}\,\delta(y)\,\delta(z)\int \delta(x - vt)e^{i\omega t}\,v\,dt = \frac{q\,e^{i\omega x/v}\,\hat{\mathbf{x}}}{2\pi}\,\delta(y)\,\delta(z) = \frac{q\,e^{ickx/nv}\,\hat{\mathbf{x}}}{2\pi}\,\delta(y)\,\delta(z). \tag{47}$$

The spatial Fourier transform of this is,

$$\mathbf{J}_{\omega,\mathbf{k}} = \frac{q\,\hat{\mathbf{x}}}{(2\pi)^4}\int\!\!\int\!\!\int e^{i\omega x/v}\,\delta(y)\,\delta(z)\,e^{-i\mathbf{k}\cdot\mathbf{x}}d^3\mathbf{x} = \frac{q\,\hat{\mathbf{x}}}{(2\pi)^3}\delta(k_x - \omega/v), \tag{48}$$

noting that $\delta(k) = \int e^{-ikx}\,dx/2\pi$. Using this in eq. (23), the temporal Fourier components of the magnetic field are given by,

$$\begin{aligned}\mathbf{B}_\omega(\mathbf{x}) &= -\frac{4\pi^2\mu}{c}\int\!\!\int \frac{q}{(2\pi)^3}\delta(k_x - \omega/v)\frac{\mathbf{k}^\pm \times \hat{\mathbf{x}}}{k_z}\,e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\,dk_x\,dk_y \\ &= -\frac{q\mu\,e^{i\omega x/v}}{2\pi c}\int \frac{\pm k_z\,\hat{\mathbf{y}} - k_y\,\hat{\mathbf{z}}}{k_z}\,e^{i(k_y y \pm k_z z)}\,dk_y,\end{aligned} \tag{49}$$

where,

$$k_z = \sqrt{k^2(1 - c^2/n^2v^2) - k_y^2} = \sqrt{\omega^2(n^2 - c^2/v^2)/c^2 - k_y^2} = i\frac{\omega}{c}\sqrt{c^2/v^2 - n^2 + l_y^2}, \tag{50}$$

and $l_y = ck_y/\omega$. Recalling that $\mathbf{E}_{\omega,\mathbf{k}^\pm} = -\mathbf{k}^\pm \times \mathbf{B}_{\omega,\mathbf{k}^\pm}/nk = -\mathbf{k}^\pm \times c\mathbf{B}_{\omega,\mathbf{k}^\pm}/n^2\omega$, the temporal Fourier expansion of the electric field follows from eq. (49) as,

$$\begin{aligned}\mathbf{E}_\omega(\mathbf{x}) &= \frac{q\mu}{2\pi n^2\omega}\int\!\!\int \delta(k_x - \omega/v)\,\mathbf{k}^\pm \times \frac{\pm k_z\,\hat{\mathbf{y}} - k_y\,\hat{\mathbf{z}}}{k_z}\,e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\,dk_x\,dk_y \\ &= \frac{q\,e^{i\omega x/v}}{2\pi\epsilon v}\int \frac{k_x(1 - n^2v^2/c^2)\,\hat{\mathbf{x}} + k_y\,\hat{\mathbf{y}} \pm k_z\,\hat{\mathbf{z}}}{k_z}\,e^{i(k_y y \pm k_z z)}\,dk_y.\end{aligned} \tag{51}$$

2.7.1 $v < c/n$

When the speed of the charge is less than the speed of light c/n in the surrounding medium, k_z is purely imaginary according to eq. (50), and all plane waves in the expansion (49) are evanescent. No radiation (to "infinity" [21]) is emitted by a charge moving uniformly at sublight speed.

The field $\mathbf{B}$ is axially symmetric, and azimuthal, with respect to the axis of motion, which is the x-axis here. The azimuthal field $B_\phi(\mathbf{x}, t)$ (about the x-axis) at distance $r_\perp$ from the x-axis can be evaluated as $-B_y(x, 0, r_\perp, t)$ using eq. (49),

