Voltage Drop, Potential Difference and $\mathcal{EMF}$

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
(September 18, 2012; updated January 28, 2018)

1 Introduction

To give the concept of voltage a well-defined meaning in time-dependent electromagnetism,[1] we advocate that it be defined as the retarded electric scalar potential [4, 5, 6] in the Lorenz gauge [7] (plus a possible additive constant),

$$V(\mathbf{r}, t) = \int \frac{\rho(\mathbf{r}', t' = t - \mathcal{R}/c)}{4\pi\epsilon_0 \mathcal{R}}\, d\mathrm{Vol}' \qquad \text{(Lorenz)}, \tag{1}$$

in SI units, where ρ is the electric charge density, c is the speed of light in the medium between $\mathbf{r}$ and $\mathbf{r}'$ (assumed in this note to be vacuum), $\mathcal{R} = |\mathbf{r} - \mathbf{r}'|$, and ϵ_0 is the permittivity of the vacuum. This convention follows Stratton [8], p. 352.[2]

A prominent application of the notion of voltage is in circuit analysis, where prescriptions exist for the "voltage drop" across various circuit elements (secs. 3.2.1-5), and where Kirchhoff's (voltage) law [9, 10] states that the sum of the "voltage drops" around any loop is zero. However, Kirchhoff's "law" is only approximately valid, and the "voltage drops" it considers are, in general, neither differences in the electric scalar potential V, nor the $\mathcal{EMF}$ $= \int \mathbf{E} \cdot d\mathbf{l}$ along some path between the ends of the circuit element.[3,4] In "ordinary" circuit analysis, as defined below, the "voltage drops" are well approximated by both the difference in the electric scalar potential (in the Lorenz gauge) and by the $\mathcal{EMF}$ along a suitable path. But, great care is required when considering examples outside "ordinary" circuit analysis where these familiar approximations can be very poor (sec. 3.4).

An "ordinary" circuit, operating at angular frequency ω, is a circuit for which:[5]

1. The size of the circuit is small compared to the wavelength $\lambda = 2\pi c/\omega$. In this case there is no spatial variation to the current in any segment of a loop between two nodes;

2. Effects of retardation (wave propagation) and radiation (flow or energy into or out of the circuit via the electromagnetic field [13] can be ignored;[6]

---

[1] The ambiguous meaning of "voltage" is noted, for example, in [1], and reflected in the debate at [2]. See also [3].

[2] In contrast, some authors consider that voltage is undefined outside electrostatics [11].

[3] Kirchhoff's (circuit loop) law is often considered to be equivalent to conservation of energy, with $q\Delta V = q\int \mathbf{E}\cdot d\mathbf{l}$ being the work done by the electric field on a charge q that moves along a segment of a circuit loop. *In this context, the magnetic field* $\mathbf{B}$ *does no work [12].* However, the energy of the charges is conserved only if the circuit elements remain at rest, and radiation is ignored.

[4] See sec. 3.5 below for discussion of the $\mathcal{EMF}$ of batteries, which differs from $\int \mathbf{E}\cdot d\mathbf{l}$.

[5] A superconducting loop, for which $\oint \mathbf{E}\cdot d\mathbf{l} = 0 = -d\dot{\Phi}_M/dt$, is not an "ordinary" circuit, as the magnetic flux Φ_M and current I cannot change (without resulting in a "quench" in which the conductor goes "normal"). A circuit that is partly superconducting can be an "ordinary" circuit if it meets the four criteria above.

[6] When retardation is ignored the Lorenz-gauge scalar potential reduces to the (instantaneous) Coulomb

3. Magnetic flux through the circuit is well localized in small inductors (coils).

4. All circuit elements are at rest.[7]

The vast success of circuit analysis leaves many people with the impression that it is an "exact" procedure, which results in considerable confusion in the realms where the approximations of "ordinary" circuit analysis are not valid (transmission lines, antennas, and even "simple" circuits where the self inductance of a loop cannot be neglected, etc.). This has led to the unfortunate impression that "voltage" is ill defined, or irrelevant, in examples outside "ordinary" circuit analysis. The present note attempts to provide some correction to this misapprehension.

2 Electrostatics

The concept of voltage V is well defined (up to an additive constant) in electrostatics, where it is identified with the electrostatic Coulomb potential ϕ, and also with the $\mathcal{EMF}$ (electromotive force),

$$V(\mathbf{r}) - V_0 = \phi(\mathbf{r}) - \phi(\mathbf{r}_0) = -\mathcal{EMF}(\mathbf{r_0}, \mathbf{r}) \qquad \text{(electrostatics)}, \tag{2}$$

where V_0 is the voltage at the reference position $\mathbf{r}_0$, the Coulomb potential is,

$$\phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0 \mathcal{R}}\, d\mathrm{Vol}' \qquad \text{(Coulomb)}, \tag{3}$$

ρ is the (static) electric charge density, the electromotive force is,[8]

$$\mathcal{EMF}(\mathbf{r_0}, \mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l} \qquad \text{(electrostatics)}, \tag{4}$$

and the electric field $\mathbf{E}$ is related to the potential ϕ by,

$$\mathbf{E} = -\boldsymbol{\nabla}\phi. \tag{5}$$

The $\mathcal{EMF}$ is a unique function of $\mathbf{r}_0$ and $\mathbf{r}$, independent of the path of integration, only if $\boldsymbol{\nabla} \times \mathbf{E} = 0$. This condition does hold in electrostatics (and magnetostatics).[9] Furthermore,

---

potential. Hence, the conceptual preference for the Lorenz-gauge potentials over those in the Coulomb gauge becomes apparent only in situations in which retardation and wave propagation are significant (sec. 3.3).

[7]When circuit elements are in motion, Faraday's law is often invoked to consider them as effective sources of $\mathcal{EMF}$, but care is needed in doing so. See, for example, sec. 17-2 of [14]. We do not pursue this topic further here; for an example by the author, see [15].

[8]Maxwell used this definition in Art. 69 of [17].

[9]A static electric field obeys $\partial\mathbf{E}/\partial t = 0$, in which case the magnetic field $\mathbf{B}$ obeys $\partial^2\mathbf{B}/\partial t^2 = 0$, as follows on taking the time derivative of Faraday's law, $\boldsymbol{\nabla} \times \mathbf{E} = -\partial\mathbf{B}/\partial t$. In principle, this is consistent with a magnetic field that varies linearly with time, $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0(\mathbf{r}) + \mathbf{B}_1(\mathbf{r})t$. However, this leads to arbitrarily large magnetic fields at early and late times, and nonzero $\mathbf{B}_1$ is excluded on physical grounds. Hence, $\partial\mathbf{E}/\partial t = 0$ implies that $\partial\mathbf{B}/\partial t = 0$ also, and $\boldsymbol{\nabla} \times \mathbf{E} = 0$ according to Faraday's law. However, the condition that $\boldsymbol{\nabla} \times \mathbf{E} = 0$ does not necessarily imply that $\partial\mathbf{E}/\partial t = 0$. See, for example, [16].

the energy stored in the electric field of an electric charge Q at rest and a static electric charge density ρ (not including Q) is,

$$U_E = Q\ \phi(\mathbf{r}_Q) = Q \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0 \mathcal{R}}\, d\mathrm{Vol}' \qquad \text{(electrostatic)}, \tag{6}$$

where $\mathcal{R}$ is the distance between the charge Q (at $\mathbf{r}_Q$) and the volume element at $\mathbf{r}'$, so the static Coulomb potential $\phi(\mathbf{r})$ can also be interpreted as the electrostatic energy of a unit test charge at $\mathbf{r}$ and the charge distribution ρ.

