What Does an AC Voltmeter Measure?

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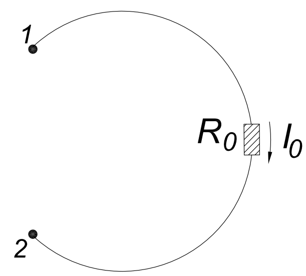
1 Problem

An AC voltmeter is a device that measures the (peak) oscillating current I_0 across a large resistor R_0 that is attached to leads whose tips, 1 and 2, may be connected to some other circuit. The reading of the voltmeter (if properly calibrated) is $V_{\rm meter} = I_0(R_0 + R_{\rm leads})$ where $R_{\rm leads} \ll R_0$. *AC voltmeters typically report the root-mean-square voltage* $V_{\rm rms} = I_0(R_0 + R_{\rm leads})/\sqrt{2}$ *rather than* $I_0(R_0 + R_{\rm leads})$.



Discuss the relation of the meter reading to the difference $V_1 - V_2$ in the scalar potential V between points 1 and 2, and to the line integral $\int_1^2 \mathbf{E} \cdot d\mathbf{l}$ along the circuit being probed, in the absence of the voltmeter.

First consider "ordinary" circuits operating at angular frequency ω for which:

1. The size of the circuit (and of the voltmeter leads) is small compared to the wavelength $\lambda = 2\pi c/\omega$, where c is the speed of light. In this case there is no spatial variation to the current in any segment of a loop between two nodes;

2. Effects of wave propagation and radiation can be ignored;

3. Magnetic flux through the circuit is well localized in small inductors (coils).

Then, consider cases in which these restrictions are relaxed.

Show that while in general the meter reading is not equal to either $V_1 - V_2$ or $\int_1^2 \mathbf{E} \cdot d\mathbf{l}$, to a good approximation the reading is $\int_1^2 \mathbf{E} \cdot d\mathbf{l}$ if the conditions 1 and 2 are satisfied, and to a similar approximation the reading is $V_1 - V_2$ if all three conditions are satisfied.[1]

To give a well-defined meaning to the potentials, work in the Lorenz gauge *[2] (and SI units) where the vector potential* $\mathbf{A}(\mathbf{r}, t)$ *is related to the scalar potential* $V(\mathbf{r}, t)$ *by,*

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \,. \tag{1}$$

[1]Circuits for which conditions 1 and 2 are satisfied can be well analyzed by Kirchhoff's circuit law. Most circuits analyzed this way also satisfy condition 3. For discussion of "paradoxical" exceptions, see [1].

2 Solution

This problem has a long and erratic history [3]-[15]. An unfortunate complication is that in the English-language literature of electrical engineering the term "voltage" is (in this author's view) inappropriately defined for time-varying situations [16].[2]

The voltmeter as modeled above reports $V_{\rm meter} = I_0(R_0 + R_{\rm leads})$, which equals the line integral,

$$\int_1^2 \mathbf{E} \cdot d\mathbf{l} \qquad \text{(along meter leads)} \tag{2}$$

along the path of its conductors. To see this, note that for a cylindrical, resistive medium of length l, radius r, and electrical conductivity σ that obeys Ohm's law $\mathbf{J} = \sigma\mathbf{E}$, where $\mathbf{J} = I\,\hat{\mathbf{l}}/\pi r^2$ and I is the (uniform) axial current, then $El = Jl/\sigma = Il/\pi r^2\sigma = IR$, and the (axial) electrical resistance is $R = l/\pi r^2\sigma$. Such line integrals were called an electromotive force ($\mathcal{EMF}$) by Faraday, which term we will use for them in this note.

In time-varying situations, particularly where there are large magnetic fields in the vicinity of the circuit that is being probed by the voltmeter, the $\mathcal{EMF}$ (2) depends on the path between points 1 and 2.[3] However, in "ordinary" circuits (ones that satisfy the three conditions given in sec. 1) there is very little magnetic flux linked by the loop that includes the voltmeter, and the integral (2) is independent of the path to a good approximation. This means that for such "ordinary" circuits the electric field between points 1 and 2 can be related to a scalar potential V according to $\mathbf{E} \approx -\boldsymbol{\nabla} V$ to a good approximation, such that

$$\int_1^2 \mathbf{E} \cdot d\mathbf{l} \approx -\int_1^2 \boldsymbol{\nabla} V \cdot d\mathbf{l} = V_1 - V_2. \tag{3}$$

That is, when an AC voltmeter is used with an "ordinary" circuit it reads, to a good approximation, the voltage drop $V_1 - V_2$ between the electric scalar potential at the tips of its leads.[4]

We confirm this result in sec. 2.3, after lengthy preliminaries in secs. 2.1 and 2.2. Sections 2.4-2.6 then illustrate how $V_{\rm meter}$ is often quite different from $V_1 - V_2$ when the meter leads are comparable in length to λ or when radiation and wave propagation are important (such that there are significant magnetic fields in the vicinity of the circuit). Some comments about alternative conventions for defining the potentials are given in the Appendix.

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[2] The ambiguous meaning of "voltage" is noted, for example, in [17].

[3] A peculiar example in which the use of a voltmeter "creates" the voltage that it measures is discussed in [18].

[4] Some people [16] then use the term "voltage drop" to describe any integral of the form (2), although only in electrostatics does this integral equal the difference in the voltage (= the value of the electric scalar potential). In time-varying examples the line integral (2) depends, in general, on the path of integration, and does not have a value that depends only on the voltage at its end points. It appears that calling the path-dependent integral (2) a "voltage drop" has the unfortunate consequence that many people infer that voltage (electric scalar potential) does not have a well-defined meaning in time-varying situations. On the other hand, in applications of Kirchhoff's circuit law (35) to electrical networks it is convenient to assume that a scalar voltage V can be assigned to each node such that $\mathcal{EMF}$ between nodes 1 and 2 equals $V_1 - V_2$, although this is only approximately true. Altogether, the term "voltage drop" has come to be too loosely interpreted.

A thoughtful commentary on the meaning of "voltage" in AC circuits is given in sec. 6.10 of [19].

2.1 Scalar and Vector Potentials

In electrostatics the electric field $\mathbf{E}$ can be related to a scalar potential V, the voltage, according to,

$$\mathbf{E} = -\boldsymbol{\nabla} V \qquad \text{(statics)}, \tag{4}$$

and inversely,

$$V_a - V_b = -\int_b^a \mathbf{E} \cdot d\mathbf{l} \qquad \text{(statics)}, \tag{5}$$

expresses the fact that a unique voltage difference $V_a - V_b$ can be defined for any pair of points a and b independent of the path of integration between them. The static electric field is said to be conservative, and eqs. (4)-(5) are equivalent to the vector calculus relation,

$$\boldsymbol{\nabla} \times \mathbf{E} = 0. \tag{6}$$

In electrodynamics Faraday discovered (as later interpreted by Maxwell) that eq. (6) must be generalized to,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{7}$$

in SI units, which implies that time-dependent magnetic fields $\mathbf{B}$ lead to additional electric fields beyond those associated with the scalar potential V. The nonexistence (so far as we know) of isolated magnetic charges (monopoles) implies that,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{8}$$

and hence that the magnetic field can be related to a vector potential $\mathbf{A}$ according to,

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}. \tag{9}$$

Using eq. (9) in (7), we can write,

$$\boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0, \tag{10}$$

which implies that $\mathbf{E} + \partial \mathbf{A}/\partial t$ can be related to a scalar potential V as $-\boldsymbol{\nabla} V$, *i.e.*,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{(dynamics)}. \tag{11}$$

We restrict our discussion to media for which the dielectric permittivity is ϵ_0 and the magnetic permeability is μ_0. Then, using eq. (11) in the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$ leads to,

$$\boldsymbol{\nabla}^2 V + \frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{\rho}{\epsilon_0}, \tag{12}$$

and the Maxwell equation $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \partial \mathbf{E}/\partial c^2 t$ leads to,

$$\boldsymbol{\nabla}^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right). \tag{13}$$

We will work in the Lorenz gauge (1), such that the potentials obeys the differential equations,

$$\boldsymbol{\nabla}^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}, \qquad \text{and} \qquad \boldsymbol{\nabla}^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \tag{14}$$

The formal solutions to eq. (14) are the retarded potentials,

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t' = t - R/c)}{R}\, d\text{Vol}', \quad \text{and} \quad \mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t' = t - R/c)}{R}\, d\text{Vol}', \tag{15}$$

where $R = |\mathbf{r} - \mathbf{r}'|$.

In situations where the charges and currents oscillate at a single angular frequency ω, we write,[5]

$$\rho(\mathbf{r},t) = \rho(\mathbf{r})\, e^{-i\omega t}, \quad \mathbf{J}(\mathbf{r},t) = \mathbf{J}(\mathbf{r})\, e^{-i\omega t}, \quad V(\mathbf{r},t) = V(\mathbf{r})\, e^{-i\omega t}, \quad \text{and} \quad \mathbf{A}(\mathbf{r}) = \mathbf{A}(\mathbf{r})\, e^{-i\omega t}. \tag{16}$$

Then, the retarded potentials (15) can be written as,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')\, e^{ikR}}{R}\, d\text{Vol}', \qquad \text{and} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')\, e^{ikR}}{R}\, d\text{Vol}', \tag{17}$$

where $k = \omega/c = 2\pi/\lambda$.

If all distances R relevant to the situation are small compared to the wavelength λ (condition 1 of sec. 1), then $kR \ll 1$, and the potentials can be calculated to a good approximation as,

$$V(\mathbf{r}) \approx \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{R}\, d\text{Vol}', \qquad \text{and} \qquad \mathbf{A}(\mathbf{r}) \approx \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{R}\, d\text{Vol}' \qquad (R \ll \lambda). \tag{18}$$

In this case the potentials in the region close to the circuit can be deduced from the instantaneous charge and current distributions, *i.e.*, effects of the finite speed of light are ignored.[6]

2.2 $\mathcal{EMF}$s in "Ordinary" Circuits with Inductance

In this problem we consider circuits that are driven by a voltage source $V = V_0 e^{-i\omega t}$, which is a (compact) device with terminals at points, say, a (low-voltage terminal or cathode) and b (high-voltage terminal or anode) such that the $\mathcal{EMF}$ of the source is,[7]

$$V = -\int_a^b \mathbf{E}\cdot d\mathbf{l} = V_b - V_a + \frac{d}{dt}\int_a^b \mathbf{A}\cdot d\mathbf{l}, \tag{19}$$

[5] We use the "physics" convention that oscillatory time dependence is written $Re(e^{-i\omega t})$, with the real part being implied. To convert to the "engineering" convention, $Re(e^{j\omega t})$, replace i by $-j$ throughout (where $i = j = \sqrt{-1}$).