$$\begin{aligned}
B_\phi &= -B_y(x,0,r_\perp,t) = -2Re\int_0^\infty B_{\omega,y}(x,0,r_\perp)\, e^{-i\omega t}\, d\omega \\
&= \frac{q\mu}{\pi c^2} Re \int dl_y \int_0^\infty \omega\, d\omega\, e^{i\omega x/v} e^{-(\omega r_\perp/c)\sqrt{c^2/v^2-n^2+l_y^2}} e^{-i\omega t} \quad \text{where} \quad l_y = ck_y/\omega \\
&= \frac{q\mu}{\pi r_\perp^2} Re \int \frac{dl_y}{\left[ic(x-vt)/r_\perp v - \sqrt{c^2/v^2-n^2+l_y^2}\right]^2} \\
&= \frac{q\mu}{\pi r_\perp^2} \int \frac{-[c(x-vt)/r_\perp v]^2 + c^2/v^2 - n^2 + l_y^2}{\left\{[c(x-vt)/r_\perp v]^2 + c^2/v^2 - n^2 + l_y^2\right\}^2}\, dl_y \\
&= \frac{q\mu}{\pi r_\perp^2} 2\pi i \frac{d}{dl_y} \left.\frac{-[c(x-vt)/r_\perp v]^2 + c^2/v^2 - n^2 + l_y^2}{\left[l_y + i\sqrt{[c(x-vt)/r_\perp v]^2 + c^2/v^2 - n^2}\right]^2}\right|_{l_y = i\sqrt{[c(x-vt)/r_\perp v]^2 + c^2/v^2 - n^2}} \\
&= \frac{q\mu v}{c} \frac{(1-n^2v^2/c^2)r_\perp}{[(x-vt)^2 + (1-n^2v^2/c^2)r_\perp^2]^{3/2}}, \qquad (52)
\end{aligned}$$

where the integral in l_y was evaluated by completing the contour at $+\infty$. This is the usual result for the magnetic field of a charge moving at constant, sublight speed.

For completeness, we also calculate the axially symmetric electric field $\mathbf{E}(x, 0, r_\perp, t)$ in the x-z plane. Comparing eqs. (49) and eq. (51), we infer that,

$$E_z(x,0,r_\perp,t) = -\frac{c}{\epsilon\mu v} B_y(x,0,r_\perp,t) = \frac{q}{\epsilon} \frac{(1-n^2v^2/c^2)r_\perp}{[(x-vt)^2 + (1-n^2v^2/c^2)r_\perp^2]^{3/2}}. \qquad (53)$$

The y-component of Faraday's law tells us that,

$$\frac{\partial E_x(x,0,r_\perp,t)}{\partial r_\perp} = \frac{\partial E_z(x,0,r_\perp,t)}{\partial x} - \frac{1}{c}\frac{\partial B_y(x,0,r_\perp,t)}{\partial t} = -\frac{3q}{\epsilon} \frac{(1-n^2v^2/c^2)^2(x-vt)r_\perp}{[(x-vt)^2 + (1-n^2v^2/c^2)r_\perp^2]^{3/2}}, \qquad (54)$$

which integrates to,

$$E_x(x,0,r_\perp,t) = \frac{q}{\epsilon} \frac{(1-n^2v^2/c^2)(x-vt)}{[(x-vt)^2 + (1-n^2v^2/c^2)r_\perp^2]^{3/2}}. \qquad (55)$$

Thus, the electric field is radial with respect to the position of the charge, $(x - vt, 0, 0)$, and is related to the magnetic field by,

$$\mathbf{E} = \frac{n\mathbf{v}}{c} \times \mathbf{B}. \qquad (56)$$

2.7.2 The Plane-Wave Spectrum of the Coulomb Field ($v = 0$)

In the limit that the speed v of the charge goes to zero the magnetic field (52) vanishes, while the electric field becomes the Coulomb field $\mathbf{E}(v = 0) = q\,\hat{\mathbf{r}}/r^2$, where $\mathbf{r}$ is the distance from the charge to the observer.

To deduce the electric field from its plane-wave expansion (51) we write,

$$\begin{aligned}
\mathbf{E}(\mathbf{x}, t) &= 2Re \int_0^\infty \mathbf{E}_\omega(\mathbf{x})\, e^{-i\omega t}\, d\omega \\
&= \frac{q}{\pi\epsilon} Re \int dk_x \int dk_y \int_0^\infty \frac{d\omega}{\omega}\, \delta(k_x - \omega/v)\, \mathbf{k}^\pm \times \frac{\pm k_z\, \hat{\mathbf{y}} - k_y\, \hat{\mathbf{z}}}{k_z}\, e^{i\mathbf{k}^\pm\cdot\mathbf{x}} e^{-i\omega t} \\
&= \frac{q}{\pi\epsilon} Re \int\!\!\int \frac{k_x(1 - n^2v^2/c^2)\, \hat{\mathbf{x}} + k_y\, \hat{\mathbf{y}} \pm k_z\, \hat{\mathbf{z}}}{k_z}\, e^{i\mathbf{k}^\pm\cdot\mathbf{x}} e^{-ik_x vt}\, dk_x\, dk_y,
\end{aligned} \tag{57}$$

where now,

$$k = \frac{\omega n}{c} = \frac{k_x n v}{c}, \qquad \text{and} \qquad k_z = \sqrt{k^2 - k_x^2 - k_y^2} = i\sqrt{k_x^2(1 - n^2v^2/c^2) + k_y^2}. \tag{58}$$