The concept of voltage has found great popular appeal in electrostatics as characterizing the $\mathcal{EMF}$ of batteries, and in Ohm's law,

$$\Delta V = IR, \tag{7}$$

where ΔV is the "voltage drop" across a resistor R that carries a steady current I.

The term "high voltage" is often associated with the possibility of electrical "breakdown" via sparks in an otherwise static situation. However, this phenomenon is more properly associated with high electric field (= high voltage gradient in a static situation).

3 Electrodynamics

The simplicity of the relations (2) does not carry over to time-dependent electrodynamics where $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t \neq 0$, and the $\mathcal{EMF}$ (4) depends on the path between the end points of the line integral. The electric field cannot be deduced only from a scalar potential (called V hereafter), but rather,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \tag{8}$$

where $\mathbf{A}$ is the vector potential, which is related to the magnetic field by,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{9}$$

Also, the (time-dependent) energy stored in the electric field can be identified as,[10]

$$U_E = \frac{\epsilon_0}{2} \int E^2\, d\mathrm{Vol} = \frac{1}{2} \int \rho V\, d\mathrm{Vol} - \frac{\epsilon_0}{2} \int V \frac{\partial \boldsymbol{\nabla} \cdot \mathbf{A}}{\partial t}\, d\mathrm{Vol} + \frac{\epsilon_0}{2} \int \frac{\partial \mathbf{A}}{\partial t} \cdot \frac{\partial \mathbf{A}}{\partial t}\, d\mathrm{Vol}, \tag{10}$$

using eq. (8) to eliminate the electric field $\mathbf{E}$ in favor of the potentials V and $\mathbf{A}$. Clearly, the "electrical" energy U_E can no longer be related only to the scalar potential V.

The potentials V and $\mathbf{A}$ in electrodynamics can have many forms, related by so-called gauge transformations that leave the fields $\mathbf{E}$ and $\mathbf{B}$ invariant [7, 18]. For example, we could set the scalar potential V to zero, as first advocated by Gibbs [19, 20].[11]

[10] If the energy stored in electrically polarized atoms is included in the "electrical" energy we have that $U_E = \int \mathbf{E} \cdot \mathbf{D}\, d\mathrm{Vol}/2$.

[11] A static electric field (with zero magnetic field) can be deduced via eq. (8) from the trivial vector potential $\mathbf{A}_{\mathrm{Gibbs}} = -\mathbf{E}\, t$. See sec. VIII of [18] for a general expression for $\mathbf{A}$ in the gauge where $V = 0$.

Alternatively, we could continue to identify the scalar potential V with the instantaneous Coulomb potential (3); this option (for which $\nabla \cdot \mathbf{A} = 0$) has technical appeal mainly in situations where the charge density in quasistatic and propagation effects related to charges (in contrast to those of currents) are negligible.

In cases where wave propagation is significant it is appealing to use potentials that exhibit wave effects similar to those of the fields $\mathbf{E}$ and $\mathbf{B}$. This leads to use of the so-called Lorenz-gauge condition,

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \qquad \text{(Lorenz)}, \tag{11}$$

and to the retarded potentials (1) and,

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t' = t - \mathcal{R}/c)}{\mathcal{R}}\, d\text{Vol}' \qquad \text{(Lorenz)}, \tag{12}$$

where $\mathbf{J}$ is the electrical current density, and μ_0 is the permeability of the vacuum.

The retarded scalar potential (1) is well-defined throughout all space, reduces to the Coulomb potential (3) in static situations, and in quasistatic situations, such as "ordinary" circuit analysis (defined in sec. 1), leads to "voltage drops" in good agreement with the approximations associated with the concepts of capacitors and inductors. We therefore advocate that the best generalization of the concept of voltage from electrostatics to electrodynamics is to consider the retarded scalar potential (1) to be the "voltage".

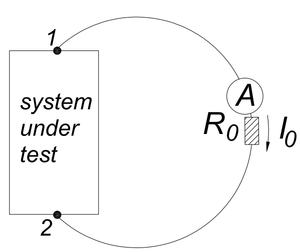
3.1 Voltage, Voltage Sources, and Voltmeters

In many practical situations the notion of "voltage" is related to a "voltage source" which has an internal feedback systems designed to deliver a specific value of "voltage" across a pair of terminals, where that "voltage" is measured by a (built-in) voltmeter. This leads many people to consider that "voltage is what is measured by a voltmeter".

So, what is a voltmeter, and what does it measure?[12]

3.1.1 Voltmeters

An (AC) voltmeter is an ammeter that measures the (oscillating) current I_0 that flows through a high-value resistor R_0 that is attached to leads whose tips, 1 and 2, may be connected to some other electrical system. The reading of the voltmeter (if properly calibrated) is $V_{\text{meter}} = I_0(R_0 + R_{\text{leads}})$ where $R_{\text{leads}} \ll R_0$.[13]



[12]This issue has a long history in pedagogic lore, [21]-[36].

[13]AC voltmeters often report the root-mean-square voltage $V_{\text{rms}} = I_0(R_0+R_{\text{leads}})/\sqrt{2}$ rather than $I_0(R_0+R_{\text{leads}})$.

In the approximation that the current does not vary spatially along the leads, the meter reading equals the $\mathcal{EMF}$ along the path of its conductors,

$$V_{\rm meter} \approx \mathcal{EMF}_{\rm meter} = \int_1^2 \mathbf{E}\cdot d\mathbf{l} = V_1 - V_2 - \frac{d}{dt}\int_1^2 \mathbf{A}\cdot d\mathbf{l} \text{ (along meter leads)}, \tag{13}$$

To see this, note that for a cylindrical, resistive medium of length l, radius r, and electrical conductivity σ that obeys Ohm's law $\mathbf{J} = \sigma\mathbf{E}$, where $\mathbf{J} = I\,\hat{\mathbf{l}}/\pi r^2$ and I is the (uniform) axial current, then $El = Jl/\sigma = Il/\pi r^2\sigma = IR$, and the (axial) electrical resistance is $R = l/\pi r^2\sigma$.

In time-varying situations, particularly where there are large magnetic fields in the vicinity of the circuit that is being probed by the voltmeter, the $\mathcal{EMF}$ depends on the path between points 1 and 2.[14] However, in "ordinary" circuits (ones that satisfy the four conditions given in sec. 1) there is very little magnetic flux linked by the loop that includes the voltmeter,[15]

$$0 \approx \Phi_M = \int \mathbf{B}\cdot d\text{Area} = \oint \mathbf{A}\cdot d\mathbf{l} = \int_1^2 \mathbf{A}\cdot d\mathbf{l} + \int_2^1 \mathbf{A}\cdot d\mathbf{l} \approx \int_1^2 \mathbf{A}\cdot d\mathbf{l} \text{ (along meter leads)}, \tag{14}$$

and the integral $\int_1^2 \mathbf{A}\cdot d\mathbf{l}$ is small (in the Lorenz gauge). This means that for such "ordinary" circuits the electric field between points 1 and 2 can be related to a scalar potential V according to $\mathbf{E} \approx -\boldsymbol{\nabla} V$ to a good approximation, such that,

$$\int_1^2 \mathbf{E}\cdot d\mathbf{l} \approx -\int_1^2 \boldsymbol{\nabla} V\cdot d\mathbf{l} = V_1 - V_2. \tag{15}$$

That is, when an AC voltmeter is used with an "ordinary" circuit it reads, to a good approximation, the voltage drop $V_1 - V_2$ between the electric scalar potential (in the Lorenz gauge) at the tips of its leads.[16]

3.1.2 Voltage Sources

Following the above discussion, we see that a voltage source which is regulated by an internal voltmeter delivers a fixed $\mathcal{EMF}$ to the extent that time-varying magnetic fields have negligible effect on the device. When such fields are negligible, the voltage source also delivers, to a good approximation, a fixed difference in the electric scalar potential (in the Lorenz gauge) between its terminals.