[6] In this case the potentials close to the circuit are the same in the Lorenz gauge and in the Coulomb gauge, such that mention of gauge conditions can be omitted in "ordinary" circuit analysis.

[7] In practice, the voltage source ensures the relation (19) only for a particular path in its voltage regulator.

recalling eq. (11). Thus, a voltage source does not necessarily deliver a difference $V_b - V_a$ in the scalar potential between its terminals whose value is equal to V. However, we will make the usual approximation that in "ordinary" circuits the term in eq. (19) involving the vector potential is negligible, so,

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \approx V_b - V_a \qquad (\text{"ordinary" voltage source}). \tag{20}$$

Greater care is required in specifying the nature of a voltage source in cases where radiation is important [20].

The elementary example of a DC circuit is a constant-voltage source V connected to a resistor R, in which case the steady current I in the circuit is given by Ohm's law,

$$V = IR. \tag{21}$$

We say that IR is the $\mathcal{EMF}$ across the resistor,

$$IR = \int_{\text{resistor}} \mathbf{E} \cdot d\mathbf{l}. \tag{22}$$

The circuit forms a loop, and an alternative formulation of the circuit law is that the sum of the $\mathcal{EMF}$s around that loop is zero. That is, the $\mathcal{EMF}$ across a resistor is IR, and the $\mathcal{EMF}$ across the voltage source is $\int_a^b \mathbf{E} \cdot d\mathbf{l} = -V$ according to eq. (19). The circuit law for this DC example follows from eq. (6), which can be expressed in integral form as,

$$IR - V = \int_{\text{resistor}} \mathbf{E} \cdot d\mathbf{l} + \int_a^b \mathbf{E} \cdot d\mathbf{l} = \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = \int_{\text{loop}} \boldsymbol{\nabla} \times \mathbf{E} \cdot d\mathbf{Area} = 0, \tag{23}$$

invoking Stokes theorem of vector calculus, and noting that eq. (7) for a DC circuit implies that $\boldsymbol{\nabla} \times \mathbf{E} = 0$.

If, however, the voltage source creates a time-dependent voltage $V(t)$, then eq. (23) must be modified according to eqs. (7) and (9) to read,

$$\begin{aligned} IR - V &= \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = \int_{\text{loop}} \boldsymbol{\nabla} \times \mathbf{E} \cdot d\mathbf{Area} = -\frac{d}{dt} \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{Area} = -\frac{d\Phi_M}{dt} \\ &= -\frac{d}{dt} \int_{\text{loop}} \boldsymbol{\nabla} \times \mathbf{A} \cdot d\mathbf{Area} = -\frac{d}{dt} \oint_{\text{loop}} \mathbf{A} \cdot d\mathbf{l}, \end{aligned} \tag{24}$$

where,

$$\Phi_M = \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{Area} = \oint_{\text{loop}} \mathbf{A} \cdot d\mathbf{l} \tag{25}$$

is the magnetic flux due to the current I that passes through the loop.

The usual approximation in circuit analysis is to ignore effects of retardation in eq. (15) (*i.e.*, to ignore effects of radiation) and write,

$$\mathbf{A}(\mathbf{r}, t) \approx \frac{\mu_0}{4\pi} \oint \frac{I(\mathbf{r}', t)}{R}\, d\mathbf{l}' \approx \frac{\mu_0 I(t)}{4\pi} \oint_{\text{loop}} \frac{d\mathbf{l}'}{R}\,, \tag{26}$$

where we also ignore possible variation of the current around the loop of the circuit. Then, the magnetic flux (25) can be written as,

$$\Phi_M \approx \frac{\mu_0 I(t)}{4\pi} \oint \oint \frac{d\mathbf{l} \cdot d\mathbf{l}'}{R} = LI(t), \tag{27}$$

where the geometric quantity,

$$L = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l} \cdot d\mathbf{l}'}{R} \tag{28}$$

is the self inductance of the loop (which is independent of time if the shape of the circuit is fixed).

The loop equation (24) can now be written as,[8]

$$IR - V \approx -L\dot{I}, \qquad \text{or} \qquad V \approx L\dot{I} + IR, \tag{29}$$

where $\dot{I} = dI/dt$.[9,10]

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[8]Equation (29) is only approximate because the path of the circuit between the terminals of the power source is not the same as the internal path along which relation (19) is maintained.

[9]It is often said that $\mathcal{EMF}$ $L\dot{I}$ is the "inductive voltage drop" across the inductance of the loop. But, it is wrong, in general, to say that $L\dot{I}$ is the voltage difference between the two ends of the inductor formed by the loop, since the two "ends" of the loop are the same point and have the same voltage V.

These basic facts are obscured by the common practice of winding part of the conductor of the circuit into a compact coil, so that most of the integral $\oint_{\rm loop} \mathbf{A} \cdot d\mathbf{l} = LI$ comes from this compact region. That region, say from points 1 to 2, is often identified as the inductor in the circuit, and the inductance L is often wrongly (but conveniently) considered to be a property of that compact portion of the circuit, rather than of the circuit as a whole. Then, the "inductive voltage drop",

$$L\dot{I} = \frac{d}{dt} \oint_{\rm loop} \mathbf{A} \cdot d\mathbf{l} \approx \frac{d}{dt} \int_1^2 \mathbf{A} \cdot d\mathbf{l} \tag{30}$$

around the loop is often (mis)identified as a voltage difference between the two ends of the inductor, with the implication that a scalar potential V can account for the behavior of inductors, as assumed in applications of Kirchhoff's circuit law to networks.

However, the error is slight for a typical inductor in the form of a coil of N turns, where N is large. In this case the vector potential of the coil is azimuthal, with an axial component due only to the current in the circuit outside the nominal coil. Then, along the axis of the coil, between points 1 and 2, we have that $\mathbf{E} \approx -\boldsymbol{\nabla} V$, such that $V_1 - V_2 = \int_1^2 \mathbf{E} \cdot d\mathbf{l}$. That is, the $\mathcal{EMF}$ between the ends of the coil is very close to the voltage drop $V_1 - V_2$ between them, with a fractional error of order $1/N$.

[10]It is also wrong, in general, o describe the $\mathcal{EMF}$ IR as the difference in voltage/scalar potential between the two ends of a resistor in a time-dependent situation. One way to see this is to use eq. (7) in eq. (22),

$$IR = \int_{\rm resistor} \mathbf{E} \cdot d\mathbf{l} = \Delta V_{\rm resistor} - \frac{d}{dt} \int_{\rm resistor} \mathbf{A} \cdot d\mathbf{l}, \tag{31}$$

where $\Delta V_{\rm resistor}$ is the magnitude of the difference in the scalar potential between the two ends of the resistor.

For another way to see this, recall that Ohm's law has the more basic form,

$$\mathbf{J} = \sigma \mathbf{E}, \tag{32}$$

where σ is the electrical conductivity of the resistive medium, and $\mathbf{E}$ is the total electric field (11). Then, the cylinder of radius r and length l has resistance,

$$R = \frac{l}{\pi r^2 \sigma} \tag{33}$$

If the circuit includes a capacitor of value C, then the $\mathcal{EMF}$ across the capacitor is Q/C where $Q(t)$ is the magnitude of the electric charge on one of the plates of the capacitor.[11] If this capacitor is in series with the resistance R, the circuit equation (29) can be generalized to the form,

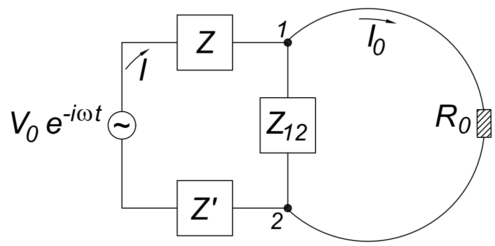
$$V \approx L\dot{I} + IR + \frac{Q}{C}\,. \tag{35}$$

Such circuit equations are then said to obey Kirchhoff's law that the sum of the $\mathcal{EMF}$s around any loop is zero.[12]

2.3 What an AC Voltmeter Measures

We can now give a fairly general discussion of what an AC voltmeter measures (besides its internal $\mathcal{EMF}$ (2)) in an "ordinary" circuit (one that satisfies conditions 1-3 of sec. 1). That is, we restrict our discussion to cases where the circuit is small compared to the wavelength $\lambda = 2\pi c/\omega$, where radiation can be ignored, and where effects of inductance are well localized in inductors.

We consider a generic circuit with three series impedances driven by a voltage source $V_0\,e^{-i\omega t}$, as sketched below.[13] The leads of the voltmeter are attached to the circuit at points 1 and 2, and the resistance R_0 of the voltmeter is large compared to the magnitudes of the impedances Z, Z' and Z_{12}, and also large compared to the resistance $R_{\rm leads}$ of the leads. The the reactance ωL_0 associated with the self inductance L_0 of the loop containing the meter is also small in magnitude compared to R_0.



for axial current flow $I = \pi r^2 J$. Then,

$$IR = \frac{Jl}{\sigma} = El = \Delta V_{\rm resistor} - l\frac{\partial \mathbf{A}}{\partial t}\,. \tag{34}$$

[11]For a typical small capacitor the electric field between its electrodes has very little contribution from the vector potential, so that $\mathbf{E} \approx -\boldsymbol{\nabla} V$, and the $\mathcal{EMF}$ across the capacitor is very close to $V_1 - V_2$, the difference in the scalar potential between the electrodes.

[12]Kirchhoff's circuit law applies whenever conditions 1 and 2 of sec. 1 hold. The inductive term $L\dot{I}$ need not be localized to an "inductor", as illustrated in sec. 2.3.4. However, if condition 3 of sec. 1 is also satisfied, then to a good approximation the $\mathcal{EMF}$ across a circuit element equals the difference in the scalar potential between the ends of that element. Thus, Kirchhoff's circuit law is not a basic law of physics, but a convenient approximation that is not accurate in all situations. Examples of this are given in secs. 2.3.3 onwards.

[13]Describing a two-terminal device as having an impedance Z implies that it can be represented in a circuit equation such as eq. (35) by a term IZ, where I is the current that flows between the two terminals, and Z is a property only of the device between the two terminals. This approximation is not accurate, for example, if one wishes to account for the effects of inductance in circuit with two parallel branches that nominally contain only resistors and capacitors. But, when the important effects of inductance are all associated with tightly wound coils, the impedance concept is a good approximation.

