Physics in the Laundromat

Kirk T. McDonald
Joseph Henry Laboratories, Princeton University, Princeton, NJ 08544
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Abstract

The spin cycle of a washing machine involves motion that is stabilized by the Coriolis force, similar to the case of the motion of shafts of large turbines. This system is an example of a stable inverted pendulum.

1 Introduction

Many of us have had the opportunity to observe clothes tumbling in a dryer at a laundromat, and perhaps have reflected how the angular velocity of the drum must be less than $\sqrt{g/a}$, where a is the radius of the drum and g is the accelerator due to gravity, if the clothes are to fall free of the drum and make improved contact with the hot air. Another common observation is the vibration of a washing machine during the start of its spin cycle. Indeed, if the load is poorly distributed the vibration becomes so violent that the washer cannot spin up until the load is redistributed. If one defeats the interlock on the door of a top-loading washer during a typical spin cycle the center of the rotor will be observed to move in a small circle, possibly off center, whose radius is a measure of the imbalance of the load.

The motion of the axis of the drum of the washer is an example of the motion of unbalanced shafts of large rotating machines, as has been well described by Landau and Kitaigorodsky in a popular book [1]. Here we analyze a simple model of a washer that contains the essential physics. The related topic of whirling of a vertical wire has been treated by Pippard [2].

2 A Model Washing Machine

The drum and the circularly symmetric part of the load of a washing machine have mass M and are constrained by a motor to rotate with angular velocity Ω (about the vertical; gravity can then be ignored). The load is not circularly symmetric in general, and we characterize the departure from symmetry by a mass m located at fixed radius a from the axis of the drum, and at fixed azimuth relative to the drum.

The axis of the drum is not, however, fixed in the frame of the laundromat. Rather, a set of springs connect the axis to the frame so as to approximate a zero-length spring of constant k. In motion, the axis of the drum can be displaced from rest by amount (r, θ) in a cylindrical coordinate system fixed in the laundromat. The azimuth of the line from the center of the drum to mass m is labeled ϕ, as shown in Fig. 1. The time derivative of ϕ is constrained by the motor to be constant: $\dot{\phi} = \Omega$.

It is interesting to consider the motion of the drum in the presence of damping. As a simple model we suppose the spatial motion of shaft of the drum is subject to a frictional

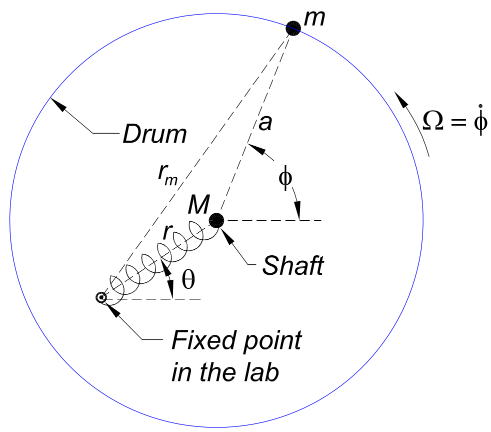

Figure 1: Model of the spin cycle of a washing machine. The drum and symmetrical part of the load have mass M. The center of the drum is at (r, θ) and is connected to the origin by a zero-length spring of constant k. An unbalanced load of mass m lies at distance a from the center of the drum and at angle ϕ with respect to a fixed direction in the laundromat. The washer motor turns the drum with constant angular velocity $\dot{\phi} = \Omega$.

force, $-\gamma\,\dot{\mathbf{r}}$, proportional to its velocity. The frictional torque that opposes the forced rotation Ω does not, however, affect the motion.

The equations of motion of this system of two degrees of freedom, r and θ, are readily deduced from a Newtonian approach,

$$m_{\rm cm}\ddot{\mathbf{r}}_{\rm cm} = M\,\ddot{\mathbf{r}} + m\,\ddot{\mathbf{r}}_m = -k\,\mathbf{r} - \gamma\,\dot{\mathbf{r}}, \tag{1}$$