3.2 Voltage and AC Circuit Analysis

Circuit analysis is a mathematical model of a network of electrical components, such as "voltage" sources (DC or AC), wires (resistive or perfectly conducting), capacitors, inductors,

[14] A peculiar example in which the use of a voltmeter "creates" the voltage that it measures is discussed in [37].

[15] If the loop includes an inductor we take the loop to follow a direct path between its ends, rather than following the winding. Very little magnetic flux is linked by the loop that includes this path, as discussed in sec. 3.2.1.

[16] Discussion by the author of cases when a voltmeter does not read the difference in the electric scalar potential between its tips and/or the $\mathcal{EMF}$ $\int_1^2 \mathbf{E}\cdot d\mathbf{l}$ are given in [38, 39], which include many references to comments by others on this theme.

as well as "active" devices such as diodes and transistors. A key assumption is that a scalar "voltage" can be assigned to each node of the network, and that there is a viable model for the "voltage drop" between any pair of nodes. An implication of this assumption is that the sum of the "voltage drops" is zero around any loop in the network/circuit.

Faraday's law tells us that the line integral of the tangential electric field around a loop is non zero when the magnetic flux through the loop is time dependent,

$$\mathcal{EMF}_{\rm loop} = \oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_M}{dt} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathrm{Area} = -\frac{d}{dt}\oint \mathbf{A}\cdot d\mathbf{l}. \tag{16}$$

Hence, $\mathcal{EMF}$'s (line integral of the tangential electric field along some path) cannot strictly be the "voltage drops" used in AC circuit analysis, although it appears that most people identify "voltage" with $\mathcal{EMF}$ in circuit analysis.[17]

In contrast, the electric scalar potential V is a scalar defined at every point (once a "gauge" is specified), and so is a candidate for the "voltage" of circuit analysis.

However, circuit analysis also assumes particular expressions for the "voltage drop" across specific circuit elements, which are only approximately equal to the difference in the Lorenz-gauge electric scalar potential between the ends of the element. That is, "voltage drops" in circuit analysis are neither exactly $\mathcal{EMF}$'s nor differences in a scalar potential function, with the discrepancies being of similar size in both cases. In general practice, it is equally accurate (or equally inaccurate) to characterize "voltage" in the analysis of "ordinary" circuits (defined in sec. 1) as related either to $\mathcal{EMF}$'s or to an electric scalar potential (in either to Lorentz or Coulomb gauges, as these are identical for "ordinary" circuits).

We now give examples for several types of circuit elements, assuming that the "voltage" and currents have time dependence $e^{j\omega t}$ where $j = -\sqrt{-1} = -i$.

3.2.1 Inductors

We begin with consideration of inductors, *i.e.*, coils of wire for which effects of inductance are more prominent than those of resistance and capacitance.

Recall that (self) inductance is a geometric property of a loop (which is independent of time if the shape of the circuit is fixed),

$$L = \frac{\mu_0}{4\pi}\oint\oint \frac{d\mathbf{l}\cdot d\mathbf{l}'}{\mathcal{R}}\,, \tag{17}$$

where $\mathcal{R}$ is the distance between line elements $d\mathbf{l}$ and $d\mathbf{l}'$. Every loop in a circuit has an inductance, which is ignored in "ordinary" circuit analysis unless the loop contains an "inductor".

Furthermore, the $\mathcal{EMF}$ associated with a loop is given by,

$$\mathcal{EMF}_{\rm loop} = \oint_{\rm loop} \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_M}{dt} = -\frac{d}{dt}\oint_{\rm loop} \mathbf{A}\cdot d\mathbf{l}, \tag{18}$$

where,

$$\Phi_M = \int_{\rm loop} \mathbf{B}\cdot d\mathbf{Area} = \oint_{\rm loop} \mathbf{A}\cdot d\mathbf{l}, \tag{19}$$

[17] One of the more thoughtful justifications of this identification is given in Chap. 22 of [14].

is the magnetic flux due to the current I that passes through the loop.

The usual approximation in "ordinary" circuit analysis is to ignore effects of retardation and write,

$$\mathbf{A}(\mathbf{r},t) \approx \frac{\mu_0}{4\pi} \oint \frac{I(\mathbf{r}',t)}{\mathcal{R}}\, d\mathbf{l}' \approx \frac{\mu_0 I(t)}{4\pi} \oint_{\text{loop}} \frac{d\mathbf{l}'}{\mathcal{R}}\,, \tag{20}$$

where we also ignore possible variation of the current around the loop of the circuit. Then, the magnetic flux (19) can be written as,

$$\Phi_M \approx \frac{\mu_0 I(t)}{4\pi} \oint \oint \frac{d\mathbf{l}\cdot d\mathbf{l}'}{\mathcal{R}} = L\, I(t), \tag{21}$$

and the $\mathcal{EMF}$ (18) around the loop can be written as,

$$\mathcal{EMF}_{\text{loop}} = \oint_{\text{loop}} \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_M}{dt} = -L\frac{dI}{dt}\,. \tag{22}$$

A common practice is to wind part of the conductor of the circuit into a compact coil, so that most of the integral $\int \mathbf{B}\cdot d\mathbf{Area} = \oint_{\text{loop}} \mathbf{A}\cdot d\mathbf{l} = L\,I$ comes from this compact region. That region, say from points 1 to 2 on the conductor, is often identified as the inductor in the circuit, and the inductance L is wrongly (but conveniently) considered to be a property of that compact portion of the circuit, rather than of the circuit as a whole. Then, the "inductive voltage drop",

$$L\dot{I} = -\oint_{\text{loop}} \mathbf{E}\cdot d\mathbf{l} = \frac{d}{dt}\oint_{\text{loop}} \mathbf{A}\cdot d\mathbf{l} \approx \frac{d}{dt}\int_1^2 \mathbf{A}\cdot d\mathbf{l}, \tag{23}$$

around the loop is (mis)identified as a voltage difference between the two ends of the inductor, with the implication that a scalar potential V can account for the behavior of inductors, as assumed in applications of Kirchhoff's voltage law to networks.[18]

[18]The notion that the (self) inductance L of a loop can be considered as localized between the ends of an "inductor" that occupies only part of the loop is an aspect of the "lumped-impedance" approximation. This is sometimes justified for inductance via energy considerations, perhaps following the lead of Maxwell, Arts. 578-584 of [40]. The most extensive discussion may be that by Heaviside [41]-[43]. See also sec. 5.4 of [45]. The energy stored in the magnetic field (and in the magnetization, if any) of an "ordinary" circuit that carries current I can be written as,

$$U_M = \int_{\text{all space}} \frac{\mathbf{B}\cdot\mathbf{H}}{2}\, d\text{Vol} \approx \int_{\text{inductor}} \frac{\mathbf{B}\cdot\mathbf{H}}{2}\, d\text{Vol} \approx \frac{L_{\text{inductor}} I^2}{2}\,. \tag{24}$$