Faraday's law for the meter loop gives,

$$\begin{aligned}
\oint_{\text{meter loop}} \mathbf{E}\cdot d\mathbf{l} &= \int_{R_0} \mathbf{E}\cdot d\mathbf{l} + \int_{\text{leads}} \mathbf{E}\cdot d\mathbf{l} + \int_2^1 \mathbf{E}\cdot d\mathbf{l} \\
&= I_0 R_0 + I_0 R_{\text{leads}} + V_2 - V_1 - i\omega \int_1^2 \mathbf{A}\cdot d\mathbf{l} \\
&= -\frac{d\Phi_0}{dt} = i\omega M I - i\omega L_0 I_0,
\end{aligned} \tag{36}$$

where the path of the integral between points 1 and 2 is through the impedance Z_{12}, and M is the mutual inductance of the two loops. In obtaining eq. (36) we have used,

$$\int_2^1 \mathbf{E}\cdot d\mathbf{l} = \int_2^1 \left(-\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}\right)\cdot d\mathbf{l} = V_2 - V_1 - \frac{d}{dt}\int_2^1 \mathbf{A}\cdot d\mathbf{l} = V_2 - V_1 + i\omega \int_2^1 \mathbf{A}\cdot d\mathbf{l}. \tag{37}$$

The premise of the measurement is that the current I_0 through the voltmeter is small compared to the current I delivered by the voltage source. In this case the vector potential $\mathbf{A}$ in the segment 1-2 is essentially that due to current I alone, and therefore,

$$\int_1^2 \mathbf{A}\cdot d\mathbf{l} \approx \frac{\mu_0}{4\pi}\int_1^2 \oint \frac{I_{\text{total}}(\mathbf{r}', t)}{R}\, d\mathbf{l}\cdot d\mathbf{l}' \approx \frac{\mu_0 I}{4\pi}\int_1^2 \oint \frac{d\mathbf{l}\cdot d\mathbf{l}'}{R} \equiv L_{12} I, \tag{38}$$

where the inductance L_{12} is that portion of the total inductance of the left loop associated with segment 1-2. Because segment 1-2 is in common to both loops, the mutual inductance M between these loops is, to a good approximation, the same as L_{12}.

Thus, the meter reading V_{meter} is,

$$V_{\text{meter}} = I_0(R_0 + R_{\text{leads}}) = \int_{R_0} \mathbf{E}\cdot d\mathbf{l} \approx V_1 - V_2 + i\omega(M - L_{12})I - i\omega L_0 I_0. \tag{39}$$

Since $\omega(M - L_{12})I - \omega L_0 I_0$ is a small quantity (if frequency ω is not too large), to a good approximation the AC voltmeter measures the difference $V_1 - V_2$ in the scalar potential between points 1 and 2 (when the meter is present). When $V_1 - V_2$ is very small (for example, if points 1 and 2 are the same) the meter reads a small value of order $|\omega(M - L_{12})I - \omega L_0 I_0|$.[14]

Referring to the figure on the preceding page, the usual circuit analysis is to say that (in the absence of the voltmeter) the $\mathcal{EMF}$ across impedance Z_{12} is,

$$I Z_{12} = I R_{12} - i\omega L_{12} I = (V_1 - V_2)_{\text{no meter}} \approx (V_1 - V_2)_{\text{meter present}} \approx V_{\text{meter}}, \tag{40}$$

where we suppose that the impedance between points 1 and 2 is due a wire segment whose portion of the total self inductance of the loop is L_{12} and whose resistance is R_{12}. We have also supposed that the resistance R_0 of the meter is so large that the voltage difference $V_1 - V_2$ is the same with and without the meter attached to the circuit under test.

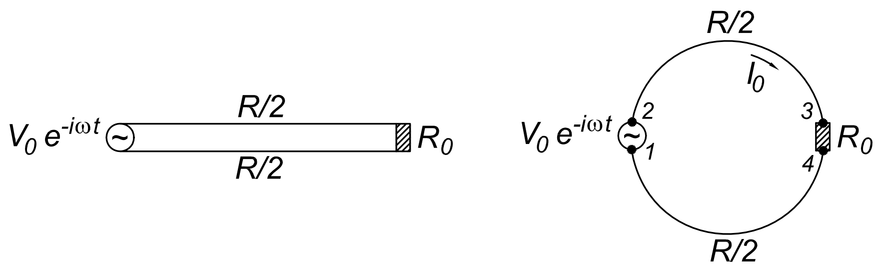
[14]If the tips of the leads are touching then $L_{12} = 0$, and noting that $I_0 \ll I$ we have $V_{\text{meter}} \approx \omega M I = \Phi_M = B\,\text{Area}_{\text{loop}}$. Thus, an AC voltmeter in this configuration is better thought of as an ammeter or a magnetic-field meter. See also secs. 2.4.3, 2.4.4 and 2.5.1.

An AC voltmeter when used in an “ordinary” circuit measures, to a good approximation, the difference $V_1 - V_2$ in the (Lorenz-gauge) scalar potential at the position of the tips of the leads in the absence of the meter. In this case the meter reading is also approximately the $\mathcal{EMF}$ $\int_1^2 \mathbf{E} \cdot d\mathbf{l}$ along the test circuit.

In secs. 2.3.1 and 2.3.2 we give two examples of this behavior for “ordinary” circuits. Section 2.3.3 considers a circuit that appears “ordinary” but is operated at a high frequency such that the circuit is no longer small compared to a wavelength. Section 2.3.4 considers a “small” circuit in which the induced $\mathcal{EMF}$ is not well localized.

2.3.1 An AC Voltmeter Connected Directly to an AC Voltage Source

As a first example we consider an AC voltmeter that is attached to a compact voltage source, $V_0\, e^{-i\omega t}$. The voltmeter is modeled as a large resistance R_0 with two leads of resistance $R/2$ each, where $R \ll R_0$. The connection of the leads to the terminals of voltage source forms a loop whose area is small if the leads are parallel, but which could be large if the leads form a circle. In any case, the self inductance L_0 of this loop is taken to satisfy $\omega L_0 \ll R_0$.



The impedance that the voltmeter presents to the voltage source is,

$$Z = R_0 + R - i\omega L_0 \approx R_0, \tag{41}$$

so the current $I_0 e^{-i\omega t}$ in the voltmeter obeys,

$$I_0 = \frac{V_0}{Z} \approx \frac{V_0}{R_0}. \tag{42}$$

The voltmeter observes this current and reports the measured voltage as,

$$V_{\rm meter} = I_0 R_0 \approx V_0. \tag{43}$$

AC voltmeters typically report the root-mean-square voltage $V_{\rm rms} = V_0/\sqrt{2}$ *rather than the magnitude* V_0*.*

We now consider details of the scalar and vector potential, and of the electric field at the loop. We restrict our discussion to the case of a circular loop for simplicity.

A key result is that the electric field in the leads equals the $\mathcal{EMF}$ divided by their length a, *i.e.*,

$$E_{\rm lead} = \frac{I_0 R}{a} \approx \frac{V_0 R}{a R_0}\,. \tag{44}$$

This field vanishes in the limit that the leads are ideal conductors. It takes some care to account for this field as partly due to the scalar potential and partly due to the vector potential.

The scalar potential V at terminal 1 of the voltage source can be taken as zero. Then, the scalar potential at terminal 2 is V_0 (times $e^{-i\omega t}$). However, the scalar potential at, say, point 3 is not simply $V_0 - I_0R/2$. Rather, we use eq. (34) to find,

$$V_3 = V_0 - \Delta V = V_0 - \frac{I_0 R}{2} - \frac{a}{2}\frac{\partial \mathbf{A}}{\partial t}\,, \tag{45}$$

where a is the circumference of the loop.

The vector potential $\mathbf{A}e^{-i\omega t}$ at the loop is uniform around the (circular) loop and parallel to the direction of the local current. Recalling eq. (25), the magnetic flux through the loop is $\Phi_M = L_0 I_0 = Aa$, where a is the circumference of the loop, so the vector potential at the loop is,

$$A = \frac{L_0 I_0}{a} \approx \frac{L_0 V_0}{a R_0}\,. \tag{46}$$

Thus, the scalar potential (46) at point 3 is,

$$V_3 \approx V_0\left(1 - \frac{R}{2R_0} + \frac{i\omega L}{2R_0}\right). \tag{47}$$

Similarly, the scalar potential at point 4 is,

$$V_4 \approx V_0\left(\frac{R}{2R_0} - \frac{i\omega L}{2R_0}\right). \tag{48}$$

The electric field at the loop has contributions from both the scalar and the vector potentials,

$$\mathbf{E} = \mathbf{E}_V + \mathbf{E}_A = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}\,. \tag{49}$$

In particular, the electric field along the wire leads has a piece,

$$E_V = \frac{V_0 - V_3}{a/2} \approx \frac{V_0}{a}\left(\frac{R}{R_0} - \frac{i\omega L_0}{R_0}\right), \tag{50}$$

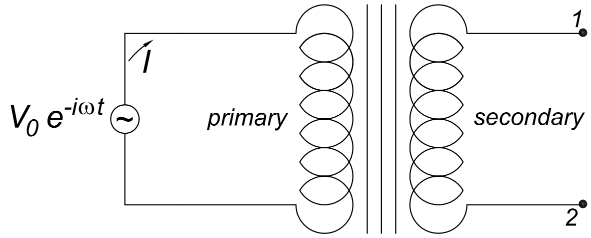
while the contribution from the vector potential is,

$$E_A = -\frac{\partial \mathbf{A}}{\partial t} \approx \frac{V_0}{a}\frac{i\omega L_0}{R_0}, \tag{51}$$

Combining eqs. (50) and (51) we recover the simple result (44).

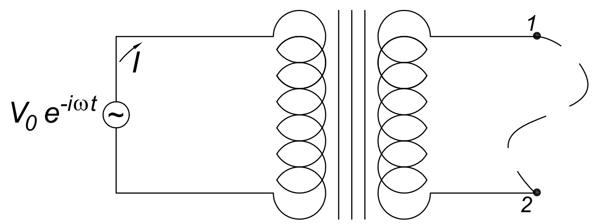
2.3.2 A 1:1 Transformer

To illustrate how an AC voltmeter reads, to a good approximation, the difference in the electric scalar potential in the absence of the voltmeter, consider the circuit shown below. An AC voltage source $V_0 e^{-i\omega t}$ is connected to the primary of a 1:1 (lossless, isolating) transformer. Initially, there is no load connected to the terminals, 1, 2, of the secondary. The conductors in the circuit shown are approximated as having zero resistance.



