Weber's Electrodynamics and the Hall Effect

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(June 5, 2020)

1 Problem

In 1846, Weber used a model of electric current as moving electric charge (of both signs) to give an instantaneous, action-at-a-distance, "microscopic" force law, from which Ampère's force law between current-carrying circuits could be deduced.[1] In Gaussian units, the force on electric charge e due to moving charge e' is, according to Weber,

$$\mathbf{F}_{e,\mathrm{Weber}} = -\frac{ee'}{r^2}\left[1 - \frac{2}{c^2}\left(\frac{\partial r}{\partial t}\right)^2 + \frac{r}{c^2}\frac{\partial^2 r}{\partial t^2}\right]\hat{\mathbf{r}}, \tag{1}$$

where $\mathbf{r} = \mathbf{r}_{e'} - \mathbf{r}_e$, and c is the speed of light in vacuum. Following Ampère, Weber supposed that the force between two moving charges (current elements) is along their line of centers, and obeys Newton's third law (of action and reaction).

This contrasts with the Lorentz force law (given below in Gaussian units), which does not obey Newton's third law in general,[2]

$$\mathbf{F}_{e,\mathrm{Lorentz}} = e\left(\mathbf{E}_{e'} + \frac{\mathbf{v}_e}{c}\times\mathbf{B}_{e'}\right), \tag{2}$$

where the fields $\mathbf{E}_{e'}$ and $\mathbf{B}_{e'}$ are given by the forms of Liénard and Wiechert.[3]

Until the late 1870's the only experiments possible with electric currents were those in which the currents flowed in complete circuits (as studied by Ampère). The first step towards a larger context of electric currents was made by Rowland [52, 65] (1876, working in Helmholtz' lab in Berlin), who showed that a rotating, charged disk produces magnetic effects. More importantly, Rowland's student Hall (1878) [55] showed that an electric conductor in a magnetic field $\mathbf{B}$ perpendicular to the current $\mathbf{I}$ develops an electric potential

[1] References to original works are given in the historical Appendix below.

[2] Momentum conservation is satisfied in Maxwellian electrodynamics in that the electromagnetic field carries momentum, with density $\mathbf{E}\times\mathbf{B}/4\pi c$ (as first noted by J.J. Thomson [85]).

[3] See, for example, sec. 63 of [70].

If the charges had uniform velocity, then (sec. 38 of [70]),

$$\mathbf{E}_{e'} = \frac{\gamma_{e'} e'\,\hat{\mathbf{r}}}{r^2\left(1 + \gamma_{e'}^2(\mathbf{v}_{e'}\cdot\hat{\mathbf{r}})^2/c^2)\right)^{3/2}}, \qquad \mathbf{B}_{e'} = \frac{\mathbf{v}_{e'}}{c}\times\mathbf{E}_{e'}, \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{3}$$

Even for $|\mathbf{v}_e| = |\mathbf{v}_{e'}| = v$, we have,

$$\begin{aligned}\mathbf{F}_{e,\mathrm{Lorentz}} &= \frac{\gamma ee'}{r^2}\left(\frac{\hat{\mathbf{r}}}{(1+\gamma(\mathbf{v}_{e'}\cdot\hat{\mathbf{r}})^2/c^2)^{3/2}} + \frac{\mathbf{v}_e}{c}\times\left(\frac{\mathbf{v}_{e'}}{c}\times\frac{\hat{\mathbf{r}}}{(1+\gamma(\mathbf{v}_{e'}\cdot\hat{\mathbf{r}})^2/c^2)^{3/2}}\right)\right) \\ &= \frac{\gamma ee'}{r^2\left(1+\gamma(\mathbf{v}_{e'}\cdot\hat{\mathbf{r}})^2/c^2)\right)^{3/2}}\left((1 - \mathbf{v}_e\cdot\mathbf{v}_{e'}/c^2)\,\hat{\mathbf{r}} + \frac{(\mathbf{v}_e\cdot\hat{\mathbf{r}})\,\mathbf{v}_{e'}}{c^2}\right),\end{aligned} \tag{4}$$

which is not along $\hat{\mathbf{r}}$, and does not equal $-\mathbf{F}_{e',\mathrm{Lorentz}}$, unless $\mathbf{v}_e$, $\mathbf{v}_{e'}$ and $\hat{\mathbf{r}}$ are all parallel.

difference in the direction $\mathbf{I} \times \mathbf{B}$. This effect follows directly from the Lorentz force law (2), which predicts a transverse force on the conduction charges, leading to a transverse charge separation, and a transverse voltage drop.

Does Weber's force law (1) predict the Hall effect?