Furthermore,

$$\begin{aligned} U_M &= \int \frac{\mathbf{H}\cdot\boldsymbol{\nabla}\times\mathbf{A}}{2}\, d\text{Vol} \approx \int \frac{\mathbf{A}\cdot\boldsymbol{\nabla}\times\mathbf{H}}{2}\, d\text{Vol} = \int \frac{\mathbf{A}\cdot(\mathbf{J}+\partial\mathbf{D}/\partial t)}{2}\, d\text{Vol} \approx \int \frac{\mathbf{A}\cdot\mathbf{J}}{2}\, d\text{Vol} \\ &\approx \frac{I}{2}\oint \mathbf{A}\cdot d\mathbf{l} = \frac{I\Phi_{M,\text{total}}}{2} = \frac{L_{\text{total}} I^2}{2} \approx \frac{I}{2}\int_{\text{inductor}} \mathbf{A}\cdot d\mathbf{l} = \frac{I\Phi_{M,\text{inductor}}}{2} = \frac{L_{\text{inductor}} I^2}{2}\,, \end{aligned} \tag{25}$$

where the first approximation requires the neglect of the surface integral at infinity, $\int \mathbf{A}\cdot\mathbf{H}\, d\text{Area}$, that arises in the integration by parts, which integral is nonzero in general due to radiation; the second approximation requires the neglect of the "displacement current" $\partial\mathbf{D}/\partial t$; and the fourth approximation (like that used in eq. (24)) supposes that most of the magnetic flux through the circuit is localized in the "inductor".

However, the error is slight for a typical inductor in the form of a coil of N turns, where N is large. In this case, the magnetic field inside the coil is axial, and its vector potential is largely azimuthal, with a small axial component primarily due to the current in the circuit outside the nominal coil. Then, along the axis of the coil, between points 1 and 2, we have that $\mathbf{E} \approx -\boldsymbol{\nabla} V$, such that $V_1 - V_2 \approx \int_1^2 \mathbf{E} \cdot d\mathbf{l}$, where now the line integral is not taken along the conductor of the coil, but along a line down its interior. That is, the $\mathcal{EMF}$ along this path between the ends of the coil is very close to the voltage drop $V_1 - V_2$ between them, with a fractional error of order $1/N$.

The geometry of a coil permits us to consider the $\mathcal{EMF}$ between its ends along a path that does not follow the helical winding, but runs straight through "empty" space between the end points. As discussed in sec. 3.2.3 below, if the coil is made from a perfect conductor, the $\mathcal{EMF}$ between its endpoints, evaluated along the path of that conductor, is zero. However, the dependence on the path of the $\mathcal{EMF}$ in time-dependent situations can be exploited to redefine the $\mathcal{EMF}$ associated with a coil to equal approximately the total $\mathcal{EMF}$ around the circuit, and also approximately equal the difference in the electric scalar potential between the ends of the coil.[19]

3.2.2 AC Voltage Source

As discussed above in sec. 3.1.2, an AC voltage source (regulated by an internal voltmeter) delivers a fixed $\mathcal{EMF}$ to the extent that time-varying magnetic fields have negligible effect on the source. When such fields are negligible, the voltage source also delivers, to a good approximation, a fixed difference in the electric scalar potential (in the Lorenz gauge) between its terminals.

3.2.3 Perfectly Conducting Wires

Perhaps the simplest circuit element is a wire, often assumed to be perfectly conducting.

The electric field at the surface of a perfect conductor must have zero tangential component (and $\mathbf{E} = 0$ inside the conductor), so the $\mathcal{EMF}$ is zero along any such wire. Hence,

$$\mathcal{EMF} = 0 = \int_1^2 \mathbf{E} \cdot d\mathbf{l} = V_1 - V_2 - \frac{d}{dt} \int_1^2 \mathbf{A} \cdot d\mathbf{l} \qquad \text{(perfect conductor)}, \tag{26}$$

for any pair of points 1 and 2 on the perfect conductor. In general, the electric scalar

[19] It is useful to consider an auxiliary loop that contains points 1 and 2, the conductor between them and the line of centers outside the conductor that joins the points . This auxiliary loop contains essentially all of the magnetic flux as that through the larger loop, and so we have that $-L\dot{I} \approx \oint_{aux\ loop} \mathbf{E} \cdot d\mathbf{l}$ $= \int_1^2 \mathbf{E}_{\text{conductor}} \cdot d\mathbf{l} - \int_1^2 \mathbf{E}_{\text{center line}} \cdot d\mathbf{l}$. The electric field inside the (good) conductor is negligible, and hence $L\dot{I} \approx \int_1^2 \mathbf{E}_{\text{center line}} \cdot d\mathbf{l} = V_1 - V_2$, where the last equality associates a "voltage drop" of an effective scalar potential with the inductor, even though eq. (23) relates the $\mathcal{EMF}$ to the vector potential. This permits an inductor to be considered like other "load" elements in Kirchhoff's loop law, rather than as a "source" of $\mathcal{EMF}$.

See, for example, Chap. 22 of [14].

potential is not constant over a perfect conductor,[20]

$$V_1 - V_2 = \frac{d}{dt}\int_1^2 \mathbf{A}\cdot d\mathbf{l} \qquad \text{(perfect conductor).} \tag{27}$$

The presence of the term in eq. (26) involving the vector potential $\mathbf{A}$ is a reminder that magnetic effects of a current-carrying wire should not be completely ignored, although the usual procedure in "ordinary" circuit analysis is to do so. Every loop in a circuit has an inductance that should, in principle, be accounted for in the circuit analysis.[21]

To get a sense of the size of the error made in ignoring this inductance for loops without an "inductor", we consider a loop of radius a made of a perfectly conducting wire of radius r_0. When that wire carries current I the magnetic field close to the wire circulates around it, and varies with distance r from the center of the wire as,

$$B \approx \frac{\mu_0 I}{2\pi r}\,. \tag{28}$$

The magnetic flux linked by the loop of radius a is approximately,

$$\Phi_M \approx 2\pi a\int_{r_0}^a B\,dr = \mu_0 a I \ln\frac{a}{r_0}\,. \tag{29}$$

Recalling eq. (21), we estimate the (self) inductance of the wire loop as,

$$L = \frac{\Phi_M}{I} \approx \mu_0 a \ln\frac{a}{r_0}\,. \tag{30}$$

The wire has length $2\pi a$, so the inductance per unit length of the wire is,

$$\frac{\mu_0}{2\pi}\ln\frac{a}{r_0} \approx \mu_0 \approx 10^{-6}\ \text{H/m} = 10\ \text{nH/cm} \qquad \text{(inductance per length of a wire).} \tag{31}$$

3.2.4 Resistive Wires

A wire of radius r and length l has resistance,

$$R = \frac{l}{\pi r^2\sigma}\,, \tag{32}$$

for axial current flow $I = \pi r^2 J$. The current density $\mathbf{J}$ is related to the electric field inside the wire by,

$$\mathbf{J} = \sigma\mathbf{E}, \tag{33}$$

where σ is the electrical conductivity of the wire. Then,

$$IR = \frac{Jl}{\sigma} = El = \Delta V_{\text{resistor}} - l\frac{\partial\mathbf{A}}{\partial t}\,. \tag{34}$$

As noted in sec. 3.2.3, a (resistive) wire loop is associated with a small self inductance (31) which should be included in the circuit analysis if accuracy is desired.

[20]This fact is disconcerting to those who identify "voltage" with a static electric scalar potential, which has a constant value everywhere in/on a perfect conductor.