The role of the 1:1 transformer is to present a voltage drop between terminals 1 and 2 equal to the voltage drop $V_0 e^{-i\omega t}$ of the power source. This voltage drop is equal, to a good approximation to the difference in the electric scalar potential between terminals 1 and 2 (as well as that at the terminals of the power source).

We digress a bit to verify the previous statements. Consider a loop that follows the path of the secondary conductor, and then proceeds through the empty space between points 1 and 2, as sketched below.



Applying Faraday's law to this loop,

$$\oint_{\text{secondary loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_{\text{loop}}}{dt} = i\omega\Phi_{\text{loop}} = i\omega M I, \tag{52}$$

where the magnetic flux $\Phi_{\text{loop}} = MI$ through the secondary loop is due to the magnetic field in the primary loop associated with its current I, as quantified by the product of the current and the mutual inductance M. For a 1:1 transformer the self inductances L_p and L_s of the primary and secondary are very nearly equal, and differ only due to slightly differing topologies to the paths that "complete the circuits".[15] Since in general $L_p L_s = M^2$ we have that $L_p \approx L_s \approx M$. When there is no load on the secondary it induces no back $\mathcal{EMF}$ on the primary, so the current in the primary is simply $I = iV_0/\omega L_p$. Thus,

$$\oint_{\text{secondary loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{M}{L_p} V_0 \approx -V_0. \tag{53}$$

In the approximation of perfect conductors in the transformer, the tangential component of the electric field is zero along the wire, so $\int_{\text{secondary wire}} \mathbf{E} \cdot d\mathbf{l} = 0$, and hence the integral along the path from point 1 to 2 is,

$$\int_1^2 \mathbf{E} \cdot d\mathbf{l} \approx -V_0 = V_1 - V_2 + i\omega \int_1^2 \mathbf{A} \cdot d\mathbf{l}, \tag{54}$$

recalling eq. (37). The vector potential $\mathbf{A}$ along the path from points 1 to 2 is essentially zero, so long as the path does not enter the core of the transformer.[16] So, for any practical

[15] For windings of 1000 turns about a core of (relative) permeability 1000, the self inductances L_p and L_s of the primary and seconary will differ by roughly a part per million. This sets the scale of the accuracy of the subsequent approximations.

[16] The transformer core is typically in the form of a torus, such that little/no magnetic field exists outside the core.

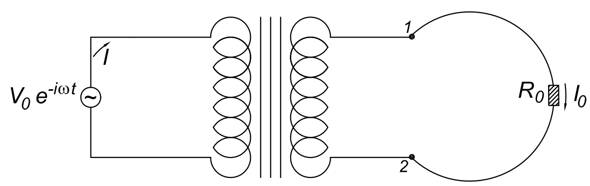
path outside the transformer we have that,

$$\int_1^2 \mathbf{E} \cdot d\mathbf{l} \approx -V_0 \approx V_1 - V_2. \tag{55}$$

This confirms that the difference in the scalar potential between points 1 and 2 of the secondary, in that absence of a load, equals the source voltage V_0 to a good approximation.

However, this result raises an interesting point. The scalar potential at points 1 and 2 oscillates in time. This implies that the charge accumulation near these points oscillates in time, which requires there to be a conduction current between points 1 and 2. This current cannot flow in the empty space; rather there is an oscillating current I' in the secondary conductor even under no-load conditions. The current I' must vanish at points 1 and 2, but it is nonzero, and roughly uniform, along the winding of the secondary. This current distribution is like that in a "linear" dipole antenna, and indeed the main physical process in the secondary under no-load conditions is the radiation asssociated with the oscillating current. This effect is extremely weak at frequencies such as 50-60 Hz, and is justifiably neglected in typical circuit analysis.[17]

Now suppose that the AC voltmeter is connected to points 1 and 2 as sketched below.



There is now a current I_0 in the secondary, which is essentially uniform along the secondary loop. The coupled-circuit analysis for currents I and I_0 is,

$$\begin{aligned} V_0 &= -i\omega L_p I + i\omega M I_0, && (56) \\ 0 &= I_0(R_0 + R_{\text{leads}}) - i\omega L_s I_0 + i\omega M I, && (57) \end{aligned}$$

In the approximation that $L_p = L_s = M$ we have simply that,

$$I_0 \approx \frac{V_0}{R_0 + R_{\text{leads}}}, \tag{58}$$

and the AC voltmeter reads,

$$V_{\text{meter}} = I_0(R_0 + R_{\text{leads}}) \approx V_0 \approx V_1 - V_2. \tag{59}$$

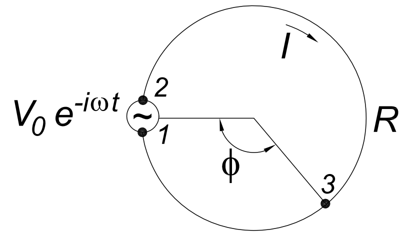
As expected, the AC voltmeter reads, to a good approximation, the difference in the electric scalar potential between points 1 and 2 in the absence of the meter.

2.3.3 A Single-Turn Inductor at High Frequency

As an illustration of how the reading of AC voltmeter can depart from the voltage drop in the electric scalar potential at high frequencies, we consider a single-turn inductor in the

[17]The 1:1 isolation transformer might be considered as part of the output stage of the AC voltage source, such that the output voltage is "floating". This reminds us that an AC voltage source with no load can/should also be considered as a small radiating system [20].

form of a circular loop of circumference a, with resistance R uniformly distributed around the loop, and self inductance L where $R \ll \omega L$.[18] This inductor is driven by a voltage source $V_0 e^{-i\omega t}$, as sketched below.



Taking the scalar potential to be zero at terminal 1 of the voltage source, and to have magnitude V_0 at angle $\phi = 2\pi - \epsilon$, the magnitude of the scalar potential at point 3 at angle ϕ is,

$$V_3 = V(\phi) = V_0 \frac{\phi}{2\pi} \, . \tag{60}$$

The impedance of this circuit is,

$$Z = R - i\omega L, \tag{61}$$

so when driven by a voltage source $V e^{-i\omega t}$ the current $I \, e^{-i\omega t}$ in the loop is,

$$I = \frac{V_0}{Z} = \frac{V_0}{R - i\omega L} \approx \frac{iV_0}{\omega L} \left(1 - \frac{iR}{\omega L}\right) \tag{62}$$

The vector potential at the loop is,

$$A = \frac{\Phi_M}{a} = \frac{LI}{a} \approx \frac{iV_0}{\omega a} \left(1 - \frac{iR}{\omega L}\right), \tag{63}$$

where a is the circumference of the loop. The electric field tangential to the loop has contribution $E_V = V_0/a$ from the scalar potential, and,

$$E_A = -\frac{dA}{dt} \approx -\frac{V_0}{a} \left(1 - \frac{iR}{\omega L}\right) . \tag{64}$$

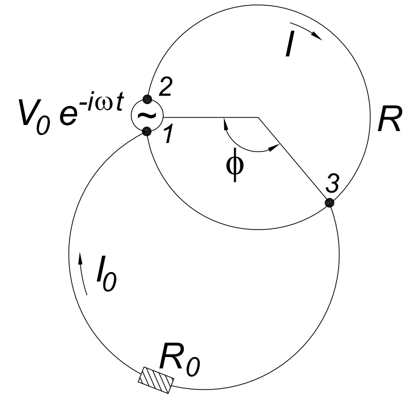
The total electric field tangential to the loop is,

$$E = E_V + E_A \approx \frac{V_0}{a} \frac{iR}{\omega L}, \tag{65}$$

which vanishes in the limit of zero resistance for the loop.

Now suppose the AC voltmeter is connected to points 1 and 3.

[18] The self inductance L is of order $\mu_0 a$ where a is the radius of the loop, so $\omega L \approx \mu_0 c a/\lambda \approx a/\lambda$. Hence, the condition that ωL is large implies that the size of the circuit is large compared to a wavelength.



The two loops in the resulting configuration have self inductances L_0 and L, and mutual inductance $M \approx L\phi/2\pi$ since the flux through meter loop due to current I in the single-turn inductor is largely due to the current on the segment 1-3, which segment is associated with self inductance $L_{13} \approx L\phi/2\pi \approx M$.

Applying Faraday's law to the meter loop, we have,

$$\oint_{\text{meter loop}} \mathbf{E} \cdot d\mathbf{l} = \int_{\text{leads}+R_0} \mathbf{E} \cdot d\mathbf{l} + \int_1^3 \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_0}{dt} = -i\omega M I + i\omega L_0 I_0. \qquad (66)$$

The reading of the meter is,

$$V_{\text{meter}} = I_0(R_0 + R_{\text{leads}}) = \int_{\text{leads}+R_0} \mathbf{E} \cdot d\mathbf{l}. \qquad (67)$$

The $\mathcal{EMF}$ in the segment 1-3 is,

$$\int_1^3 \mathbf{E} \cdot d\mathbf{l} = (I_0 - I)R_{13} \approx -IR\frac{\phi}{2\pi} \approx -\frac{iR}{\omega L}V_3, \qquad (68)$$

which is tiny compared to $\omega M I$. Altogether,

$$I_0(R_0 + R_{\text{leads}} - i\omega L_0) \approx -i\omega M I \approx V_0\frac{M}{L} = V_0\frac{\phi}{2\pi} = V_3, \qquad (69)$$

and the meters reads,

$$V_{\text{meter}} \approx V_3 \frac{R_0 + R_{\text{leads}}}{R_0 + R_{\text{leads}} - i\omega L_0}\,. \qquad (70)$$

If $\omega L_0 \ll R_0$, then the meter would read approximately the scalar potential V_3 at point 3. But for high enough frequencies the impedance $R_0 + R_{\text{leads}} - i\omega L_0$ of the meter is almost purely reactive, and the meter would read $iR_0V_3/\omega L_0$, which is essentially zero.

This illustrates the claim that an AC voltmeter will read the difference in the electric scalar potential only if the frequency is not too high. For high frequencies, both the meter reading and $\int_1^3 \mathbf{E} \cdot d\mathbf{l}$ are small, but the meter reading is roughly R_0/R times the line integral, since $L_0 \approx L$.

