Accelerating Through a Resonance on a "Washboard" Road

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1 Problem

Estimate the minimum horizontal acceleration a of a car such that it can pass through the resonance without significant excitation of (damped) vertical oscillations when it drives on a "washboard" road whose height y_R varies with horizontal position x according to $y_R = A\sin(2\pi x/\lambda)$. The car can be approximated as a mass m at height y that can oscillate vertically at natural (angular) frequency ω_0 subject to a velocity-dependent damping force $-\gamma m\, d(y - y_R)/dt$.

A simpler version of this problem is Prob. 3 of
http://kirkmcd.princeton.edu/examples/ph205set7.pdf

2 Solution

2.1 Quick Estimate

If the resonance (angular) frequency for vertical oscillations of the car is ω_0, and the horizontal period of the "washboard" road is λ, then the oscillations will be maximally excited when the car has horizontal velocity $v = f_0\lambda = \omega_0\lambda/2\pi$.

If the oscillations are damped by a velocity-dependent frictional force $-\gamma m\, d(y - y_R)/dt$, then the damping time is $1/\gamma$, which is the characteristic time required for oscillations to build up (or die out). Also, the range of frequencies for which the oscillations are excited to at least half the maximum amplitude is roughly $\omega_0 - \gamma < \omega < \omega_0 + \gamma$. Hence, the car must pass through this range of frequencies in time much less than $1/\gamma$ to avoid significant excitation of the resonance.

If the car has acceleration a, then its velocity is $v = v_0 + at$, and the angular frequency ω of the force of the "washboard" road on the car is,

$$\omega = \frac{2\pi}{\lambda} v = \frac{2\pi}{\lambda}(v_0 + at). \tag{1}$$

The time T required for ω to pass through the resonance of width $\Delta\omega \approx 2\gamma$ is,

$$T = \frac{\gamma\lambda}{\pi a}\,, \tag{2}$$

which must be small compared to the damping time $1/\gamma$. Altogether, the acceleration must satisfy,

$$a \gg \frac{\gamma^2\lambda}{\pi}\,. \tag{3}$$

Apparently, shock absorbers on cars have $\gamma \approx 0.5$, so if the period of the "washboard" is $\lambda = 10$ m, the acceleration must be large compared to $10/4\pi \approx 1$ m/s^2 $= 0.1$ g, where g is the acceleration due to gravity. Thus, the acceleration must be a substantial fraction of g for the car to pass through the "washboard" resonance with little noticeable effect.

2.2 Further Details

Suppose the car moves in the $+x$-direction with initial speed v_0 when at time $t = 0$ it encounters a washboard road that occupies the region $x > 0$. Thereafter, the car accelerates with constant acceleration a for time T, during which its horizontal position is $x = v_0 t + at^2/2$.

The height y_R of the surface of the washboard road is,

$$y_R = A \sin \frac{2\pi x}{\lambda}\,, \tag{4}$$

so the height $y_r(t)$ of the road at the position of the car varies as,

$$y_R(t) = A \sin \frac{2\pi(v_0 t + at^2/2)}{\lambda}\,. \tag{5}$$

Approximating the car by a mass m at position (x, y) that is connected to a spring of constant k and rest length L, such that the spring only exerts a vertical force, the equation of vertical motion is,

$$m\ddot{y} = -k(y - y_R - L) - mg - \gamma m \frac{d}{dt}(y - y_R), \tag{6}$$

assuming a velocity-dependent frictional force proportional to the vertical velocity of the car relative to the road. Clearly, the equilibrium height of mass m is $y_0 = L - mg/k$. In the rest of the problem we measure height y relative to y_0, so the equation of motion can be written as,

$$\begin{aligned}
\ddot{y} + \gamma\dot{y} + \omega_0^2 y &= \omega_0^2 y_R + \gamma \dot{y}_R = \omega_0^2 A \sin \frac{2\pi(v_0 t + at^2/2)}{\lambda} + \frac{2\pi\gamma A}{\lambda}(v_0 + at) \cos \frac{2\pi(v_0 t + at^2/2)}{\lambda} \\
&= Re\left\{ A e^{-i\omega(t)t} \left[i\omega_0^2 + \frac{2\pi\gamma}{\lambda}(v_0 + at) \right] \right\},
\end{aligned} \tag{7}$$

where $\omega_0 = \sqrt{k/m}$ and,

$$\omega(t) = \frac{2\pi(v_0 + at/2)}{\lambda}\,. \tag{8}$$

The relation (7) is a second-order linear differential equation, whose solution can be expressed as the sum of a particular solution plus the general solution to the homogeneous equation,

$$\ddot{y} + \gamma\dot{y} + \omega_0^2 y = 0, \tag{9}$$

subject to a specific set of initial conditions. For the present example we take to the latter to be $y(0) = 0 = \dot{y}(0)$, supposing that the car first encounters the washboard road at time $t = 0$.