In the limit that $v = 0$, $k_z = i\sqrt{k_x^2 + k_y^2}$, $k^2 = 0 = (\mathbf{k}^\pm)^2$ and we obtain the expansion,

$$\begin{aligned}
\mathbf{E}(\mathbf{x}, v = 0) &= \frac{q}{\pi\epsilon} Re \int\!\!\int \frac{k_x\, \hat{\mathbf{x}} + k_y\, \hat{\mathbf{y}} \pm k_z\, \hat{\mathbf{z}}}{k_z}\, e^{i\mathbf{k}^\pm\cdot\mathbf{x}}\, dk_x\, dk_y \\
&= -\frac{q}{\pi\epsilon} Re \int\!\!\int \frac{i\mathbf{k}^\pm\, e^{i\mathbf{k}^\pm\cdot\mathbf{x}}}{\sqrt{k_x^2 + k_y^2}}\, dk_x\, dk_y.
\end{aligned} \tag{59}$$

The Fourier components $\mathbf{E}_{\mathbf{k}^\pm}$ of this expansion obey the relation $\mathbf{k}^\pm \cdot \mathbf{E}_{\mathbf{k}^\pm} = 0$ as v and k go to zero. So we can say that the expansion (59) expresses the Coulomb field in terms of zero-frequency electromagnetic plane waves, all of which are evanescent since k_z is pure imaginary. This expansion is therefore conceptually superior to that of eq. (3) (although eq. (59) suffers from the arbitrariness of the choice of the plane $z = 0$.)

An expansion of the Coulomb field in evanescent plane waves was perhaps first given in [22].

Presumably the classical gravitational field can be decomposed into gravitational plane waves. Then, a static gravitational field consists of zero-frequency plane waves ("virtual gravitons"), which are not affected by the gravitational field since they have zero frequency (zero energy in the quantum view). Hence, the exterior gravitational field of a black hole can exist without being "sucked" into it. Similarly, a black hole can have an exterior static electric field consisting of zero-frequency (zero energy) "virtual photons."

2.7.3 Čerenkov Radiation: $c/n < v < c$

When a charge q moves with speed v greater than that of light, c/n, in a medium (but with $v < c$, of course), the plane-wave expansion (49) contains both homogeneous and inhomogeneous waves, and radiation is therefore emitted. This is sometimes considered to be paradoxical in that the charge is not obviously accelerating. However, the radiation exists only when the charge moves through a medium with index of refraction greater than 1, in which case the charges in the medium are accelerated by the passing charge q, and we can say that is the medium, rather than the charge itself, which emits the radiation. Of course, the radiated energy must come from the charge itself, so there must be a (small) back reaction of the medium on the passing charge, which decelerates the latter.

The temporal expansion of the magnetic field is, from eq. (49),

$$\mathbf{B}_\omega(\mathbf{x}) = -\frac{q\mu\, e^{i\omega x/v}}{2\pi c} \int \frac{\pm k_z\,\hat{\mathbf{y}} - k_y\,\hat{\mathbf{z}}}{k_z}\, e^{i(k_y y \pm k_z z)}\, dk_y, \tag{60}$$

where,

$$k_x = \frac{\omega}{v} = \frac{ck}{nv} \qquad \text{and} \qquad k_z = \sqrt{k^2(1 - c^2/n^2v^2) - k_y^2}. \tag{61}$$

