Wagon in the Rain

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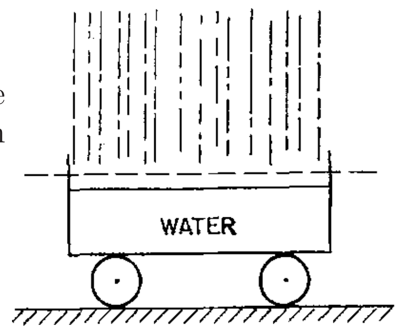
1 Problem



Discuss the motion of a wagon that moves on a horizontal plane without friction but which collects rain that falls vertically (in the "lab" frame) at constant rate $dm/dt = k$.

2 Solution

This problem was discussed by Tiersten [1] in a thoughtful commentary on variable-mass problems, and may be the inspiration for the subtler example of the motion of a leaky tank car [2].

Suppose for simplicity that the water collected in the wagon has no motion relative to the latter, and that the wagon moves along a straight line with velocity v_0 at time $t = 0$, when the mass of the wagon plus water therein is m_0 and its position is $x_0 = 0$.[1]

The rate of collection of rain in the wagon is independent of its speed v; $m(t) = m_0 + kt$.

The horizontal momentum p is constant, in that the falling rain has no horizontal momentum,[2]

$$p = m_0 v_0 = m(t)v(t) = (m_0 + kt)v(t), \tag{1}$$

which immediately tells us that the wagon slows down according to,

$$v(t) = \frac{m_0}{m_0 + kt} v_0 = \frac{m_0}{m(t)} v_0. \tag{2}$$

The wagon takes an infinite time to come to rest, and travels logarithmically infinite distance,

$$x(t) = \int_0^t v\, dt = m_0 v_0 \int_0^t \frac{dt}{m_0 + kt} = \frac{m_0 v_0}{k} \ln \frac{m_0 + kt}{m_0} = \frac{m_0 v_0}{k} \ln \frac{m(t)}{m_0}\, . \tag{3}$$

If the rain lasts only for time T, the final velocity of, and distance x traveled during time T by, the wagon are obtained from the above with $t = T$.

3 Comments

Taking the time derivative of eq. (1), we find,

$$(m_0 + kt)\frac{dv}{dt} = m(t)a = -k\, v, \tag{4}$$

[1]We also ignore the impact of the rain on the vertical sides of the wagon.

[2]If the rain happened to have the same horizontal velocity v as the wagon, then the falling rain would increase the mass/momentum of the wagon + water, while the velocity of the wagon would not change. See also [3], and the Appendix below.

where a is the acceleration of the wagon plus collected water. This form is suggestive, but its interpretation is problematic, as noted by Tiersten [1].

One can note that the horizontal velocity of the rain relative to the wagon is $-v$, so that in the instantaneous rest frame of the wagon the rain transfers momentum to the wagon at rate $dp'/dt' = -kv$, which effect can be interpreted as a force $F' = -kv$ on the wagon.

The wagon has no acceleration in its rest frame, so the total force on the wagon in this frame is $F'_{\rm total} = -m(t')a - kv = 0$, where the term $-m(t')a$ is a "fictitious" (coordinate) force.

In the "lab" frame (in which the rain falls vertically) there are no "fictitious" forces, so we are led to say that the rain exerts horizontal force $F = F' = -kv$ on the wagon plus the collected rain, which is a reaction to the wagon giving momentum to the collected rain at rate kv.

The difficulty with this association is that in the lab frame there appears to be no net "external" force on the system consisting of the wagon and water therein. We can say that in the "lab" frame there are two equal and opposite (horizontal) forces, kv and $-kv$, but it is delicate to say what these forces act on. And, it seems that the system has acceleration without (net) force, according to the "lab"-frame analysis.

An interpretation is that the force kv acts on the water that is "newly collected" in the wagon, while the force $-kv$ acts on the wagon plus water already collected. However, the notion of "newly collected" water is ill defined in the continuum limit; if the mass of the wagon plus collected water is $m(t) = m_0 + kt$, then the mass of the "newly collected" water goes to zero in the continuum limit, and the acceleration of the "newly collected" water goes to infinity.

Instead, if we think of individual drops of mass Δm of rain hitting the wagon and being accelerated from horizontal velocity zero to v during some small but finite time Δt, each drop experiences a large but finite force $v\Delta m/\Delta t$ from the water already collected in the wagon; and if N drops hit the wagon per second, the average force on the drops is $N\ \Delta t \cdot v\Delta m/\Delta t = kv$ with $k = N\Delta m$. The reaction force of the drops on the water already in the wagon is $-kv$. These statements are logically consistent, but they invoke an implausible physical limit as $N \to \infty$ and $\Delta m,\ \Delta t \to 0$ with $N\Delta m = k$.

It seems more straightforward just to do the analysis (1)-(3) based on the fact that in the "lab" frame $F_{\rm total} = 0 = dp_{\rm total}/dt$, without interpreting the results in terms of $F = ma$.

A Appendix: The Rain Has a Horizontal Velocity u

Feb. 11, 2022. This Appendix was suggested by Johan Otto.

If the falling rain has horizontal velocity u, positive in the direction of v_0, when it is captured in the wagon, the momentum of the wagon changes at rate,

$$\frac{dp}{dt} = \frac{dm}{dt}u = ku = \frac{d(mv)}{dt} = \dot{m}v + m\dot{v} = kv + (m_0 + kt)\dot{v}, \tag{5}$$

ignoring the effect of the impact of rain on the vertical walls of the wagon. In this approximation, the velocity of the wagon remains constant if $u = v$, while the velocity of the wagon increases if $u > v$. If $u < v$ the velocity of the wagon drops to zero, after which (if friction is negligible), the wagon would accelerate in the direction of u.

References

[1] M.S. Tiersten, *Force, Momentum Change and Motion*, Am. J. Phys. **37**, 82 (1969),
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[3] C.P. Pesce and L. Casetta, *Variable Mass Systems Dynamics in Engineering Mechanics Education*, 19$^{\text{th}}$ Int. Cong. Mech. Eng. (Brasilia, Nov. 5-9, 2007),
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