2.3.4 A Surprising "DC" Circuit

Even in circuits where the currents of interest are constant it is possible that a voltmeter does not read the difference in the electric scalar potential between the tips of its leads. We illustrate this with an example given by W. Lewin of MIT.[19]

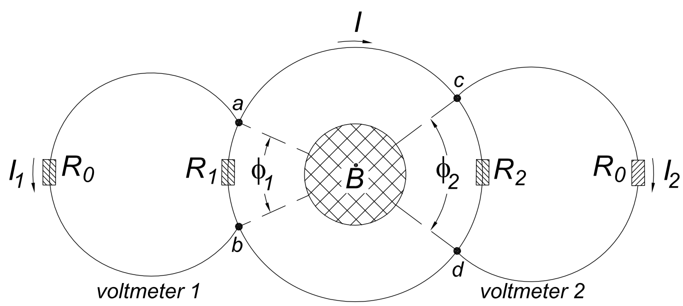
The circuit is illustrated in the figure below. The central loop contains two resistors, R_1 and R_2. A solenoid magnet inside the central loop produces a time-varying magnetic field $B(t) = B_0 t$ through the loop,[20] and creates an $\mathcal{EMF}$ around the loop,

$$\mathcal{E}_0 = -\dot{\Phi} = -B_0 \text{Area}. \tag{71}$$

As a result, DC current,

$$I = \frac{\mathcal{E}_0}{R_1 + R_2} \tag{72}$$

flows in the central loop.



Two identical voltmeters with internal resistance $R_0 \gg R_1, R_2$ probe the central circuit, the first connecting to points a and b, and the second connecting to points c and d. The positive leads of the voltmeters are attached to points a and c, which defines the sense of currents I_1 and I_2 to be as shown. We suppose that no magnetic flux from the solenoid passes through outer loops 1 and 2.

Kirchhoff's circuit equations for the three loops are,

$$\mathcal{E}_0 = (I + I_1)R_1 + (I - I_2)R_2 + L\dot{I} \approx (R_1 + R_2)I + R_1 I_1 - R_2 I_2, \tag{73}$$

$$0 = I_1 R_0 + (I_1 + I)R_1 \approx R_1 I + R_0 I_1, \tag{74}$$

$$0 = I_2 R_0 + (I_2 - I)R_2 \approx -R_2 I + R_0 I_2, \tag{75}$$

on neglect of the small $\mathcal{EMF}$ $L\dot{I}$. Solving these three simultaneous linear equations for the currents, we find,

$$I \approx \frac{\mathcal{E}_0}{R_1 + R_2}, \qquad I_1 \approx -\frac{\mathcal{E}_0 R_1}{R_0(R_1 + R_2)}, \qquad I_2 \approx \frac{\mathcal{E}_0 R_2}{R_0(R_1 + R_2)}. \tag{76}$$

The meter readings are therefore,

$$V_{\text{meter1}} = I_1 R_0 \approx -\frac{\mathcal{E}_0 R_1}{(R_1 + R_2)}, \qquad V_{\text{meter2}} = I_2 R_0 \approx \frac{\mathcal{E}_0 R_2}{(R_1 + R_2)}. \tag{77}$$

[19] http://www.youtube.com/watch?v=eqjl-qRy71w&NR=1
http://www.youtube.com/watch?v=1bUWcy8HwpM&feature=related

[20] The current in the solenoid varies linearly with time, so taken as a whole, this circuit is not strictly DC.

The meter readings do not depend on where the leads are connected, and in particular if the two meters are connected at the same points, $a = c$ and $b = d$, their readings are different.[21] This is surprising in that we might have expected that the (perfectly conducting) wires are equipotentials.

However, the proper assumption is that the electric field tangential to the wires is zero (in the limit of perfectly conducting wires). Indeed, Since $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, we have from Stoke's theorem that $\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{Area} = \Phi$, so the azimuthal component A_ϕ of the vector potential is given by,

$$A_\phi = \frac{\Phi}{2\pi r}, \tag{78}$$

where r is the radius of the central loop. In the present example the magnetic flux Φ through the central loop has contributions from the magnetic fields due to the currents I, I_1 and I_2, as well as from the solenoid. However, only the contribution from the solenoid is significant.

Furthermore, the azimuthal electric field E_ϕ is related to the potentials V and $\mathbf{A}$ by,

$$E_\phi = -\frac{1}{r}\frac{\partial V}{\partial \phi} - \frac{\partial A_\phi}{\partial t}, \tag{79}$$

Applying this to wire segments in the central loop, for which $E_\phi = 0$, we have that,

$$\frac{\partial V}{\partial \phi} = -r\frac{\partial A_\phi}{\partial t} = -\frac{\dot{\Phi}}{2\pi} = \frac{\mathcal{E}_0}{2\pi}, \tag{80}$$

recalling eq. (71), and so the scalar potential along a wire segment has the form,

$$V(\phi) = V_0 + \frac{\mathcal{E}_0 \phi}{2\pi} \tag{81}$$

where ϕ increases for counterclockwise movement around the loop.

The voltage drops across resistors R_1 and R_2 are,

$$\Delta V_1 \approx I R_1 \approx \frac{\mathcal{E}_0 R_1}{(R_1 + R_2)}, \qquad \Delta V_2 \approx I R_2 \approx \frac{\mathcal{E}_0 R_2}{(R_1 + R_2)}, \tag{82}$$

if the azimuthal extent of the resistors is negligible, and we now move in a clockwise sense around the loop. In this convention the voltage drops along each of the wire segments of the central loop are $-\mathcal{E}_0/2$, so the total voltage drop around the loop is zero, as expected for the scalar potential. Finally, the voltage drops between the points where the voltmeters are attached to the central loop are,

$$V_a - V_b \approx \frac{\mathcal{E}_0 \phi_1}{2\pi} - \frac{\mathcal{E}_0 R_1}{(R_1 + R_2)}, \qquad V_c - V_d \approx -\frac{\mathcal{E}_0 \phi_2}{2\pi} + \frac{\mathcal{E}_0 R_2}{(R_1 + R_2)}. \tag{83}$$

Only if the meter leads are connected directly to the ends of resistors R_1 and R_2 (as would be good practice) do the meter readings equal the voltage differences between the tips of the leads.

[21] If both voltmeters were connected to the main loop from the left, or both from the right, then the two meter readings would be the same; namely $-\mathcal{E}_0 R_1/(R_1 + R_2)$ when on the left and $\mathcal{E}_0 R_2/(R_1 + R_2)$ when on the right. They differ only when one meter is connected from the left and the other from the right.

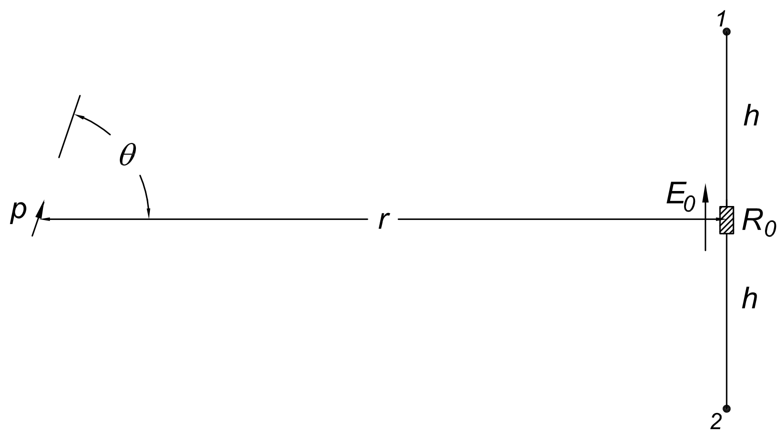
For further discussion of this example, see [1], which contains reference to works by others that consider what a voltmeter measures. A special case is considered in sec. 2.2.2 of [1] in which the central loop consists of a single resistive wire, and the scalar potential V can be taked as zero everywhere.

2.4 An AC Voltmeter Far from a Dipole Antenna

2.4.1 Straight Leads and an Electric Dipole Antenna

In the preceding examples (except for sec. 2.3.3) we have considered the use of voltmeters in situations in which the leads are short compared to the wavelength $\lambda = 2\pi c/\omega$ and in which wave propagation and radiation can be ignored. We now consider a case where the leads are still short but effects of wave propagation are important. In particular, consider an AC voltmeter with large resistance R_0 and straight leads of length $h \ll \lambda$, as sketched on the following page.

This voltmeter is placed at distance $r \gg \lambda$ (*i.e.*, in the far zone) from an antenna whose oscillating electric dipole moment is $\mathbf{p}\, e^{-i\omega t}$, and the leads of the antenna are oriented to be perpendicular to the vector $\mathbf{r}$ from the antenna to the voltmeter. Thus, the electric field $\mathbf{E}_0$ due to the dipole antenna (in the absence of the voltmeter) is parallel to the leads.



The voltmeter is in effect a dipole receiving antenna with a high resistance between its terminals. The $\mathcal{EMF}$ across the terminals will be essentially the same as if the resistance R_0 were infinite, *i.e.*, the open-circuit voltage. This voltage has been discussed elsewhere [21], and the simple result is that,

$$V_{\rm meter} \approx E_0 h. \tag{84}$$

The resulting current $I_0 = V_{\rm meter}/R_0$ through the meter is so small that the fields produced by this current, and that in the leads of the voltmeter, have negligible effect on the distant dipole antenna.

What is the relation between $V_{\rm meter}$ and the potentials of the distant antenna?

In the far zone of a dipole antenna with dipole moment $\mathbf{p}\, e^{-i\omega t}$ the electric and magnetic fields are [22],

$$\mathbf{E} \approx k^2(\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}} \frac{e^{i(kr-\omega t)}}{r}, \qquad \mathbf{B} \approx \frac{k^2}{c}(\hat{\mathbf{r}} \times \mathbf{p}) \frac{e^{i(kr-\omega t)}}{r} = \boldsymbol{\nabla} \times \mathbf{A} \approx ik\, \hat{\mathbf{r}} \times \mathbf{A}, \tag{85}$$

and the corresponding potentials (in the Lorenz gauge) are,

$$\mathbf{A} \approx -i\frac{k}{c}\mathbf{p}\frac{e^{i(kr-\omega t)}}{r}\,, \qquad V = \frac{-ic}{k}\boldsymbol{\nabla}\cdot\mathbf{A} \approx -c\,\hat{\mathbf{r}}\cdot\mathbf{A} \approx -ik\mathbf{p}\cdot\hat{\mathbf{r}}\frac{e^{i(kr-\omega t)}}{r}\,. \tag{86}$$

The electric field strength due of the oscillating dipole $\mathbf{p}$ at the voltmeter is,

$$E_0 = \frac{k^2 p\sin\theta}{r}\,, \tag{87}$$

and the magnitude of scalar potential at points 1 or 2 on the voltmeter is,

$$V_{1,2} \approx \frac{kp\cos\theta_{1,2}}{r} \approx \frac{E_0}{k\sin\theta}\left(\cos\theta \pm \frac{h}{r}\sin\theta\right), \tag{88}$$

noting that $\theta_{1,2} \approx \theta \mp h/r$. Thus, the difference in the scalar potential between those two points is,

$$V_1 - V_2 \approx \frac{2E_0 h}{kr} \ll E_0 h \approx V_{\rm meter}, \tag{89}$$

since $kr = \lambda/2\pi r \ll 1$ in the far zone.