[21]A complete, perfectly conducting loop must have $\oint_{\text{loop}} \mathbf{E}\cdot d\mathbf{l} = 0$, so the magnetic flux Φ_M through the loop cannot change with time. The practical realization of a perfect conductor is a superconductor – which will cease to be a superconductor (quench) if the magnetic flux through a superconducting loop is force to change.

3.2.5 Capacitors

If the circuit includes a capacitor of value C, then the $\mathcal{EMF}$ across the capacitor is Q/C where $Q(t)$ is the magnitude of the electric charge on one of the plates of the capacitor. For a typical capacitor the electric field between its electrodes has very little contribution from the vector potential, so that $\mathbf{E} \approx -\boldsymbol{\nabla} V$, and the $\mathcal{EMF}$ across the capacitor is very close to $V_1 - V_2$, the difference in the scalar potential between the electrodes.

We have now verified, element by element, that the circuit analysis of a loop that contains a voltage source V, and inductor L, a resistor R and a capacitor C, all in "series", can be described by a set of "voltage drops" each of which is very close to the difference in the electric scalar potential (in the Lorenz gauge) between the ends of the element. The sum of these "voltage drops" around the loop is zero (Kirchhoff's voltage law), which fact is usually summarized in the form,

$$V = L\dot{I} + IR + \frac{Q}{C} = j\omega LI + IR - \frac{jI}{\omega C}\,, \tag{35}$$

where the latter form holds when the voltage source is sinusoidal with angular frequency ω.[22]

3.3 Voltage and Transmission Lines

A transmission line is a pair of parallel, linear conductors, often approximated as perfect conductors, that support transverse electromagnetic (TEM) waves in the direction of the conductors.

The electric scalar potential provides a useful, "exact" description of the TEM waves on a transmission line (although most people are not aware of this). The scalar potential is defined everywhere, and can be written as a traveling wave,[23]

$$V(x, y, z, t) = V_\perp(x, y)\, e^{j(\omega t - kz)}, \tag{36}$$

where the z-axis is in the direction of the conductors, and ω/k is the speed c of electromagnetic waves in the medium outside the conductors. Similarly, the electric and magnetic fields, which have no z-components for TEM waves, can be written as,

$$\mathbf{E}(x, y, z, t) = \mathbf{E}_\perp(x, y)\, e^{j(\omega t - kz)}, \qquad \mathbf{B}(x, y, z, t) = \mathbf{B}_\perp(x, y)\, e^{j(\omega t - kz)}, \tag{37}$$

and the vector potential can be written as,

$$\mathbf{A}(x, y, z, t) = A_z(x, y)\, e^{j(\omega t - kz)}\, \hat{\mathbf{z}}, \tag{38}$$

[22]Kirchhoff's "law" is not an exact law of physics, but rather is a useful approximation. The "voltage drops" which appear in this "law" are not strictly $\mathcal{EMF}$'s along the conductors of the circuit, nor are they exactly equal to the differences in the electric scalar potential between the ends of the various circuit elements. In practice, there is very little error in identifying the "voltage drops" with either the $\mathcal{EMF}$'s or the differences in electric scalar potential. However, this useful approximation tends to leave many people without a crisp understanding of the distinction between "voltage drop", potential difference, and $\mathcal{EMF}$, which can be large outside "ordinary" circuit analysis.

[23]The presence of the wavefunction $e^{j(\omega t - kz)}$ in eq. (36) indicates that the potential is in the Lorenz gauge.

since the currents that generate the vector potential have only a z-component. As the electric field has no z-component, eq. (8) tells us that,

$$E_z = 0 = -\frac{\partial V}{\partial z} - \frac{\partial A_z}{\partial t} = jkV_\perp(x,y)\, e^{j(\omega t - kz)} + j\omega A_z\, e^{j(\omega t - kz)}, \tag{39}$$

and hence,[24]

$$A_z = \frac{k}{\omega} V_\perp = \frac{V_\perp}{c}\,. \tag{40}$$

Since $B_z = 0$, Faraday's law tells us that,

$$0 = -\frac{\partial B_z}{\partial t} = (\boldsymbol{\nabla} \times \mathbf{E})_z = \boldsymbol{\nabla}_\perp \times \mathbf{E}_\perp\, e^{j(\omega t - kz)}, \tag{41}$$

where $\boldsymbol{\nabla}_\perp = (\partial/\partial x, \partial/\partial y)$. Hence, the 2-dimensional, time-independent field $\mathbf{E}_\perp$ has zero curl, and so can be deduced from a (static) 2-dimensional scalar potential $V_\perp$ according to,

$$\mathbf{E}_\perp = -\boldsymbol{\nabla}_\perp V_\perp. \tag{42}$$

In the approximation of perfect conductors, $\mathbf{E}_\perp$ vanishes on the surfaces of the conductors of the transmission line, we recognize $V_\perp$ and $\mathbf{E}_\perp$ as the electrostatic potential and static electric field that could be supported by the conductors of the transmission line.[25]

The electric field can now be deduced from the potentials as,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}_\perp V_\perp\, e^{j(\omega t - kz)} + jkV_\perp\, e^{j(\omega t - kz)}\,\hat{\mathbf{z}} - j\omega A_z\, e^{j(\omega t - kz)}\,\hat{\mathbf{z}} = \mathbf{E}_\perp\, e^{j(\omega t - kz)}. \tag{43}$$

Note that for two points, 1 and 2, with the same z-coordinate,

$$\mathcal{EMF}_{12} = \int_1^2 \mathbf{E}\cdot d\mathbf{l} = -\int_1^2 \boldsymbol{\nabla}_\perp V_\perp\, e^{j(\omega t - kz)}\cdot d\mathbf{l} = (V_{\perp,1} - V_{\perp,2})\, e^{j(\omega t - kz)} = V_1 - V_2. \tag{44}$$

for any path in the plane of constant z. However, if points 1 and 2 have different z-coordinates the $\mathcal{EMF}_{12}$ depends on the path (because of the factor e^{-jkz} in the integrand of eq. (44)).

Most discussions of transmission lines imply that the scalar potential is defined only on the conductors, and omit discussion of the potential in the space around them. Often, the potential difference between the two conductors at a given z is related to a line integral of $\mathbf{E}\cdot d\mathbf{l}$ at fixed z, as in eq. (44), which can give the impression that the potential difference is also an $\mathcal{EMF}$, and perhaps that the potential is a static potential. That is, many discussions do not distinguish between the full scalar potential (wave) $V = V_\perp\, e^{j(\omega t - kz)}$ and its time-independent amplitude $V_\perp(x,y)$.

[24]Equation (40) also follows from the Lorenz-gauge condition (11).
[25]Further details for a coaxial transmission line are given in [46].

3.4 Voltage and Antennas

Antennas are conductors designed to guide the flow of electromagnetic energy from an energy source into electromagnetic waves that travel away from the antenna into the "empty" space surrounding it. The flow of energy is described by the Poynting vector,

$$\mathbf{S} = \mu_0 \mathbf{E} \times \mathbf{B}, \tag{45}$$

and, in examples where the time dependence is purely sinusoidal, by its time average,

$$\langle \mathbf{S} \rangle = \frac{\mu_0}{2} Re(\mathbf{E} \times \mathbf{B}^\star). \tag{46}$$

The energy source is usually characterized as a "voltage" source that delivers a known "voltage drop" across the "terminals" of the antenna. For example, if the source "voltage drop" is $V_{\rm source}$ and the distance between the terminals is $d\,\hat{\mathbf{z}}$, we suppose that the electric field along the line between the terminals is $\mathbf{E}_{\rm source} = V_{\rm source}\,\hat{\mathbf{z}}/d$.