where the position $\mathbf{r}_m$ of mass m is,

$$\begin{aligned}
\mathbf{r}_m &= \mathbf{r} + \mathbf{a} = r\,\hat{\mathbf{r}} + a\cos(\phi-\theta)\,\hat{\mathbf{r}} + a\sin(\phi-\theta)\,\hat{\boldsymbol{\theta}}, && (2)\\
\dot{\mathbf{r}}_m &= \dot{\mathbf{r}} + \dot{\mathbf{a}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + a\Omega\,\hat{\boldsymbol{\phi}} \\
&= \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + a\Omega[-\sin(\phi-\theta)\,\hat{\mathbf{r}} + \cos(\phi-\theta\,\hat{\boldsymbol{\theta}}] && (3)\\
\ddot{\mathbf{r}}_m &= \ddot{\mathbf{r}} + \ddot{\mathbf{a}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\boldsymbol{\theta}} \\
&\quad -a\Omega(\Omega-\dot{\theta})[\cos(\phi-\theta)\,\hat{\mathbf{r}} + \sin(\phi-\theta)\,\hat{\boldsymbol{\theta}}] - a\Omega\dot{\theta}[\sin(\phi-\theta)\,\hat{\boldsymbol{\theta}} + \cos(\phi-\theta)\,\hat{\mathbf{r}}] \\
&= [\ddot{r} - r\dot{\theta}^2 - a\Omega^2\cos(\phi-\theta)]\,\hat{\mathbf{r}} + [r\ddot{\theta} + 2\dot{r}\dot{\theta} - a\Omega^2\sin(\phi-\theta)]\,\hat{\boldsymbol{\theta}}, && (4)
\end{aligned}$$

where $\hat{\boldsymbol{\theta}}$ is perpendicular to $\hat{\mathbf{r}}$, $\dot{\hat{\mathbf{r}}} = \dot{\theta}\,\hat{\boldsymbol{\theta}}$ and $\dot{\hat{\boldsymbol{\theta}}} = -\dot{\theta}\,\hat{\mathbf{r}}$.

Then, the equation of motion associated with coordinate r is,

$$\ddot{r} = r\,\dot{\theta}^2 + b\,\Omega^2\cos(\phi-\theta) - \omega_0^2\,r - \Gamma\,\dot{r}, \tag{5}$$

and that with coordinate θ is,

$$0 = r\,\ddot{\theta} + 2\dot{r}\,\dot{\theta} - b\,\Omega^2\sin(\phi-\theta) + \Gamma\,r\,\dot{\theta}, \tag{6}$$

where we have introduced the notation,

$$\omega_0 = \sqrt{\frac{k}{m+M}}. \tag{7}$$

for the natural frequency of vibration of the washing machine,

$$b = \frac{m}{m+M} a, \tag{8}$$

for the distance of the center of mass from the shaft and,

$$\Gamma = \frac{\gamma}{m+M}. \tag{9}$$

These equations can be interpreted in a frame rotating with angular velocity $\dot{\theta}$. Equation (5) tells us that the total mass times the radial acceleration of mass M equals the spring force plus the radial component of the centrifugal force and friction. Equation (6) indicates that the azimuthal coordinate forces plus friction sum to zero; the term $2\dot{r}\dot{\theta}$ is the Coriolis acceleration.

The equations of motion with the neglect of friction are also readily deduced from the Lagrangian,

$$L = \frac{1}{2}(m+M)(\dot{r}^2+r^2\dot{\theta}^2) - ma\dot{r}\,\Omega\sin(\phi-\theta) + mar\,\dot{\theta}\Omega\cos(\phi-\theta) + \frac{1}{2}(I+ma^2)\Omega^2 - \frac{1}{2}kr^2, \tag{10}$$

where I is the moment of inertia of the drum plus symmetric part of the load. The rotational kinetic energy is constant by assumption, so the moment of inertia does not appear further in the analysis.

3 Steady Motion

We first discuss steady motion in which $\dot{r} = 0$, $\ddot{r} = 0$ and $\ddot{\theta} = 0$. The shaft of the drum moves in a circle of radius r_0 and the mass m is at constant azimuth $\phi_0 = \phi - \theta$ relative to the azimuth of the shaft. Then, eq. (5) tells us that,

$$r_0 = \frac{b\,\Omega^2\cos\phi_0}{\omega_0^2 - \Omega^2}, \tag{11}$$

while eq. (6) indicates,

$$r_0 = \frac{b\,\Omega\sin\phi_0}{\Gamma}. \tag{12}$$

Together,

$$\cos\phi_0 = \frac{\omega_0^2 - \Omega^2}{\sqrt{(\omega_0^2-\Omega^2)^2 + \Gamma^2\Omega^2}}, \qquad \sin\phi_0 = \frac{\Gamma\,\Omega}{\sqrt{(\omega_0^2-\Omega^2)^2 + \Gamma^2\Omega^2}}, \tag{13}$$

and,

$$r_0 = \frac{b\,\Omega^2}{\sqrt{(\omega_0^2-\Omega^2)^2 + \Gamma^2\Omega^2}}. \tag{14}$$

For a balanced load ($m = 0$) distance b is zero, so the equilibrium displacement is zero also.