As usual, we consider a trial solution to the homogeneous equation (9) of the form,

$$y(t) = Re[y_0 e^{-i\alpha t}]. \tag{10}$$

Then, eq. (9) leads to the quadratic equation,

$$\alpha^2 + i\alpha\gamma - \omega_0^2 = 0, \tag{11}$$

such that,

$$\alpha = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \gamma^2/4}\,. \tag{12}$$

Hence, the general solution to the homogenous equation (9) can be written as,

$$y = e^{-\gamma t/2}\left[y_1 \cos\sqrt{\omega_0^2 - \gamma^2/4}\,t + y_2 \sin\sqrt{\omega_0^2 - \gamma^2/4}\,t\right], \tag{13}$$

where the real constants y_1 and y_2 are still to be determined.

If the acceleration a is zero such that $\omega = 2\pi v_0/\lambda$ is constant, then the steady-state vertical oscillations of the car (for $t > 0$) are described by,

$$y(t) = Re\left[\frac{Ae^{-i\omega t}(i\omega_0^2 + \gamma\omega)}{\omega_0^2 - \omega^2 - i\gamma\omega}\right] = A\frac{-\gamma\omega^3\cos\omega t + [\omega_0^2(\omega_0^2 - \omega^2) + \gamma^2\omega^2]\sin\omega t}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\,. \tag{14}$$

The energy stored in the steady oscillatory motion is,

$$U = \frac{k}{2}\left|y\right|^2, \tag{15}$$

and the time-average power consumed by the damping is,

$$P = \frac{1}{2}Re(F\dot{y}^\star) = \frac{\gamma m}{2}\left|\dot{y}\right|^2 = \frac{\gamma m\omega^2}{2}\left|y\right|^2. \tag{16}$$

The ratio of the stored energy to the energy consumed per cycle at resonance is,

$$\frac{Q}{2\pi} = \frac{U}{P(2\pi/\omega_0)} = \frac{k}{2\pi\gamma m\omega} = \frac{\omega_0}{2\pi\gamma}\,, \tag{17}$$

which confirms the well-known result that $Q = \omega_0/\gamma$.

The full width at half maximum of the resonance curve $|y(\omega)| = \omega_0^2 A/\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \approx \omega_0 A/\sqrt{4(\omega_0 - \omega)^2 + \gamma^2}$ is $\Delta\omega = \sqrt{3}\gamma \approx 2\gamma$.

Returning to the full differential equation (7), its solution is the sum of eqs. (13) and (14), again assuming that $a = 0$,

$$y(t) = A\frac{-\gamma\omega^3\cos\omega t + [\omega_0^2(\omega_0^2 - \omega^2) + \gamma^2\omega^2]\sin\omega t}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} + e^{-\gamma t/2}\left[y_1 \cos\sqrt{\omega_0^2 - \gamma^2/4}\,t + y_2 \sin\sqrt{\omega_0^2 - \gamma^2/4}\,t\right]. \tag{18}$$

The initial conditions are,

$$y(0) = 0 = -\frac{\omega^3\gamma A}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} + y_1\,, \qquad i.e., \qquad y_1 = \frac{\omega^3\gamma A}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\,, \tag{19}$$