Typical antenna analysis involves computation of the Poynting vector (*i.e.*, the far-field radiation pattern $r^2 \langle \mathbf{S} \rangle$, which is independent of r at large r) from the fields $\mathbf{E}$ and $\mathbf{B}$, with these fields being computed from the retarded potentials V and $\mathbf{A}$,[26] the potentials being computed from the charge and current distributions, ρ and $\mathbf{J}$, and these distributions being computed from the source voltage/field.[27] These computations are simplified by use of the equation of continuity, $\boldsymbol{\nabla} \cdot \mathbf{J} = -\partial\rho/\partial t$, and/or the Lorenz-gauge condition (11), to eliminate ρ and/or V in favor of $\mathbf{J}$ and/or $\mathbf{A}$.[28]

Such computations are nontrivial, and are now mainly performed by Numerical Electromagnetic Codes, such as NEC4 [48] (for pure sinusoidal time dependence), which report the radiation pattern (*i.e.*, the Poynting vector), the terminal impedance (ratio of terminal "voltage" to terminal current), and, if requested, the fields $\mathbf{E}$ and $\mathbf{B}$ close to the antenna. The potentials V and $\mathbf{A}$ are not reported, which can give the impression that they are irrelevant, and perhaps even undefined.

Use of a voltmeter with an antenna system generally does not lead to results that are closely related to the scalar potential V (or even to the $\mathcal{EMF}$ along the meter leads), and which are very dependent on the path of the leads of the voltmeter.[29,30] If one considers that "voltage is what a voltmeter measures", this behavior gives the impression that "voltage" is not well-defined near antennas. We take the attitude that it is better to consider "voltage" to be well-defined as the scalar potential V in the Lorenz gauge, and to accept that a voltmeter measures this "voltage" reliably only in "ordinary" circuits.

[26]The behavior of the fields far from most antennas is well described in terms of their electric and magnetic dipole moments $\mathbf{p}$ and $\mathbf{m}$. The potentials in the far zone are then $V \approx jk\mathbf{p} \cdot \hat{\mathbf{r}}\, e^{j(\omega t - kr)}/4\pi\epsilon_0 r$ and $\mathbf{A} \approx j\omega\mu_0(\mathbf{p} + \mathbf{m} \times \hat{\mathbf{r}})\, e^{j(\omega t - kr)}/4\pi r$. Both $\boldsymbol{\nabla} V$ and $\partial A/\partial t$ have longitudinal components that vary as $1/r$ at large r, but $\mathbf{E} = -\boldsymbol{\nabla} V - \partial A/\partial t$ is purely transverse.

[27]Antenna computations are examples of the electromagnetic fields throughout all space being deduced from partial information as to the fields in a very limited volume.

[28]Sec. 11 of [47] is notable for its prominent use of both V and $\mathbf{A}$ when discussing radiation.

[29]See, for example, secs. 2.4-5 of [38].

[30]A voltmeter whose leads are short compared to a wavelength, and not connected to anything, is a kind of receiving antenna whose open-circuit terminal voltage is the $\mathcal{EMF}$ $\mathbf{E} \cdot \mathbf{l}$ where $\mathbf{E}$ is the electric field of the antenna at the voltmeter and $\mathbf{l}$ is the (vector) length of the leads [49].

3.5 $\mathcal{EMF}$ and Batteries (July 22, 2022)

When Volta developed the first electric batteries around 1799 [50], he associated them with an internal *force motrice électrique* (see p. 254 of [51]), which is now called the electromotive force ($\mathcal{EMF}$) of the battery.

When a battery is not connected to anything, its "positive" terminal is positively charged, and its "negative" terminal is negatively charged, forming a kind of capacitor. Hence, there is a static electric field inside the battery that points from the positive to the negative terminal.

When a battery is connected to, say, a resistor, current flows out of the positive terminal of the battery, through the resistor, and back into the negative terminal of the battery. This current requires a force on positive charges in the battery in the direction from the negative terminal to the positive terminal, which is opposite to the direction of the electric field inside the battery before it was connected to the resistor. We infer, along with Volta, that there must exist some nonelectrostatic force/field inside the battery that can drive the current (once the battery is connected to the resistor).[31]

This theme was not much further developed in the 1800's. Rather, Faraday and others developed an understanding that contact potentials play an important role in the generation of currents by a battery, as reviewed in [54]-[60].

A revival of the spirit of Volta's views was made by Abraham in 1904 [61]-[63],[32] who supposed that a battery is associated with a nonelectrostatic, internal field $\mathbf{E}'$ in addition to the electrostatic field $\mathbf{E}$, such that the current density $\mathbf{J}$ inside the battery is related by,

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}'), \tag{47}$$

where σ is the electrical conductivity inside the battery (which may vary with position).

When the battery is not connected to a circuit, the current is zero and $\mathbf{E}' = -\mathbf{E}$ inside the battery, such that $\mathbf{E}'$ points from the negative terminal of the battery to its positive terminal.[33]

When the battery is connected to a circuit, $\mathbf{E}'$ remains roughly the same while the internal electrostatic field $\mathbf{E}$ is reduced in magnitude, and the current density $\mathbf{J}$ points from the negative terminal of the battery to its positive terminal.

The $\mathcal{EMF}$ associated with the battery is taken to be,

$$\mathcal{EMF}_{\text{batt}} = \int_{-}^{+} \mathbf{E}' \cdot d\mathbf{l}, \tag{48}$$

and not $\int_{-}^{+} \mathbf{E} \cdot d\mathbf{l}$, which can cause confusion in view of eq. (4).[34]

---

[31] Ohm's law [52] was originally written by him (1827) as the equivalent of $\mathbf{J} = \sigma\mathbf{E}$, and seems to have been first cast in the form $\mathcal{E} = IR$, where $\mathcal{E}$ is the $\mathcal{EMF}$ of a battery that drives current I through an electrical resistance R, by Fechner (1831), p. 225 of [53].

[32] See also [64].

[33] The field $\mathbf{E}'$ is zero outside the battery at all times. Outside the battery, when not connected to anything, there exists the electrostatic "fringe" field $\mathbf{E}$ of the battery/capacitor.

[34] In Art. 49 of [17], Maxwell noted that: *Electromotive force cannot in all cases be expressed in the form of a difference in potentials. These cases, however, are not treated of in Electrostatics. We shall consider them when we come to heterogeneous circuits, chemical actions, motions of magnets, inequalities of temperature, &c.* Maxwell's *chemical actions* include batteries. In Arts. 241 and 249 of [17], Maxwell considered *heterogeneous circuits* that include batteries, and wrote of the *external electromotive force* of the batteries (with "internal" referring to the other element of the circuits).

This concept of a nonelectrostatic field $\mathbf{E}'$ can be used in a derivation of Poynting's theorem for the energy density, and flow of energy, in the electromagnetic field. See, for example, sec. 10-1 of [65], and [66].