For low spin ($\Omega \ll \omega_0$) an unbalanced load finds itself at relative azimuth $\phi_0 \approx 0$, while near resonance ($\Omega \approx \omega_0$) the azimuth is $\approx \pi/2$, and for high spin ($\Omega \gg \omega$) the azimuth approaches π. In the latter case the system is a kind of inverted pendulum.

The center of mass of the system is at distance,

$$r_{\rm cm} = \frac{b\,\omega_0^2}{\sqrt{(\omega_0^2 - \Omega^2)^2 + \Gamma^2\Omega^2}}. \tag{15}$$

Thus the center of mass approaches the origin as the spin Ω becomes large, even though the shaft is at radius $r_0 \approx b$. The system can be called self-centering as the spin Ω increases, once it successfully passes through the resonance region.

4 Stability

Is the desirable self-centering motion found above stable against small perturbations? If angle $\theta(t)$ were locked at $\phi - \phi_0 = \Omega\, t - \phi_0$, *i.e.*, if only radial oscillations of the axis of the drum were permitted, and $\Omega > \omega_0$ answer would be no!

To see this we refer to eq. (5), which for the locked hypothesis reads,

$$\ddot{r} = (\Omega^2 - \omega_0^2)r + b\,\Omega^2\cos\phi_0 - \Gamma\,\dot{r}. \tag{16}$$

For oscillatory radial motion the coefficient of the term in r must be negative. Hence, the locked motion would be stable only for low spin, $\Omega < \omega_0$.

However, we will find that the motion is stable when both radial and azimuthal oscillations are considered. The linked system of masses m and M forms a kind of double pendulum. The motion in which $\phi \approx \theta + \pi$ that arises when the drive frequency Ω exceeds the resonant frequency ω_0 is an example of a stable inverted pendulum.

To demonstrate this we perform a perturbation analysis, seeking solutions of the form,

$$r = r_0(1 + \epsilon), \qquad \theta = \phi - \phi_0 + \delta, \tag{17}$$

where the perturbations are desired to be small and oscillatory with angular frequency ω,

$$\epsilon = \epsilon_0\, e^{i\omega t}, \qquad \text{and} \qquad \delta = \delta_0\, e^{i\omega t} \qquad \text{with} \qquad \epsilon_0,\ \delta_0 \ll 1. \tag{18}$$

The constants ϵ_0, δ_0 and ω are complex in general, and, of course, the physical motion is described by the real parts of eq. (17). Both the real and imaginary parts of ω should be positive; the real part is the frequency of oscillation and the imaginary part is the damping decay constant.

In the first approximation we now have,

$$\cos(\phi - \theta) = \cos\phi_0 + \delta\sin\phi_0, \qquad \text{and} \qquad \sin(\phi - \theta) = \sin\phi_0 - \delta\cos\phi_0. \tag{19}$$

Then, using (17-19) in (5) and keeping terms only of first order of smallness, we find,

$$-\,\omega^2\epsilon_0 = \Omega^2 + 2i\Omega\delta_0 + \frac{b\,\Omega^2\sin\phi_0}{r_0}\delta_0 - \omega_0^2\epsilon_0 - i\omega\Gamma\epsilon_0. \tag{20}$$

With eq. (12) this tells us that,

$$\epsilon_0 = -\frac{\Gamma\omega + 2i\omega\Omega}{\omega^2 - \omega_0^2 + \Omega^2 - i\omega\Gamma}\delta_0. \tag{21}$$

Similarly, eq. (6) leads to,

$$0 = -\omega^2\delta_0 + 2i\Omega\omega\epsilon_0 + \frac{b\,\Omega^2\cos\phi_0}{r_0}\delta_0 + \Gamma\Omega_0\epsilon_0 + i\omega\Gamma\delta_0, \tag{22}$$

which together with eq. (11) tells us that,

$$\delta_0 = \frac{\Gamma\Omega + 2i\Omega\omega}{\omega^2 - \omega_0^2 + \Omega^2 - i\omega\Gamma}\epsilon_0. \tag{23}$$