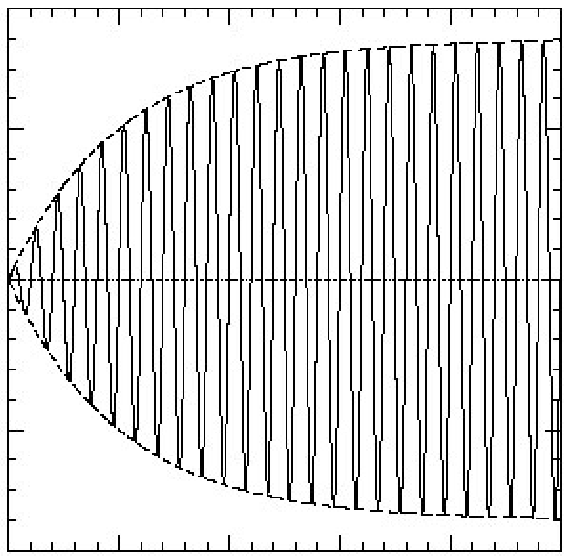
and,

$$\dot{y}(0) = 0 \;=\; \frac{[\omega_0^2\omega(\omega_0^2-\omega^2)+\gamma^2\omega^3]A}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} - \frac{\gamma}{2}y_1 + \sqrt{\omega_0^2-\gamma^2/4}\,y_2$$
$$= \frac{[\omega_0^2\omega(\omega_0^2-\omega^2)+\gamma^2\omega^3/2]A}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2} + \sqrt{\omega_0^2-\gamma^2/4}\,y_2\,. \tag{20}$$

Finally, the vertical motion of the car (for fixed ω) is given by,

$$y(t) = \frac{A}{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}\left\{\omega^3\gamma\left[e^{-\gamma t/2}\cos\sqrt{\omega_0^2-\gamma^2/4}\,t - \cos\omega t\right]\right. \tag{21}$$
$$\left. - \left[\frac{\omega_0^2\omega(\omega_0^2-\omega^2)+\gamma^2\omega^3/2}{\sqrt{\omega_0^2-\gamma^2/4}}e^{-\gamma t/2}\sin\sqrt{\omega_0^2-\gamma^2/4}\,t - [\omega_0^2(\omega_0^2-\omega^2)+\gamma^2\omega^2]\sin\omega t\right]\right\},$$

as sketched below for $\omega = \omega_0 = 16\gamma$.



We see that the motion has very nearly reached its full amplitude after $Q = \omega_0/\gamma = 16$ cycles. This confirms the argument in sec. 2.1 that the acceleration of the car must be sufficient that the angular frequency ω of the driving force due to the washboard road must be within $\pm\gamma$ of ω_0 only for a time small compared to $1/\gamma$, *i.e.*, for a time much less than Q cycles.

A Appendix: No Damping as an Illustration of the Higgs Mechanism

If the damping constant γ is zero, and a steady state could still somehow be achieved, then the vertical oscillations have amplitude,

$$y(t) = A\frac{\omega_0^2\sin\omega t}{\omega_0^2-\omega^2}\,, \tag{22}$$

according to eq. (14), where now $\omega = 2\pi v/\lambda$. Nonzero energy,

$$U_{\rm osc} = k \left\langle y(t) \right\rangle^2 = \frac{kA^2}{2} \frac{\omega_0^4}{(\omega_0^2 - \omega^2)^2} < \frac{m v_0^2}{2}\,, \tag{23}$$

is associated with these oscillations. If the car simply coasted onto the washboard road with initial velocity v_0 (and no oscillations), conservation of energy implies that it would slow down to velocity v given by.

$$\frac{m v_0^2}{2} = \frac{m v^2}{2} + U_{\rm osc}. \tag{24}$$

If we take a view that ignores the tranverse oscillations and emphasizes only the longitudinal velocity of the car, we might say that when on the washboard road it has effective mass $\overline{m}$ related by.

$$\frac{m v_0^2}{2} = \frac{\overline{m} v^2}{2}\,, \tag{25}$$

and hence,

$$\overline{m} = m + \frac{2U_{\rm osc}}{v^2} = m + \frac{2U_{\rm osc}}{m v_0^2 - 2U_{\rm osc}}\,. \tag{26}$$

In language more common to quantum theory, we might say that the car has become a "quasicar" with effective mass $\overline{m}$. The "quasicar" has been "given" (additional) mass by its interaction with the washboard road, which is a kind of a "background field."

In the quantum realm, we say that the Higgs background field "gives mass" to otherwise massless elementary (quasi)particles by a mechanism somewhat analogous to the case of a car on a washboard road. See, for example, [1].

References

[1] K.T. McDonald, *A Simplified View of the Higgs/Yukawa Mechanism* (July 17, 2013), http://kirkmcd.princeton.edu/examples/higgs.pdf