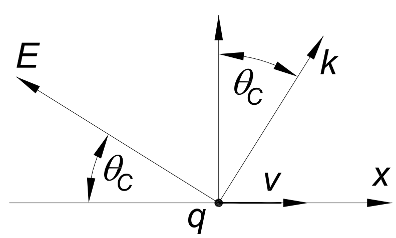
For plane waves in the x-y plane, $k_z = 0$ and $k_y = k\sqrt{1 - c^2/n^2v^2} = k_x(nv/c)\sqrt{1 - c^2/n^2v^2} = k_x\sqrt{n^2v^2/c^2 - 1}$, which is real, so these waves are homogeneous, and carry energy away from the charge q. Similarly, for plane waves in the x-z plane, $k_y = 0$ and $k_z = k_x\sqrt{n^2v^2/c^2 - 1}$. The wave vector $\mathbf{k}$ for the homogeneous waves (radiation field) does not have a continuous angular distribution, but always makes angle θ_C to the y-z plane, where,

$$\tan\theta_C = \frac{k_x}{k_y(k_z = 0)} = \frac{k_x}{k_z(k_y = 0)} = \frac{1}{\sqrt{n^2v^2/c^2 - 1}}\,, \tag{62}$$

so that,

$$\cos\theta_C = \frac{1}{\sqrt{1 + \tan^2\theta_C}} = \frac{c}{nv}\,. \tag{63}$$

The angle θ_C is the famous Čerenkov angle.



Since $\mathbf{k}^\pm \cdot \mathbf{E}_{\omega,\mathbf{k}^\pm} = 0$, the electric field points only in a single direction, namely at the Čerenkov angle θ_C to the negative x-axis (and the magnetic field circles about the x-axis). *This field configuration was first depicted by Heaviside [18].*

The temporal Fourier expansion of the electric field follows from eq. (51) as,

$$\mathbf{E}_\omega(\mathbf{x}) = \frac{q\, e^{i\omega x/v}}{2\pi\epsilon v} \int \frac{-k_x(n^2v^2/c^2 - 1)\,\hat{\mathbf{x}} + k_y\,\hat{\mathbf{y}} \pm k_z\,\hat{\mathbf{z}}}{k_z}\, e^{i(k_y y \pm k_z z)}\, dk_y. \tag{64}$$

The electric field in, say, the x-z plane for $z > 0$ consists of plane waves with $k_y = 0$, so we have that,

$$\begin{aligned}
\mathbf{E}(x, 0, z > 0, t) &= 2Re\int_0^\infty \mathbf{E}_\omega(x, 0, z > 0)\, e^{-i\omega t}\, d\omega \\
&= \frac{q}{\pi\epsilon v}\, Re\int_0^\infty [-\tan\theta_C(n^2v^2/c^2 - 1)\,\hat{\mathbf{x}} + \hat{\mathbf{z}}]\, e^{i\omega[(x + z/\tan\theta_C)/v - t]}\, d\omega \\
&= -\frac{2q}{\epsilon v}\,[\tan\theta_C(n^2v^2/c^2 - 1)\,\hat{\mathbf{x}} - \hat{\mathbf{z}}]\,\delta\left(\frac{x + z/\tan\theta_C}{v} - t\right). \qquad (65)
\end{aligned}$$

At time $t = 0$ the electric field in the x-z plane for $z > 0$ is nonzero only along the line $z = -x \tan\theta_C$, as shown in the figure. By a similar argument the magnetic field in the z-z plane is nonzero only along this line. The electric and magnetic fields are is azimuthally symmetric, so the fields are nonzero only on the Čerenkov cone. The present argument predicts infinite fields on this cone, whereas in reality the index n exceeds unity for only a finite range of frequency, and the fields extend slightly outside the cone, and are finite.

To deduce the frequency spectrum of the radiated power, we first note that the total energy d^2U that crosses an area element $d\mathbf{Area}$, integrated over all time, is,

$$\begin{aligned} d^2U &= \int_{-\infty}^{\infty} \mathbf{S} \cdot d\mathbf{Area}\, dt = \frac{c}{4\pi\mu}\, d\mathbf{Area} \cdot \int_{-\infty}^{\infty} \mathbf{E} \times \mathbf{B}\, dt \\ &= \frac{c}{4\pi\mu}\, d\mathbf{Area} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{E}_\omega \times \mathbf{B}\, e^{-i\omega t}\, d\omega\, dt \\ &= \frac{c}{2\mu}\, d\mathbf{Area} \cdot \int_{-\infty}^{\infty} \mathbf{E}_\omega \times \mathbf{B}_\omega^\star\, d\omega = \frac{c}{\mu}\, d\mathbf{Area} \cdot Re \int_0^{\infty} \mathbf{E}_\omega \times \mathbf{B}_\omega^\star\, d\omega, \end{aligned} \tag{66}$$

since $\mathbf{E}_\omega(-\omega) = \mathbf{E}_\omega^\star(\omega)$ and $\mathbf{B}_\omega(-\omega) = \mathbf{B}_\omega^\star(\omega)$. Equal amounts of energy cross any plane at $z > 0$ or at $z < 0$, so the total energy radiated is twice that which crosses a plane at $z > 0$,

$$U = \frac{2c}{\mu} \int\int dx\, dy\, \hat{\mathbf{z}} \cdot Re \int_0^{\infty} (\mathbf{E}_\omega \times \mathbf{B}_\omega^\star)_{z>0}\, d\omega. \tag{67}$$