In sum, when an AC voltmeter is used as a receiving antenna in the far zone of a dipole transmitting antenna, the reading of the voltmeter is very large compared to the voltage difference between the tips of the leads in the absence of the meter. The AC voltmeter does measure the local electric field strength of the distant antenna, and if the distance r and wavelength λ are known the local potential difference can be calculated from the meter reading according to,

$$V_1 - V_2 \approx \frac{\lambda}{\pi r}V_{\rm meter}. \tag{90}$$

2.4.2 Straight Leads and a Magnetic Dipole Antenna

The distant dipole antenna might well be a loop antenna with oscillating magnetic dipole moment $\mathbf{m}\,e^{-i\omega t}$. In this case the electric and magnetic fields in the far zone is given by,

$$\mathbf{E} \approx k^2(\mathbf{m}\times\hat{\mathbf{r}})\frac{e^{i(kr-\omega t)}}{r}\,, \qquad \mathbf{B} \approx \frac{k^2}{c}(\hat{\mathbf{r}}\times\mathbf{m})\times\hat{\mathbf{r}}\frac{e^{i(kr-\omega t)}}{r} = \boldsymbol{\nabla}\times\mathbf{A} \approx ik\,\hat{\mathbf{r}}\times\mathbf{A}, \tag{91}$$

and the corresponding potentials (in the Lorenz gauge) are,

$$\mathbf{A} \approx -ik(\mathbf{m}\times\hat{\mathbf{r}})\frac{e^{i(kr-\omega t)}}{r}\,, \qquad V = \frac{-ic}{k}\boldsymbol{\nabla}\cdot\mathbf{A} \approx -c\,\hat{\mathbf{r}}\cdot\mathbf{A} \approx 0. \tag{92}$$

That is, there is no scalar potential in the far zone of a magnetic dipole transmitting antenna.

If the AC voltmeter is oriented so that its leads are parallel to the local electric field $\mathbf{E}_0$ of the antenna, whose magnitude is

$$E_0 = \frac{k^2 m\sin\theta}{r}\,, \tag{93}$$

the meter reading will be,

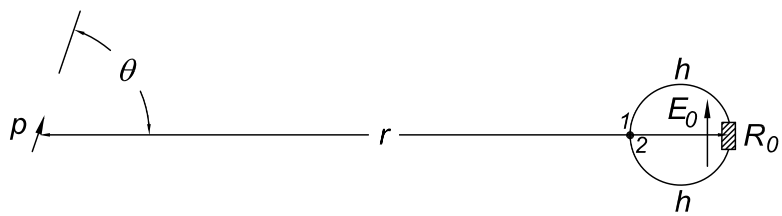
$$V_{\rm meter} \approx E_0 h. \tag{94}$$

just if the transmitting antenna were an electric dipole antenna. Since there is no scalar potential in the far zone of a magnetic dipole antenna, we learn that a nonzero reading on an AC voltmeter does not necessarily imply a nonzero voltage difference between the tips of the meter leads.

Indeed, from a single reading of the AC voltmeter in the far zone of an antenna, we cannot tell whether that antenna was an electric or a magnetic dipole antenna. Only if we are able to move the voltmeter around enough to map out the pattern of the vector polarization of the electric field can we distinguish the two types of antennas.

2.4.3 Looped Leads and an Electric Dipole Antenna

We now consider the case that the leads of the voltmeter form a circular loop, with points 1 and 2 being the same, as shown in the figure below.



Clearly the potential difference $V_1 - V_2$ is zero, but the meter can have a nonzero reading. We orient the loop formed by the leads such that the local electric field $\mathbf{E}_0$ of the distant electric dipole antenna lies in the plane of the loop. Then, the magnetic field, whose magnitude is,

$$B_0 = \frac{k^2 p \sin\theta}{cr} = \frac{E_0}{c}\,, \tag{95}$$

according to eqs. (85) and (87), is perpendicular to the loop. Faraday's law tells us that the $\mathcal{EMF}$ around the loop is,

$$\left| -\frac{d\Phi_m}{dt} \right| = \frac{\omega B_0 h^2}{\pi} = E_0 h \frac{kh}{\pi} = I_0 R_0 = V_{\rm meter}. \tag{96}$$

2.4.4 Looped Leads and a Magnetic Dipole Antenna

If the transmitter is a magnetic dipole antenna, and the leads of the voltmeter form a circular loop whose plane is parallel to the local electric field $\mathbf{E}_0$, then the meter reading is again,

$$V_{\rm meter} = \left| -\frac{d\Phi_m}{dt} \right| = \frac{\omega B_0 h^2}{\pi} = E_0 h \frac{kh}{\pi}\,, \tag{97}$$

where,

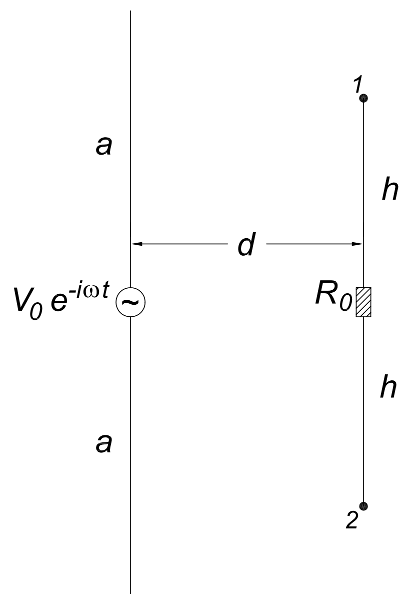
$$B_0 = \frac{k^2 m \sin\theta}{cr} = \frac{E_0}{c}\,. \tag{98}$$

Not only does $V_1 - V_2 = 0$ in this case, but $V_1 = V_2 = 0$, as discussed in sec. 2.3.2.

2.5 An AC Voltmeter Near a Half-Wave Dipole Antenna

We now consider an example that contrasts to the preceeding ones by having the leads of the voltmeter be comparable in length to the wavelength $\lambda = 2\pi c/\omega$. A half-wave dipole antenna has arms of length $a \approx \lambda/2$ such that the impedance Z presented by the antenna to the voltage source $V_0\, e^{-i\omega t}$ is purely real and approximately 70 Ω. The AC voltmeter of impedance $R_0 \gg 70\ \Omega$ is positioned a distance d away from the antenna, with leads of length $h \approx a$ parallel to the arms of the antenna, as shown in the figure on the next page.

We recognize this configuration as similar to that of a Yagi antenna. That is, even though very little current I_0 will flow through resistor R_0, significant standing-wave currents will exist in the leads of the voltmeter, which cause substantial changes in the radiation pattern of the antenna. The reading $V_{\rm meter} = I_0 R_0$ will not readily be expressible in terms of the scalar potential or the currents in the antenna in the absence of the voltmeter.



To predict the behavior of the voltmeter, I used the NEC4 simulation [23] with the LD *command to generate a resistance R_0 of 1 megohm. Results are given in Table 1 for a drive voltage $V_0 = 1$ volt.*

The first line of Table 1 simulates a voltmeter with short leads connected close to the voltage source, which makes a small perturbation in the antenna impedance Z and results in a reading of $V_{\rm meter} = V_0$. As longer leads are used to connect the meter to points farther out on the arms of the antenna, the effect on the antenna impedance grows strong, and the meter reading rises to 2 V and then falls back to 1 V. Because the input impedance is affected when the voltmeter is connected to the antenna, it is clear that the meter reading does not represent the voltage difference $V_1 - V_2$ in the absence of the voltmeter.

If the leads of the meter do not make contact with the antenna, but they remain extended to form a kind of receiving antenna, there is little effect on the antenna impedance, and the meter reading falls off as the meter is moved away from the antenna. The last six lines of Table 1 show that for a voltmeter with very short leads ($h = 0.1$ m $= \lambda/500$) that are oriented along the direction of the electric field, the meter readings are close to $E_0 h$, where E_0 is the field strength at the position of resistor R_0 in the absence of the meter. This

Table 1: NEC4 [23] simulations of an AC voltmeter with $R_0 = 1$ megohm near a half-wave dipole antenna with $a = 5$ m, driven by a 1-volt rf source. The antenna impedance Z is calculated in the presence of the voltmeter. The meter voltage $V_{\rm meter}$ is the current I_0 in the meter segment times R_0. The electric field E is calculated at the position of resistor R_0 but when the meter is absent.

h	d	Z	$V_{\rm meter}$		E
0.3 m	0.02 m	$79 + 3.5i\ \Omega$	1.0 V	connected	
1.25	0.02	$72 + 1.5i$	1.5	connected	
2.5	0.02	$65 - 36i$	2.0	connected	
3.75	0.02	$67 - 176i$	1.3	connected	
5	0.02	$78 - 965i$	1.0	connected	
0.3	0.02	73	0.32		
1.25	0.02	75	0.59		
2.5	0.02	74	0.73		
3.75	0.02	72	0.86		
5	0.02	72	1.0		
5	1	70	0.72		
5	2	72	0.46		
5	4	72	0.26		
0.01	0.05	72	0.034		2.96 V/m
0.01	0.25	72	0.0076		0.67
0.01	0.5	72	0.0038		0.34
0.01	1	72	0.0024		0.22
0.01	2	72	0.0017		0.16
0.01	4	72	0.0014		0.13

behavior was anticipated by eq. (84) of sec. 2.4.1.

NEC4 does not at present have an option to display the scalar and vector potentials in the near zone, so we cannot immediately relate the meter readings to the potentials. However, the examples of sec. 2.3 alert us that we should not expect $V_{\rm meter}$ to equal $V_1 - V_2$, although the meter reading will be of the same order as the potential difference.