Equations (21) and (23) are consistent only if,

$$\frac{\Gamma\Omega + 2i\Omega\omega}{\omega^2 - \omega_0^2 + \Omega^2 - i\omega\Gamma} = \pm i, \tag{24}$$

which leads to the quadratic equation,

$$\omega^2 - 2\omega(\pm\Omega - i\Gamma/2) - \omega_0^2 + \Omega^2 \pm i\Gamma\Omega = 0. \tag{25}$$

The roots of this with positive real parts are,

$$\omega = \begin{cases} \sqrt{\omega_0^2 - (\Gamma/2)^2} \pm \Omega + i\Gamma/2, & \Omega < \sqrt{\omega_0^2 - (\Gamma/2)^2}, \\ \Omega \pm \sqrt{\omega_0^2 - (\Gamma/2)^2} + i\Gamma/2, & \Omega > \sqrt{\omega_0^2 - (\Gamma/2)^2}. \end{cases} \tag{26}$$

In the above we have assumed that the damping is weak enough that $\omega_0 > \Gamma/2$. Then, perturbations die out with characteristic time $2/\Gamma$ which is greater than the natural period of oscillation, $1/\omega_0$.

Thus stable motion exists for all values of the spin Ω. Referring to eq. (22), we note that the key coupling between the radial and azimuthal perturbations (ϵ and δ) is provided by the Coriolis force.

As the spin frequency Ω approaches the resonant frequency ω_0 the lower frequency of the perturbed motion goes to zero. If the amplitude of the perturbation is large it will be noticeable throughout the laundromat.

For high spin, eqs. (23) and (26) yield the relation,

$$\delta_0 = \frac{i\epsilon_0}{1 - \frac{i\Gamma}{2\Omega} - \frac{\Gamma^2}{8\Omega\left(\Omega \pm \sqrt{\omega_0^2 - (\Gamma/2)^2}\right)}} \approx i\epsilon_0, \tag{27}$$

which indicates that the radial and azimuthal perturbations are 90° out of phase. The angular velocity of the motion of the center of the drum is,

$$\dot\theta = \Omega + \epsilon_0 \sin\left(\Omega \pm \sqrt{\omega_0^2 - (\Gamma/2)^2}\right), \tag{28}$$

which is Ω on average. This implies that the perturbed motion of the shaft of the drum is still a circle of radius r_0 to first approximation. However, since the frequency ω of the perturbation differs from the average rotation frequency Ω, the orbit of the center of the drum is not closed but precesses at angular frequency ω_0, as sketched in Fig. 2. In the limit that $\Omega \gg \omega_0$ the motion of the center of the drum is essentially a circle of radius r_0 displaced by distance $\epsilon_0 r_0$ from the origin, as shown in Fig. 3. In practice this displacement can be quite noticeable, as the reader can confirm on his or her next trip to the laundromat.

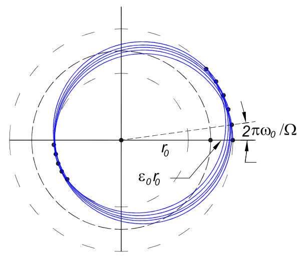

Figure 2: The steady motion of the axis of the drum is at angular velocity Ω in a circle of radius r_0 about the origin. The perturbed orbit is nearly circular, but precesses with angular velocity ω_0 and lies in the annulus $r_0(1-\epsilon_0) < r < r_0(1+\epsilon_0)$.

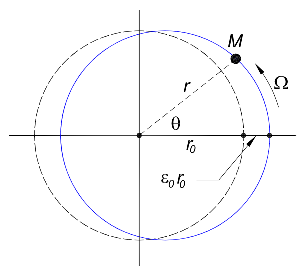

Figure 3: If the drive frequency Ω is large compared to the resonant frequency ω_0 the perturbed motion is a circle of radius r_0 whose center is displaced by ϵr_0.

The author wishes to thank A.B. Pippard for a gracious reading of the manuscript, and for the suggestion to add damping to the analysis. I would also like to salute the (unknown to me) inventor of the washing machine, who likely solved this problem without the benefit of a formal analysis.

References

[1] L. Landau and A. Kitaigorodsky, *Physics for Everyone*, Book 1 (Mir Publisher, Moscow, 1978), pp. 176-180, http://kirkmcd.princeton.edu/examples/mechanics/landau_motion_74.pdf

[2] A.B. Pippard, *The Physics of Vibration*, Vol. 1 (Cambridge U. Press, 1978), pp. 212-216, http://kirkmcd.princeton.edu/examples/mechanics/pippard_v1_78.pdf