References

[1] D. Roller, *Misuse of the Names of Physical Units*, Am. J. Phys. **14**, 340 (1946), http://kirkmcd.princeton.edu/examples/EM/roller_ajp_14_340_46.pdf

[2] J. Slepian, *Electrostatic or Electromagnetically Induced Electric Field?* Elec. Eng. **68**, 877 (1949), http://kirkmcd.princeton.edu/examples/EM/slepian_ee_68_877_49.pdf

[3] C.J. Papachristou and A.N. Magoulas, *Electromotive Force: A Guide for the Perplexed* (June 6, 2013), https://arxiv.org/abs/1211.6463
Does the electromotive force (always) represent work? (June 26, 2014), https://arxiv.org/abs/1405.7474

[4] B. Reimann, *Ein Beitrag zur Elektrodynamik*, Ann. Phys. **207**, 237 (1867) [written 1858], http://kirkmcd.princeton.edu/examples/EM/riemann_ap_207_237_67.pdf

[5] L. Lorenz, *Mémoire sur la théorie de l'élasticité des corps homgènes à élasticité constante*, J. Ann. Phys. **58**, 329 (1861), http://kirkmcd.princeton.edu/examples/mechanics/lorenz_jram_58_329_61.pdf

[6] L. Lorenz, *Ueber die Identität der Schwingungen des Lichts mit den elektrischen Strömen*, Ann. Phys. **207**, 243 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_ap_207_243_67.pdf
On the Identity of the Vibrations of Light with Electrical Currents, Phil. Mag. **34**, 287 (1867), http://kirkmcd.princeton.edu/examples/EM/lorenz_pm_34_287_67.pdf

[7] J.D. Jackson and L.B. Okun, *Historical roots of gauge invariance*, Rev. Mod. Phys. **73**, 663 (2001), http://kirkmcd.princeton.edu/examples/EM/jackson_rmp_73_663_01.pdf

[8] J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, 1941), http://kirkmcd.princeton.edu/examples/EM/stratton_electromagnetic_theory.pdf

[9] G. Kirchhoff, *Ueber den Durchgang eines elektrischen Stromes durch eine Ebene, insbesondere durch eine kreisförmige*, Ann. d. Phys. Chem. **64**, 497 (1845), http://kirkmcd.princeton.edu/examples/EM/kirchhoff_apc_64_497_45.pdf

[10] G. Kirchhoff, *Ueber die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird*, Ann. d. Phys. Chem. **72**, 497 (1847), http://kirkmcd.princeton.edu/examples/EM/kirchhoff_apc_72_497_47.pdf
On the Solution of the Equations Obtained from the Investigation of the Linear Distribution of Galvanic Currents, IRE Trans. Circuit Theory **5**-3, 4 (1958), http://kirkmcd.princeton.edu/examples/EM/kirchhoff_apc_72_497_47_english.pdf

[11] R.M. Fano, L.J. Chu and R.B. Adler, *Electromagnetic Fields, Energy, and Forces*, (Wiley, 1960), kirkmcd.princeton.edu/examples/EM/fano_chu_adler_60.pdf

[12] K.T. McDonald, *Magnetic Forces Can Do Work* (Apr. 10, 2011), http://kirkmcd.princeton.edu/examples/disk.pdf

[13] K.T. McDonald, *On the Definition of Radiation by a System of Charges* (Sep. 6, 2010), http://kirkmcd.princeton.edu/examples/radiation.pdf

[14] R.P. Feynman, R.B. Leighton and M. Sands, *The Feynman Lectures on Physics*, Vol. 2 (Addison-Wesley, 1964), http://www.feynmanlectures.caltech.edu/II_17.html#Ch17-S2 http://www.feynmanlectures.caltech.edu/II_22.html

[15] K.T. McDonald, *Blondel's Experiment* (June 11, 2010), http://kirkmcd.princeton.edu/examples/blondel.pdf

[16] K.T. McDonald, *An Electrostatic Wave* (July 28, 2002), http://kirkmcd.princeton.edu/examples/bernstein.pdf

[17] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 1, 3rd ed. (Clarendon Press, 1892), http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v1_04.pdf

[18] J.D. Jackson, *From Lorenz to Coulomb and other explicit gauge transformations*, Am. J. Phys. **70**, 917 (2002), http://kirkmcd.princeton.edu/examples/EM/jackson_ajp_70_917_02.pdf

[19] J.W. Gibbs, *Velocity of Propagation of Electrostatic Forces*, Nature **53**, 509 (1896), http://kirkmcd.princeton.edu/examples/EM/gibbs_nature_53_509_96.pdf

[20] K.T. McDonald, *Potentials of a Hertzian Dipole in the Gibbs Gauge* (Aug. 23, 2012), http://kirkmcd.princeton.edu/examples/gibbs.pdf

[21] O. Lodge, *On an Electrostatic Field Produced by Varying Magnetic Induction*, Phil. Mag. **27**, 469 (1889), http://kirkmcd.princeton.edu/examples/EM/lodge_pm_27_116_89.pdf

[22] J. Slepian, *Energy Flow In Electric Systems – the Vi Energy-Flow Postulate*, Trans. Am. IEE **61**, 835 (1942), http://kirkmcd.princeton.edu/examples/EM/slepian_taiee_61_835_42.pdf

[23] P. Hammond, *P.D. vs. E.M.F.*, Students Quart. J. **30**, 3 (1959), http://kirkmcd.princeton.edu/examples/EM/hammond_sqj_30_3_59.pdf

[24] J.W. Buchta, *Electromotive Force and Faraday's Law of Electromagnetic Induction*, Phys. Teacher **1**, 133 (1963), http://kirkmcd.princeton.edu/examples/EM/buchta_pt_1_133_63.pdf

[25] M. Philips, *Electromotive Force and the Law of Induction*, Phys. Teacher **1**, 155 (1963), http://kirkmcd.princeton.edu/examples/EM/philips_pt_1_155_63.pdf

[26] D.R. Moorcroft, *Faraday's Law – Demonstration of a Teaser*, Am. J. Phys. **37**, 221 (1969), http://kirkmcd.princeton.edu/examples/EM/moorcroft_ajp_37_221_69.pdf

[27] D.R. Moorcroft, *Faraday's Law, Potential and Voltage – Demonstration of a Teaser*, Am. J. Phys. **38**, 376 (1970),
http://kirkmcd.princeton.edu/examples/EM/moorcroft_ajp_38_376.pdf

[28] W. Klein, *Discovering Induction*, Am. J. Phys. **49**, 603 (1981),
http://kirkmcd.princeton.edu/examples/EM/klein_ajp_49_603_81.pdf

[29] F. Reif, *Generalized Ohm's law, potential difference, and voltage measurements*, Am. J. Phys. **50**, 1048 (1982), http://kirkmcd.princeton.edu/examples/EM/reif_ajp_50_1048_82.pdf

[30] R.H. Romer, *What do voltmeters measure?: Faraday's law in a multiply connected region*, Am. J. Phys. **50**, 1089 (1982),
http://kirkmcd.princeton.edu/examples/EM/romer_ajp_50_1089_82.pdf

[31] P.C. Peters, *The role of induced emf's in simple circuits*, Am. J. Phys. **52**, 208 (1983),
http://kirkmcd.princeton.edu/examples/EM/peters_ajp_52_208_83.pdf

[32] W. Gough and J.P.G. Richards, **Electro***magnetic or electro***magnetic** *induction?*, Eur. J. Phys. **7**, 195 (1986), http://kirkmcd.princeton.edu/examples/EM/gough_ejp_7_195_86.pdf

[33] J. Roche, *Explaining electromagnetic induction: a critical re-examination*, Phys. Ed. **22**, 91 (1987), http://kirkmcd.princeton.edu/examples/EM/roche_pe_22_91_87.pdf

[34] E. Lanzara and R. Zangara, *Potential difference measurements in the presence of a varying magnetic field*, Phys. Ed. **30**, 85 (1995),
http://kirkmcd.princeton.edu/examples/EM/lanzara_pe_30_85_95.pdf