2.5.1 A Small Voltmeter Connected to a Dipole Antenna

The preceding discussion indicates that when the length of the leads of the voltmeter is a significant fraction of a wavelength the fields and potentials are significantly perturbed. If the voltmeter and leads are small compared to a wavelength they will not perturb the fields and potentials very much. If the voltmeter is connected to the antenna can it then report differences in the scalar potential at the surface of the antenna conductors?

Recalling the discussion around eqs. (36)-(39), the voltmeter will report a reliable measure of the difference $V_1 - V_2$ in the scalar potential between the contact points 1 and 2 provided the mutual inductance M of the loop formed by the voltmeter leads and the segment 12 of the test circuit is equal to the self inductance L_{12} of that segment.[22] In more detail, we need that,

$$MI = L_{12}I = \int_1^2 \mathbf{A} \cdot d\mathbf{l}, \tag{99}$$

according to eq. (38). If the antenna conductor has radius R, and the distance between points 1 and 2 along the antenna conductor is d_{12}, and the voltmeter is at a (small) distance D from that conductor, then the mutual inductance between the antenna and the voltmeter loop is related by,

$$MI = \Phi_{\rm loop} \approx \int_R^D \frac{\mu_0 I}{2\pi r}\, d_{12}\, dr = \frac{\mu_0 d_{12} I}{2\pi} \ln \frac{D}{R}\,. \tag{100}$$

The vector potential A_z at the surface of the antenna conductor can be estimated from the thin-wire approximation (see eqs. (13)-(14) of [24]) as,

$$A_z(z) \approx \frac{\mu_0 Z \tan ka\, I(z)}{8\pi} \approx \frac{\mu_0 V_0}{8\pi} \frac{\sin[k(a - |z|)]}{\cos ka}\,, \tag{101}$$

where a is the length of each arm of the dipole antenna, Z is its input impedance, and V_0 is the drive voltage. Then,

$$\int_1^2 \mathbf{A} \cdot d\mathbf{l} \approx \frac{\mu_0 d_{12} Z \tan ka\, I}{8\pi}\,, \tag{102}$$

which bears little relation to the product MI of eq. (100). Hence, the voltmeter will not report the difference $V_1 - V_2$ in the scalar potential.

Rather, for an antenna made with good conductors, $\int_2^1 \mathbf{E} \cdot d\mathbf{l} \approx 0$, so eq. (36) indicates that,

$$V_{\rm meter} = I_0 R_0 \approx \omega M I. \tag{103}$$

That is, the meter will sample the current distribution,

$$I(z) \approx \frac{V_0}{Z} \frac{\sin[k(a - |z|)]}{\sin ka}, \tag{104}$$

which decreases with increasing z, rather than the scalar potential (eq. (74) of [24]) at the surface of the antenna conductor,

$$V(z) \approx \frac{V_0}{2} \frac{\cos[k(a - |z|)]}{\cos ka}\,, \tag{105}$$

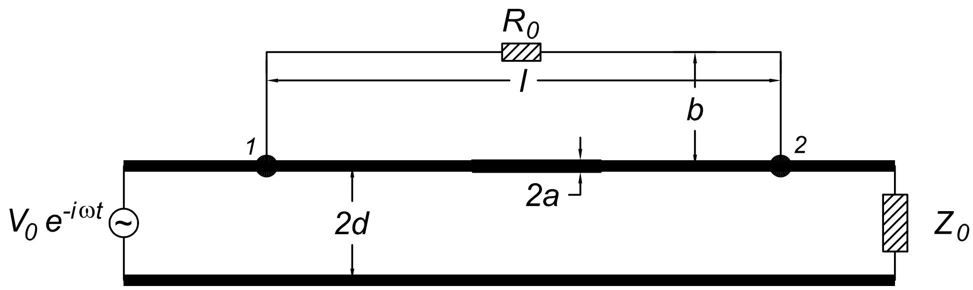
which increases with z and is very large for a half-wave dipole ($ka \approx \pi/2$).

Using NEC4, I have simulated the response of a small voltmeter with $R_0 = 1$ MΩ, $d_{12} = 0.1$ m and $D = 0.01$ m when connected to a half-wave dipole antenna with $a = 5$ m, $R = 0.001$ m and $\omega = 91$ MHz. The meter readings decrease with distance from the feedpoint, and are roughly equal to $\omega M I(z)$ where $M \approx 46$ nH according to eq. (100).

[22]If the leads of the voltmeter lie on, or inside of, the antenna conductor, then $\mathbf{E} \cdot d\mathbf{l} = 0$ everywhere along the conducting path of the voltmeter circuit. In this case $I_0 R_0 = 0$ according to eq. (36), and the voltmeter will read zero, even though $V_1 - V_2$ is nonzero.

2.6 An AC Voltmeter Connected to a Two-Wire Transmission Line

We now consider a long two-wire transmission line made of wires of radius a whose centers are separated by $2d$, as sketched below.



The capacitance per unit length is [25],

$$C = \frac{\pi\epsilon_0}{\ln \frac{d+\sqrt{d^2-a^2}}{a}}\,, \tag{106}$$

and the inductance per unit length is [26],

$$L = \frac{1}{c^2 C} = \frac{\epsilon_0\mu_0}{C} = \frac{\mu_0}{\pi}\ln\frac{d+\sqrt{d^2-a^2}}{a} \approx \frac{\mu_0}{\pi}\ln\frac{2d}{a}\,. \tag{107}$$

The line is driven at one end by voltage source $V_0 e^{-i\omega t}$, and terminated at its other end by a resistor whose resistance equals the characteristic impedance Z_0 of the line,

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu_0}{\pi\epsilon}}\ln\frac{d+\sqrt{d^2-a^2}}{a} = 120\Omega\,\ln\frac{d+\sqrt{d^2-a^2}}{a}\,. \tag{108}$$

Then, the scalar potential at the two wires is,

$$V_{\rm wire}(z,t) = \pm\frac{V_0}{2}e^{i(kz-\omega t)}, \tag{109}$$

where $k = \omega/c = 2\pi/\lambda$, which varies with position along the wires even though they are assumed to be perfect conductors. The scalar potential outside the wires is [27],

$$V(x,y,z,t) = \frac{V_0}{4}\frac{\ln\frac{\left(x+\sqrt{d^2-a^2}\right)^2+y^2}{\left(x-\sqrt{d^2-a^2}\right)^2+y^2}}{\ln\frac{d+a+\sqrt{d^2-a^2}}{d+a-\sqrt{d^2-a^2}}}e^{i(kz-\omega t)}, \tag{110}$$

in a rectangular coordinate system where the wires are centered at $(x,y) = (\pm d, 0)$. The vector potential $\mathbf{A}$ has only a z component (for a very long transmission line) which follows from the Lorenz gauge condition (1) as,

$$A_z(x,y,z,t) = \frac{V(x,y,z,t)}{c}\,. \tag{111}$$

Can an AC voltmeter measure differences in the scalar potential (109) when the leads are connected to points 1 and 2 on one of the wires of the transmission line that are separated by a substantial fraction of a wavelength, as shown in the figure above?

As when a voltmeter is connected to an antenna, the leads in effect become antennas that perturb the current flow in the transmission line, and cause the system to radiate. The reading of the meter will not have a well-defined relation to the scalar potential along the transmission line in the absence of the voltmeter.

To illustrate this I ran a NEC4 simulation of a transmission line made from wires of 1 mm radius with 10 mm separation, so the characteristic impedance of the line is nominally 275 Ω. NEC4 found the impedance to be purely real when the line was driven at 300 MHz and terminated in 260 Ω, in which case only 2% of the input power was radiated. Then, when a voltmeter (with resistance of 1 megohm and leads of length 2λ each) was connected to the transmission line at points $\lambda/2$ apart, the input impedance of the system jumped to $790 + 120i$ Ω, and 70% of the input power was radiated away. Furthermore, the reading on the voltmeter was $1.8V_0$, rather than $V_0/2$ as desired.

Appendix: Other Choices of the Gauge

An interesting review of gauge conditions is given in [29]. Here, we summarize a few facts about the so-called Coulomb gauge, and we introduce a variant that we label as the static-voltage gauge.

Coulomb Gauge

The relations,

$$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \text{and} \qquad \mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \tag{112}$$

between the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ and the potentials V and $\mathbf{A}$ permits various conventions (gauges) for the potentials. In the preceding sections of this note we have always used the Lorenz gauge,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial V}{\partial t} \qquad \text{(Lorenz)}. \tag{113}$$

In situations with steady charge and current distributions (electrostatics and magnetostatics), $\partial V/\partial t = 0$, so the condition (113) reduces to,

$$\boldsymbol{\nabla} \cdot \mathbf{A} = 0 \qquad \text{(Coulomb)}. \tag{114}$$

Even in time-dependent situations it is possible to define the vector potential to obey eq. (114), which has come to be called the Coulomb gauge condition. Then, eq. (12) becomes Poisson's equation,

$$\boldsymbol{\nabla}^2 V = -\frac{\rho}{\epsilon_0}, \tag{115}$$

which has the formal solution,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t)}{R}\, d\text{Vol}' \qquad \text{(Coulomb)}, \tag{116}$$

where $R = |\mathbf{r} - \mathbf{r}'|$, in which changes in the charge distribution ρ instantaneously affect the potential V at any distance.

It is possible to choose gauges for the electromagnetic potentials such that some of their components appear to propagate at any specified velocity v [30, 31, 32].

For completeness, a formal solution for the vector potential in the Coulomb gauge is,

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_t(\mathbf{r}', t' = t - R/c)}{R}\, d\mathrm{Vol}' \qquad \text{(Coulomb)}, \tag{117}$$

where the transverse current density is defined by,

$$\mathbf{J}_t(\mathbf{r}, t) = \frac{1}{4\pi} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \int \frac{\mathbf{J}(\mathbf{r}', t)}{R}\, d\mathrm{Vol}'. \tag{118}$$

Unless the geometry of the problem is such that the transverse current density $\mathbf{J}_t$ is easy to calculate, use of the Coulomb gauge is technically messier than the use of the Lorenz gauge, in which case the potentials are given by eq. (15).

Analysis of circuits is often performed in the quasistatic approximation that effects of wave propagation and radiation can be neglected. In this case, the speed of light is taken to be infinite, so that the Lorenz gauge condition (113) is equivalent to the Coulomb gauge condition (114), and the potentials are calculated from the instantaneous values of the charge and current distributions. As a consequence, gauge conditions are seldom mentioned in "ordinary" circuit analysis.