2 Solution

Weber had no concept of the magnetic field, but he did associate magnetic effects with electric currents, which he took to be electric charges in motion. So, we consider a magnetic field in the z-direction at the origin as due to n electric charges e' moving counterclockwise in a circle of radius a in the x-y plane, centered on the origin, with angular velocity ω. Then, according to the Biot-Savart law, $\mathbf{B} = \oint I\, d\mathbf{l} \times \hat{\mathbf{r}}/c\, r^2$, the magnetic field at the origin is $\mathbf{B}_{e'} = n\, e'\omega\, \hat{\mathbf{z}}/c\, a$. The test current is associated with charges e moving along the x-axis with velocity $\mathbf{v} = v\, \hat{\mathbf{x}}$. We desire the force on the latter charges when they are at the origin.

2.1 Force on a Charge Moving at the Center of a Current Loop

According to the Lorentz force law (2), the force on moving charge e is,

$$\mathbf{F}_{e,\mathrm{Lorentz}} = e\frac{\mathbf{v}_e}{c} \times \mathbf{B}_{e'} = -\frac{n\, ee' v_e\, \omega}{c^2 a}\, \hat{\mathbf{y}}. \tag{5}$$

We now consider the Weber force (1) on test charge e at the origin at time $t = 0$ due to charge e' at $(x, y, z) = (a\cos\phi, a\sin\phi, 0)$.

The position of the test charge e varies with time as $\mathbf{x}_{\mathrm{test}} = (v_e t, 0, 0)$, while the charge e' has time-dependent position $(a\cos(\phi + \omega t), a\sin(\phi + \omega t), 0)$.

The distance r between these two charges is,

$$r = \sqrt{(a\cos(\phi + \omega t) - v_e t)^2 + (a\sin(\phi + \omega t))^2} = \sqrt{a^2 + v_e^2 t^2 - 2a\, v_e t\cos(\phi + \omega t)}, \tag{6}$$

$$\frac{\partial r}{\partial t} = \frac{2v_e^2 t - 2a\, v_e\cos(\phi + \omega t) + 2a\, v_e\, \omega t\sin(\phi + \omega t)}{2\sqrt{a^2 + v_e^2 t^2 - 2a\, v_e t\cos(\phi + \omega t)}}, \tag{7}$$

$$\frac{\partial^2 r}{\partial t^2} = \frac{2v_e^2 + 4a\, v_e\, \omega\sin(\phi + \omega t) + 2a\, v_e\, \omega^2 t\cos(\phi + \omega t)}{2\sqrt{a^2 + v_e^2 t^2 - 2a\, v_e t\cos(\phi + \omega t)}} - \frac{(2v_e^2 t - 2a\, v_e\cos(\phi + \omega t) + 2a\, v_e\, \omega t\sin(\phi + \omega t))^2}{4(a^2 + v_e^2 t^2 - 2a\, v_e t\cos(\phi + \omega t))^{3/2}}. \tag{8}$$

Our calculation is at time $t = 0$, when,

$$r(0) = a, \qquad \frac{\partial r(0)}{\partial t} = -v_e\cos\phi, \qquad \frac{\partial^2 r(0)}{\partial t^2} = \frac{v_e^2 + 2a\, v_e\, \omega\sin\phi}{a} - \frac{v_e^2\cos^2\phi}{a}. \tag{9}$$

The Weber force on charge e at the origin due to charge e' at azimuth ϕ at time $t = 0$ is,

$$\mathbf{F}_{e,\mathrm{Weber}} = -\frac{ee'}{a^2}\left[1 - \frac{2v_e^2\cos^2\phi}{c^2} + \frac{a}{c^2}\frac{v_e^2\sin^2\phi + 2a\, v_e\, \omega\sin\phi}{a}\right](\cos\phi, \sin\phi, 0). \tag{10}$$

The number density of charge around the loop of radius a is uniform, $dn/d\phi = n/2\pi$, so integrating eq. (10) over ϕ implies that the total Weber force has only a y-component,

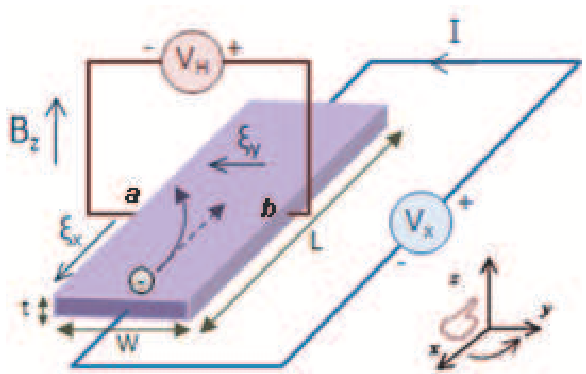
$$F_{e,\text{Weber},y} = -\frac{ee'}{a^2}\int_0^{2\pi} \frac{2a\,v_e\,\omega\sin^2\phi\,n\,d\phi}{c^2\qquad 2\pi} = -\frac{ee'}{a^2}\frac{2a\,v_e\,\omega}{c^2}\frac{n}{2\pi}\pi = -\frac{n\,ee'v_e\,\omega}{c^2\,a}\,, \tag{11}$$

in agreement with the Lorentz force (5).