[35] D. Goldberg and O. Zik, *Topologically dependent voltage measurement*, Phys. Ed. **26**, 256 (1991), http://kirkmcd.princeton.edu/examples/EM/goldberg_pe_26_256_91.pdf

[36] D. Iencinella and G. Matteucci, *An introduction to the vector potential*, Eur. J. Phys. **25**, 249 (2004), http://kirkmcd.princeton.edu/examples/EM/iencinella_ejp_25_249_04.pdf

[37] K.T. McDonald, *The Hooper-Monstein Paradox* (May 3, 2021),
http://kirkmcd.princeton.edu/examples/monstein.pdf

[38] K.T. McDonald, *What Does an AC Voltmeter Measure?* (March 16, 2008),
http://kirkmcd.princeton.edu/examples/voltage.pdf

[39] K.T. McDonald, *Lewin's Circuit Paradox* (May 7, 2010),
http://kirkmcd.princeton.edu/examples/lewin.pdf

[40] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, Vol. 2 (Clarendon Press, 1873),
http://kirkmcd.princeton.edu/examples/EM/maxwell_treatise_v2_73.pdf

[41] O. Heaviside, *Current Energy.—IV*, Electrician **10**, 437 (1883),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_10_437_83.pdf
see also p. 247 of [44].

[42] O. Heaviside, *Current Energy.—V*, Electrician **11**, 104 (1883),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_11_104_83.pdf
see also p. 277 of [44].

[43] O. Heaviside, *The Induction of Currents in Cores*, Electrician **12**, 583 (1884),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electrician_12_583_84.pdf
see also p. 353 of [44].

[44] O. Heaviside, *Electrical Papers*, Vol. I (Macmillan, 1894),
http://kirkmcd.princeton.edu/examples/EM/heaviside_electrical_papers_1.pdf

[45] R.H. Dicke, *General Microwave Circuit Theorems*, p. 130 of *Principles of Microwave Circuits*, C.G. Montgomery, R.H. Dicke and E.M. Purcell, eds. (McGraw-Hill, 1948),
http://kirkmcd.princeton.edu/examples/EM/mit_v8_montgomery.pdf

[46] K.T. McDonald, *Analysis of TEM Waves in a Coaxial Cable via the Scalar Potential* (May 12, 2010), http://kirkmcd.princeton.edu/examples/coax_potential.pdf

[47] D.J. Griffiths, *Introduction to Electrodynamics*, 3rd ed. (Prentice Hall, 1999),
http://kirkmcd.princeton.edu/examples/EM/griffiths_em3.pdf

[48] G.J. Burke, *Numerical Electromagnetic Code – NEC4*, UCRL-MA-109338 (January, 1992), http://www.llnl.gov/eng/ee/erd/ceeta/emnec.html
http://kirkmcd.princeton.edu/examples/NEC_Manuals/NEC4TheoryMan.pdf
http://kirkmcd.princeton.edu/examples/NEC_Manuals/NEC4UsersMan.pdf

[49] K.T. McDonald, *Voltage Across the Terminals of a Receiving Antenna* (June 27, 2007),
http://kirkmcd.princeton.edu/examples/receiver.pdf

[50] A. Volta, *On the Electricity excited by the mere Contact of conducting Substances of different kinds*, (in French) Phil. Trans. Roy. Soc. London **90**, 403 (1800),
http://kirkmcd.princeton.edu/examples/EM/volta_ptrsl_90_403_00.pdf
English version, Phil. Mag. **7**, 289 (1800),
http://kirkmcd.princeton.edu/examples/EM/volta_pm_7_289_00.pdf

[51] A. Volta, *De la électricité dite galvanique*, Ann. Chim. **4**, 225 (1801),
http://kirkmcd.princeton.edu/examples/EM/volta_ac_4_225_01.pdf

[52] G. Ohm, *Die galvanishe Kette* (Berlin, 1827),
kirkmcd.princeton.edu/examples/EM/ohm_galvanische_kette.pdf
The Galvanic Circuit Mathematically Investigated (Van Nostrand, 1891),
kirkmcd.princeton.edu/examples/EM/ohm_galvanic_circuit.pdf

[53] G.T. Fechner, *Massbestimmungen über die galvanishe Kette* (Leipzig, 1831),
http://kirkmcd.princeton.edu/examples/EM/fechner_31.pdf

[54] L.H. Fisher and R.N. Varney, *Contact potentials between metals: History, concepts and persistent misconceptions*, Am. J. Phys. **44**, 464 (1976),
http://kirkmcd.princeton.edu/examples/EM/fisher_ajp_44_464_76.pdf

[55] D. Roberts, *How batteries work: a gravitational analog*, Am. J. Phys. **51**, 829 (1983), http://kirkmcd.princeton.edu/examples/EM/roberts_ajp_51_829_83.pdf

[56] W.M. Saslow, *Voltaic cells for physicists: Two surface pumps and an internal resistance*, Am. J. Phys. **67**, 574 (1999), http://kirkmcd.princeton.edu/examples/EM/saslow_ajp_67_574_99.pdf

[57] H. Kragh, *Confusion and Controversy: Nineteenth-Century Theories of the Voltaic Pile*, in *Nuova Voltiana: Studies in Volta and His Times*, Vol. I, eds. F. Bevilacqua and L. Fragonese (Hoepli, 2000), pp. 133-167, http://kirkmcd.princeton.edu/examples/EM/kragh_00.pdf

[58] W.M. Saslow, *Dynamical theory for the battery's electromotive force*, Am. J. Phys. **76**, 218 (2008), http://kirkmcd.princeton.edu/examples/EM/saslow_ajp_76_218_08.pdf

[59] W.M. Saslow, *Voltaic Cells: The Good (Faraday), the Bad (Volta), and the Ugly (Galvani)*, Phys. Teach. **59**, 22 (2021), http://kirkmcd.princeton.edu/examples/EM/saslow_pt_59_22_21.pdf

[60] R. Alicki *et al.*, *Voltaic Cells: The Good (Faraday), the Bad (Volta), and the Ugly (Galvani)*, Phys. Chem. Chem. Phys. **23**, 9428 (2021), http://kirkmcd.princeton.edu/examples/EM/alicki_pccp_23_9428_21.pdf

[61] M. Abraham and A. Föppl, *Theorie der Elektrizität*, Vol. 1 (Teubner, 1904), Secs. 50-52, http://kirkmcd.princeton.edu/examples/EM/abraham_foppl_elektrizitat_v1_04.pdf

[62] M. Abraham and R. Becker, *The Classical Theory of Electricity and Magnetism* (Hafner, 1932), Chap. 6, Secs. 3-4, http://kirkmcd.princeton.edu/examples/EM/abraham_becker_32.pdf

[63] R. Becker, *Electromagnetic Fields and Interactions* (Dover, 1964), Sec. 41, http://kirkmcd.princeton.edu/examples/EM/becker_64_chap41.pdf

[64] R.N. Varney and L.H. Fisher, *Electromotive force: Volta's forgotten concept*, Am. J. Phys. **48**, 405 (1980), http://kirkmcd.princeton.edu/examples/EM/varney_ajp_48_405_80.pdf

[65] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd ed. (Addison-Wesley, 1962), Sec. 10-1, kirkmcd.princeton.edu/examples/EM/panofsky-phillips.pdf

[66] K.T. McDonald, *Energy Considerations*, http://kirkmcd.princeton.edu/examples/ph501/ph501lecture10.pdf