The Helmholtz Decomposition and the Coulomb Gauge

Helmholtz (1858) showed how any (well-behaved) vector field, say $\mathbf{E}$, that vanishes at infinity obeys the mathematical identity [28],

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla} \int \frac{\boldsymbol{\nabla}' \cdot \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}' + \boldsymbol{\nabla} \times \int \frac{\boldsymbol{\nabla}' \times \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}', \tag{119}$$

where $R = |\mathbf{r} - \mathbf{r}'|$. Time does not appear in this identity, which indicates that the vector field $\mathbf{E}$ at some point $\mathbf{r}$ (and some time t) can be reconstructed from knowledge of its vector derivatives, $\boldsymbol{\nabla} \cdot \mathbf{E}$ and $\boldsymbol{\nabla} \times \mathbf{E}$, over all space (at the same time t). The main historical significance of this identity was in showing that Maxwell's equations, which give prescriptions for the vector derivatives $\boldsymbol{\nabla} \cdot \mathbf{E}$ and $\boldsymbol{\nabla} \times \mathbf{E}$, are mathematically sufficient to determine the field $\mathbf{E}$.

The Helmholtz decomposition (119) can be rewritten as,

$$\mathbf{E} = -\boldsymbol{\nabla} V + \boldsymbol{\nabla} \times \mathbf{A}, \tag{120}$$

where,

$$V(\mathbf{r}) = \int \frac{\boldsymbol{\nabla}' \cdot \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}', \qquad \text{and} \qquad \mathbf{A} = \int \frac{\boldsymbol{\nabla}' \times \mathbf{E}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}'. \tag{121}$$

It is consistent with usual nomenclature to call V a scalar potential and $\mathbf{A}$ a vector potential. That is, Helmholtz decomposition lends itself to an interpretation of fields as related to derivatives of potentials.

Can we use the Helmholtz decomposition as a practical tool for calculating the electric field?

Not in its basic form as given in eqs. (119)-(121), because to use these forms without other input one would have to know the field $\mathbf{E}$ everywhere, so as to be able to calculate $\boldsymbol{\nabla}\cdot\mathbf{E}$ and $\boldsymbol{\nabla}\times\mathbf{E}$ everywhere, so that one could carry out the integrals in eq. (119) to deduce the field $\mathbf{E}$ at point $\mathbf{r}$. However, if one already knows $\mathbf{E}$ everywhere, there is no need to carry out the Helmholtz decomposition to determine $\mathbf{E}$.

Can we use the Helmholtz decomposition + Maxwell's equations to calculate the field $\mathbf{E}$?

Maxwell tells us that,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}, \qquad \text{and} \qquad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{122}$$

where ρ is the electric charge density and $\mathbf{B}$ is the magnetic field.

If we insert these physics relations into eq. (121), we have,

$$V(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0 R}\, d\mathrm{Vol}', \tag{123}$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\partial}{\partial t}\int \frac{\mathbf{B}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}'. \tag{124}$$

The scalar potential (123) is calculated from the instantaneous charge density, which is exactly the prescription (116) of the Coulomb gauge. That is, Helmholtz + Maxwell implies use of the Coulomb-gauge prescription for the scalar potential.

However, eq. (124) for the vector potential $\mathbf{A}$ does not appear to be that of to the usual procedures associated with the Coulomb gauge. Comparing eqs. (120)-(121) and (124), we see that we could redefine the symbol $\mathbf{A}$ to mean,

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \boldsymbol{\nabla}\times\int \frac{\mathbf{B}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}' = \int \boldsymbol{\nabla}\frac{1}{R}\times\frac{\mathbf{B}(\mathbf{r}')}{4\pi}\, d\mathrm{Vol}' = -\int \boldsymbol{\nabla}'\frac{1}{R}\times\frac{\mathbf{B}(\mathbf{r}')}{4\pi}\, d\mathrm{Vol}' \\
&= \int \frac{\boldsymbol{\nabla}'\times\mathbf{B}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}' + \int \boldsymbol{\nabla}'\times\frac{\mathbf{B}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}' = \int \frac{\boldsymbol{\nabla}'\times\mathbf{B}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}' + \oint d\mathbf{Area}'\times\frac{\mathbf{B}(\mathbf{r}')}{4\pi R} \\
&= \int \frac{\boldsymbol{\nabla}'\times\mathbf{B}(\mathbf{r}')}{4\pi R}\, d\mathrm{Vol}',
\end{aligned} \tag{125}$$

provided $\mathbf{B}$ vanishes at infinity. Then, we have,

$$\mathbf{E} = -\boldsymbol{\nabla}V - \frac{\partial \mathbf{A}}{\partial t}, \tag{126}$$

which is the usual way the electric field is related to a scalar potential V and a vector potential $\mathbf{A}$. Note also that eq. (125) obeys the Coulomb gauge condition (114) that $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$.

In view of the Maxwell equation $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$, we recognize eq. (125) as the Helmholtz decomposition $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ for the magnetic field.

We can go further by invoking the Maxwell equation,

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \tag{127}$$

where $\mathbf{J}$ is the current density vector, so that,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{R}\, d\text{Vol}' + \frac{\partial}{\partial t}\int \frac{\mathbf{E}(\mathbf{r}')}{4\pi c^2 R}\, d\text{Vol}'. \tag{128}$$

This is still not a useful prescription for calculation of the vector potential, because the second term of eq. (128) requires us to know $\mathbf{E}(\mathbf{r}')/c^2$ to be able to calculate $\mathbf{E}(\mathbf{r})$. But, c^2 is a big number, so $\mathbf{E}/c^2$ is only a "small" correction, and perhaps can be ignored. If we do so, then,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{R}\, d\text{Vol}', \tag{129}$$

which is the usual instantaneous prescription for the vector potential due to steady currents. Thus, it appears that practical use of the Helmholtz decomposition + Maxwell's equations is largely limited to quasistatic situations, where eqs. (123) and (129) are sufficiently accurate.

Of course, we exclude wave propagation and radiation in this approximation. We can include radiation and wave propagation if we now invoke the usual prescription, eqs. (117)-(118), for the vector potential in the Coulomb gauge. However, this prescription does not follow very readily from the Helmholtz decomposition, which is an instantaneous calculation.

Note that in the case of practical interest when the time dependence of the charges and currents is purely sinusoidal at angular frequency ω, i.e., $e^{-i\omega t}$, the Lorenz gauge condition (1) becomes,

$$V = -\frac{ic}{k}\boldsymbol{\nabla}\cdot\mathbf{A}. \tag{130}$$

In this case it suffices to calculate only the vector potential $\mathbf{A}$, and then deduce the scalar potential V, as well as the fields $\mathbf{E}$ and $\mathbf{B}$, from $\mathbf{A}$.

However, neither the Coulomb gauge condition, $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, nor the Lorenz gauge condition (1) suffices, in general, for a prescription in which only the scalar potential V is calculated, and then $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ are deduced from this. Recall that the Helmholtz decomposition tells us how the vector field $\mathbf{A}$ can be reconstructed from knowledge of both $\boldsymbol{\nabla}\cdot\mathbf{A}$ and $\boldsymbol{\nabla}\times\mathbf{A}$. The gauge conditions tell us only $\boldsymbol{\nabla}\cdot\mathbf{A}$, and we lack a prescription for $\boldsymbol{\nabla}\times\mathbf{A}$ in terms of V.

Static-Voltage Gauge

It is also possible to (re)define the scalar potential V to have no time dependence, such that the time-varying part of the electric field is entirely due to the vector potential $\mathbf{A}$ [33].

Suppose that the charge and current densities ρ and $\mathbf{J}$ consist of time-independent terms plus terms with time dependence $e^{-i\omega t}$. That is,

$$\rho = \rho_0 + \rho_\omega e^{-i\omega t}, \qquad \text{and} \qquad \mathbf{J} = \mathbf{J}_0 + \mathbf{J}_\omega e^{-i\omega t}. \tag{131}$$

We can choose that the scalar potential $V = V_0 + V_\omega\, e^{-i\omega t}$ obeys the static relation,

$$\boldsymbol{\nabla}^2 V = -\frac{\rho_0}{\epsilon_0}, \qquad V_\omega = 0, \tag{132}$$

provided the vector potential $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_\omega e^{-i\omega t}$ obeys the gauge condition,

$$\frac{\partial}{\partial t}\boldsymbol{\nabla} \cdot \mathbf{A} = -i\omega \boldsymbol{\nabla} \cdot \mathbf{A}_\omega e^{-i\omega t} = -\frac{\rho_\omega e^{-i\omega t}}{\epsilon_0}\,, \tag{133}$$

i.e.,

$$\boldsymbol{\nabla} \cdot \mathbf{A}_\omega = -\frac{i\rho_\omega}{\epsilon_0 \omega}\,. \tag{134}$$

We also choose that the time-independent part $\mathbf{A}_0$ of the vector potential satisfies the usual condition of magnetostatics,

$$\boldsymbol{\nabla} \cdot \mathbf{A}_0 = 0, \tag{135}$$

in which case the vector potentials obeys the relations,

$$\boldsymbol{\nabla}^2 \mathbf{A}_0 = -\mu_0 \mathbf{J}_0, \qquad \text{and} \qquad \boldsymbol{\nabla}^2 \mathbf{A}_\omega + k^2 \mathbf{A}_\omega = -\mu_0 \mathbf{J}_\omega - \frac{i\boldsymbol{\nabla}\rho_\omega}{\epsilon_0 \omega}\,. \tag{136}$$

The formal solutions to equations (132) and (136) are,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_0(\mathbf{r}')}{R}\, d\mathrm{Vol}', \tag{137}$$

$$\mathbf{A}_0(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_0(\mathbf{r}')}{R}\, d\mathrm{Vol}', \tag{138}$$

and,

$$\mathbf{A}_\omega(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_\omega(\mathbf{r}')e^{ikR}}{R}\, d\mathrm{Vol}' + \frac{i}{4\pi\epsilon_0\omega}\int \frac{\boldsymbol{\nabla}\rho_\omega(\mathbf{r}')e^{ikR}}{R}\, d\mathrm{Vol}'\,, \tag{139}$$

where $R = |\mathbf{r} - \mathbf{r}'|$.

While the forms (137)-(139) are not used in practice, they show how it is possible to define the scalar potential V to be purely static, such that the time-dependent voltage V_ω is always zero.

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