The energy radiated per unit frequency interval and per unit path length of the charge's motion along the x-axis is independent of x. Since $B_x = 0$, we have,

$$\begin{aligned} \frac{d^2U}{d\omega\, dx} &= \frac{2c}{\mu} \int dy\, Re(E_{\omega,x} B_{\omega,y}^\star)_{z>0} \\ &= \frac{2c}{\mu} \frac{q}{2\pi\epsilon v} \frac{q\mu}{2\pi c}\, Re \int\int\int \frac{k_x(n^2v^2/c^2 - 1)}{k_z} e^{i(k_y y + k_z z)} e^{-i(k'_y y + k'^{\star}_z z)}\, dy\, dk_y\, dk'_y \\ &= \frac{2c}{\mu} \frac{q\omega}{2\pi\epsilon v^2} \frac{q\mu}{2\pi c} (n^2v^2/c^2 - 1)\, Re \int\int 2\pi\delta(k_y - k'_y) \frac{e^{i(k_z - k'^{\star}_z)z)}}{k_z}\, dk_y\, dk'_y \\ &= \frac{q^2\omega}{\pi\epsilon v^2} (n^2v^2/c^2 - 1)\, Re \int \frac{e^{-2Im(k_z)z}}{k_z}\, dk_y \\ &= \frac{q^2\omega n^2}{\pi\epsilon c^2} (1 - c^2/n^2v^2) \int_{-(\omega n/c)\sqrt{1-c^2/n^2v^2}}^{(\omega n/c)\sqrt{1-c^2/n^2v^2}} \frac{dk_y}{\sqrt{(\omega^2 n^2/c^2)(1 - c^2/n^2v^2) - k_y^2}} \\ &= \frac{q^2\mu\omega}{c^2} \left(1 - \frac{c^2}{n^2v^2}\right), \end{aligned} \tag{68}$$

where we note that in the fourth line the integrand is real only when k_z is real. Equation (68) is the standard result for the energy spectrum of Čerenkov radiation [23], which has the surprising feature (of little practical import) that a magnetic medium of index n emits μ times as much Čerenkov radiation as does a dielectric medium of the same index. As usual, we note that the index n can be greater than unity for only a finite range of frequencies, so that the total power radiated over all frequencies is finite.

The x-component of the electric field at the charge is, using eq. (64),

$$\begin{aligned}
E_x(vt,0,0,t) &= 2Re\int_0^\infty E_{\omega,x}(vt,0,0)\,e^{-i\omega t}d\omega \qquad (69)\\
&= -\int_0^\infty d\omega\,\frac{q\omega}{\pi\epsilon v^2}(n^2v^2/c^2-1)\,Re\int\frac{1}{k_z}\,dk_y = -\int_0^\infty d\omega\,\frac{q\mu\omega}{c^2}\left(1-\frac{c^2}{n^2v^2}\right).
\end{aligned}$$

This is a peculiar result in that we might have expected the electric field to diverge at the charge.[9] The field (69) acts opposite to the direction of the charge's velocity and decelerates it. The work done by the electron per unit path length is $-qE_x$, whose Fourier component at frequency ω equals the energy radiated per unit path length. That is, the work done by the electron on the Čerenkov field is transformed into the Čerenkov radiation.

For additional discussion of the relation of radiation by moving charges to the plane-wave decomposition of their fields, see [24].

2.8 TEM Waves in a Coaxial Transmission Line

We consider a (vacuum) coaxial cable centered on the x-axis (so that the currents are close to the plane $z=0$) with perfect conductors of radii a and $b>a$. This cable supports TEM waves with currents in a cylindrical coordinate system $(r=\sqrt{y^2+z^2},\theta,x)$,

$$I(r=a,x) = -I(r=b,x) = I_0\,e^{i(kx-\omega_0 t)}, \qquad (70)$$

where $\omega_0 = kc$ is the angular frequency of the waves, and fields,

$$\mathbf{E}(a<r<b) = \frac{2I_0}{cr}\,e^{i(kx-\omega_0 t)}\,\hat{\mathbf{r}}, \qquad \mathbf{B}(a<r<b) = \frac{2I_0}{cr}\,e^{i(kx-\omega_0 t)}\,\hat{\boldsymbol{\theta}}. \qquad (71)$$

The TEM fields are, of course, possible static fields multiplied by the waveform $e^{i(kx-\omega_0 t)}$.

Because the fields are zero for $r<a$ and $r>b$ they are not simply plane waves of the form $e^{i(kx-\omega_0 t)}$.

To display the plane-wave expansion (23)-(24) of the TEM fields, we first note that the nonzero Fourier components of the current density are,

$$\begin{aligned}
\mathbf{J}_{\omega_0,\mathbf{k}^\pm} &= \frac{1}{(2\pi)^3}\int\int\int \mathbf{J}_{\omega_0}(\mathbf{x})\,e^{-i\mathbf{k}^\pm\cdot\mathbf{x}}\,d^3\mathbf{x}\\
&= \frac{1}{(2\pi)^3}\frac{2I_0}{c}\int\int\int e^{ikx}\left(\frac{\delta(r-a)}{2\pi a}-\frac{\delta(r-b)}{2\pi b}\right)e^{-i\mathbf{k}^\pm\cdot\mathbf{x}}\,d^3\mathbf{x}\\
&= \frac{1}{(2\pi)^3}\frac{2I_0}{c}\int e^{i(k-k_x)x}\frac{dx}{2\pi}\int\int\left(\frac{\delta(r-a)}{a}-\frac{\delta(r-b)}{b}\right)e^{-i(k_y r\cos\theta\pm k_z r\sin\theta)}\,r\,dr\,d\theta\\
&= \frac{1}{(2\pi)^3}\frac{2I_0}{c}\delta(k-k_x)\int_0^{2\pi}\left(e^{-i(k_y a\cos\theta\pm k_z a\sin\theta)}-e^{-i(k_y b\cos\theta\pm k_z b\sin\theta)}\right)d\theta. \qquad (72)
\end{aligned}$$

Thus, $k_x = k$. Then according to eq. (5),

$$k_z = \sqrt{k^2-k_x^2-k_y^2} = ik_y, \qquad (73)$$

[9] For $v<c/n$, k_z is pure imaginary and $E_x(vt,0,0,t)=0$ at the charge according to eq. (69).

so that the electromagnetic plane-wave decomposition of the TEM wave in a coaxial cable consists only of inhomogeneous, plane electromagnetic waves. This is consistent with the usual view that an (infinite) coaxial cable transmits waves along its axial direction but does not "radiate". Also, the waves in the transmission line are closely associated with the conductors of the line, and are not "free" in the sense of homogeneous, plane electromagnetic waves.

The analysis for TEM waves on an arbitrary two-conductor transmission line parallel to the x-axis differs from the above only in the form of the double integral in eq. (72) over the transverse (y-z) plane; eq. (73) holds in all cases, and the plane-wave expansion involves only inhomogeneous ("evanescent") electromagnetic plane waves.[10] *Similarly, the plane-wave expansion for waves in/on cylindrical waveguides (including optical fibers as wall as hollow metallic guides) only involves inhomogeneous plane electromagnetic waves.*

2.9 Waves in a Rectangular Cavity/Waveguide (Dec. 2, 2020)

The decomposition of waves inside a rectangular cavity or waveguide with perfectly conducting walls is readily accomplished without the formalism of sec. 2 above.

The standing-wave solutions for electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ of angular frequency ω inside a perfectly-conducting, rectangular cavity of extent $0 < x < d_x$, $0 < y < d_y$, $0 < z < d_z$ can be written (in Gaussian units) as,[11]

$$\begin{aligned} E_x &= E_{0x} \cos k_x x \, \sin k_y y \, \sin k_z z \, e^{-i\omega t}, \\ E_y &= E_{0y} \sin k_x x \, \cos k_y y \, \sin k_z z \, e^{-i\omega t}, \\ E_z &= E_{0z} \sin k_x x \, \sin k_y y \, \cos k_z z \, e^{-i\omega t}, \end{aligned} \tag{74}$$

$$\begin{aligned} B_x &= B_{0x} \sin k_x x \, \cos k_y y \, \cos k_z z \, e^{-i\omega t}, \\ B_y &= B_{0y} \cos k_x x \, \sin k_y y \, \cos k_z z \, e^{-i\omega t}, \\ B_z &= B_{0z} \cos k_x x \, \cos k_y y \, \sin k_z z \, e^{-i\omega t}, \end{aligned} \tag{75}$$

where the wave vector $\mathbf{k}$ is,

$$\mathbf{k} = (k_x, k_y, k_z) = \left(\frac{l\pi}{d_x}, \frac{m\pi}{d_y}, \frac{n\pi}{d_z} \right), \qquad \text{and} \qquad k = \frac{\omega}{c}, \tag{76}$$

for any set of integers $\{l, m, n\}$, and c is the speed of light in vacuum (and inside the cavity). Nontrivial cavity modes exist only with two or three of $\{l, m, n\}$ nonzero, so there is no cavity mode in which the wave vector $\mathbf{k}$ is parallel to a wall of the cavity.

Further, Faraday's law,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \qquad \text{implies that} \qquad i\mathbf{B}_0 = \hat{\mathbf{k}} \times \mathbf{E}_0, \tag{77}$$

[10]In the quantum view, these inhomogeneous waves correspond to virtual photons, so the plane-wave photons associated with TEM waves on a transmission line are not "real," even when the wave velocity (of the total wave) is c as for a vacuum transmission line.

[11]See, for example, http://kirkmcd.princeton.edu/examples/ph501/ph501lecture14.pdf

so,

$$\mathbf{B}_0 \cdot \mathbf{E}_0 = 0, \quad |B_0| = |E_0|\,, \quad \text{and} \quad i\mathbf{B}_0 = \frac{(k_y E_{0z} - k_z E_{0y}, k_z E_{0x} - k_x E_{0z}, k_x E_{0y} - k_y E_{0x})}{k}\,. \tag{78}$$

The free-space Maxwell equations $\boldsymbol{\nabla} \cdot \mathbf{B} = 0 = \boldsymbol{\nabla} \cdot \mathbf{E}$ imply that,

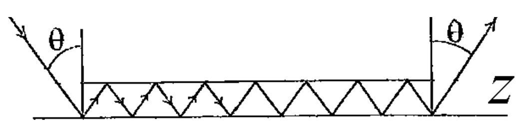
$$\hat{\mathbf{k}} \cdot \mathbf{E}_0 = 0 = \hat{\mathbf{k}} \cdot \mathbf{B}_0. \tag{79}$$

For each set of integers $\{l, m, n\}$ there are two orthogonal polarizations of the vectors $\mathbf{E}_0$ and $\mathbf{B}_0$. As usual, the physical fields are the real parts of eqs. (74)-(75).

The sines and cosines in eqs. (74)-(75) can be rewritten as sums of exponentials, leading to a decomposition into eight plane waves with the eight wave vectors $(\pm k_x, \pm k_y, \pm k_z)$, that each propagate with phase velocity c. For example,

$$\begin{aligned} E_x &= -\frac{E_{0x}}{8}\left(e^{ik_x x} + e^{-ik_x x}\right)\left(e^{ik_y y} - e^{-ik_y y}\right)\left(e^{ik_z z} - e^{-ik_z z}\right) e^{-i\omega t} \\ &= -\frac{E_{0x}}{8}\left(e^{i(k_x x + k_y y + k_z z - \omega t)} + e^{i(-k_x x + k_y y + k_z z - \omega t)} - e^{i(k_x x - k_y y + k_z z - \omega t)} - e^{i(-k_x x - k_y y + k_z z - \omega t)}\right. \\ &\quad \left. - e^{i(k_x x + k_y y - k_z z - \omega t)} - e^{i(-k_x x + k_y y - k_z z - \omega t)} + e^{i(k_x x - k_y y - k_z z - \omega t)} - e^{i(-k_x x - k_y y - k_z z - \omega t)}\right). \end{aligned} \tag{80}$$

For a rectangular waveguide with, say, no walls perpendicular to the z-axis, the modes that propagate in the $+z$ direction are given by the first four terms in the second form of eq. (80), as first noted by Brillouin [25]. However, k_z is no longer $n\pi/d_z$, but rather $k_z = \sqrt{\omega^2/c^2 - k_x^2 - k_y^2}$, as required for the guided waves to satisfy the wave equation $\nabla^2 \mathbf{E} = \partial^2 \mathbf{E}/\partial(ct)^2$.[12] The four plane waves of eq. (80) zig-zag down the guide, as shown in the figure below from [25].[13]



2.10 DC Magnets

2.10.1 Solenoid

We consider a solenoid magnet of length l and radius a that carries uniform azimuthal current per unit axial length $I = B_0/\mu_0$, such that the interior, axial magnetic field is b_0 in the limit of large l. Taking the z-axis to be that of the solenoid, which extends over $0 < z < l$, the current density is,

$$\mathbf{J}(\mathbf{x}, t) = \frac{B_0}{\mu_0}\delta(r - a)\,\hat{\boldsymbol{\phi}} \qquad (0 < z < l), \tag{81}$$

[12] The phase velocity, $v_p = \omega/k_z = c\sqrt{1 + (k_x^2 + k_y^2)/k_z^2}$, of the guided wave (written as $E_x = E_{0x} \cos k_x x \sin k_y y\, e^{i(k_z z - \omega t}$, *etc.*) is greater than c, but the group velocity, $v_g = d\omega/dk_z = c^2/v_p$ is less than c. This is the subject of an amusing puzzler by Slepian [27].

[13] The zig-zag character for guided waves is also discussed in Chap. 24 of [28], and in sec. 9.5.2 of [29].

in a cylindrical coordinate system (r, θ, z). The temporal Fourier transform of this dc current density is

$$\mathbf{J}_\omega(\mathbf{x}) = \frac{B_0}{\mu_0}\delta(r-a)\delta(\omega)\,\hat{\boldsymbol{\phi}} = \frac{B_0}{\mu_0}\delta(r-a)\delta(\omega)(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}) \qquad (0 < z < l). \tag{82}$$

That is, $\mathbf{J}_\omega$ has only a zero-frequency component. When computing the 4-dimensional Fourier transform of the azimuthally symmetric function (82), we can define the (half)plane $\phi = 0$ to contain the wave vector $\mathbf{k}$, which can then be written as $\mathbf{k} = k_x\,\hat{\mathbf{x}} + k_z\,\hat{\mathbf{z}}$. Thus,

$$\begin{aligned}
\mathbf{J}_{\omega,\mathbf{k}^\pm} &= \frac{1}{(2\pi)^3}\int\!\!\int\!\!\int \mathbf{J}_\omega(\mathbf{x})\,e^{-i\mathbf{k}^\pm\cdot\mathbf{x}}\,d^3\mathbf{x} \\
&= \frac{B_0\delta(\omega)}{(2\pi)^3\mu_0}\int\!\!\int r\,dr\,d\phi\int_0^l dz\,\delta(r-a)(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})\,e^{-ik_x r\cos\phi}e^{-ik_z z} \\
&= \frac{2aB_0\delta(\omega)\,e^{-ik_z l/2}\sin(k_z l/2)}{(2\pi)^3\mu_0 k_z}\int_0^{2\pi} d\phi\,(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}})\,e^{-ik_x a\cos\phi} \\
&= -\frac{iaB_0\delta(\omega)J_1(k_x a)e^{-ik_z l/2}\sin(k_z l/2)}{2\pi^2\mu_0 k_z}\,\hat{\mathbf{y}}.
\end{aligned} \tag{83}$$

The spectrum of plane waves of the (dc) solenoid magnet has only $\omega = 0$, so $k = 0$ and all waves are inhomogeneous (virtual photons). The wave vector $(k_x, 0, k_z)$ has $k_x \approx 1/2a$ from the Bessel function $J_1(k_x a)$, while $|k_z|$ ranges between 0 and $\approx 2/l$ according to the factor $\sin(k_z l/2)/k_z$.

2.10.2 Helical Wiggler

A so-called helical wiggler [30] made from a double helix winding on a cylinder of radius a with period $2\pi/k_0$ has a purely transverse magnetic field along its axis given by,[14]

$$\mathbf{B}(0,0,z) = B_0(\hat{\mathbf{x}}\cos k_0 z + \hat{\mathbf{y}}\sin k_0 z), \tag{84}$$

where the current I in each helical winding is

$$I = \frac{\pi B_0}{k_0^2 a K_0(k_0 a) + k_0 K_1(k_0 a)}\,. \tag{85}$$

With some effort, the Fourier transform $\mathbf{J}_{\omega,\mathbf{k}^\pm}$ could be computed "exactly," but it is clear that the virtual photons of the static magnetic field have zero frequency/energy and have $k_z = k_0$ in the limit of a long wiggler.

2.11 DC Current

As a final example (which could well have been the first, given its simplicity), we consider a DC current density of any form. This current density may or may not be electrically neutral. In any case, the temporal Fourier transform of the DC current has only a zero-frequency

[14] See, for example, sec. 2.3 of [31].

component, so the plane-wave decomposition contains only inhomogeneous waves (virtual photons). That is, the DC current does not radiate, although in a model of the current due to moving charges, these charges are accelerated as they traverse the current loop.[15]

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