Hence, the Hall effect does not distinguish the Lorentz force law from that of Weber.

2.2 The Hall Voltage

The preceding analysis tacitly assumed that the charge e was "free", but in practice it is part of the electric current I inside a conductor connected to a battery, as sketched below.



Then, in the steady state, the total (Lorentz) force on charge e is zero,

$$0 = e\left(\mathbf{E} + \frac{v_e}{c}\times\mathbf{B}\right). \tag{12}$$

Neglecting the small magnetic field due to the current I, we infer that the electric field inside the conductor includes a y-component,[4] such that a nonzero (Hall) voltage difference $\Delta V_\text{H} = V_b - V_a$ is observed across the conductor,

$$E_y = -\frac{v_e}{c}B_{e'} = \frac{V_a - V_b}{W}\,, \qquad \Delta V_\text{H} = V_b - V_a = \frac{v_e B_{e'} W}{c}\,. \tag{13}$$

For metallic conductors, the Hall voltage is positive. If the conduction charges e were positive, then positive charges would accumulate along edge b. such that edge b would become negatively charged, and the Hall voltage ΔV_H would be negative. Thus, the observed positive Hall voltage implies that the conduction charges in metals are negative.

Weber generally assumed that the conduction charges were half positive and half negative, moving with equal and opposite velocities. In this case, the Hall voltage would be zero, in contrast to the observed positive value. That is, the Hall effect rules out the symmetric form of electric currents favored by Weber. However, if the conduction charges in metals are taken to be negative in Weber's electrodynamics, then it does predict the Hall effect.

[4]In addition, there is an x-component, $E_x = J/\sigma = I/\sigma tW$, where σ is the electrical conductivity and tW is the cross sectional area of the conductor of length L, thickness t and width W.

2.3 The Hall Constant $R_{\rm H}$

Hall reported the quantity,

$$R_{\rm H} = \frac{E_y}{J_x B_{e'}} = -\frac{1}{c\, n\, e}\, , \tag{14}$$

recalling eq. (13) and noting that the x-component of the current density $\mathbf{J}$ can be written as $J_x = n\, e\, v_e$, where n is the number density of the conduction charges e. For copper, Hall measured that $R_{\rm H} = 6 \times 10^{-25}$ Gaussian units.

We can also write $J_x = \sigma E_0$, where E_0 is the electric field strength inside the conductor due to the battery. Then, we have that,

$$E_y = J_x B_{e'} R_{\rm H} = \sigma E_0 B_{e'} R_{\rm H} = -\frac{v_e}{c} B_{e'}, \qquad v_e = -c\, \sigma E_0 R_{\rm H}. \tag{15}$$

For example, with E_0 = 1 V/m = 0.01 V/cm = $10^6/c$ Gaussian units, v_e = -0.5 cm/s. Thus, Hall's experiment also gave the first indication of the very small (drift) velocity of the conduction charges in metals for circuits driven by batteries of order 1 volt.

A Appendix: Historical Comments

These comments are extracted from the longer survey in Appendix A of [89].

A.1 Ørsted

In 1820, Ørsted [2]-[5],[16] published decisive evidence that electric currents exert forces on permanent magnets and *vice versa*, indicating that electricity and magnetism are related.[5] Ørsted's term "electric conflict", used in his remarks on p. 276 of [3], is a precursor of the later concept of the magnetic field:
It is sufficiently evident from the preceding facts that the electric conflict is not confined to the conductor, but dispersed pretty widely in the circumjacent space. From the preceding facts we may likewise infer that this conflict performs circles.

A.2 Biot and Savart

Among the many rapid responses in 1820 to Ørsted's discovery was an experiment by Biot and Savart [6, 9] on the force due to an electric current I in a wire on one pole, p, of a long, thin magnet. The interpretation given of the result was somewhat incorrect, which

[5] Reports have existed since at least the 1600's that lightning can affect ship's compasses (see, for example, p. 179 of [49]), and an account of magnetization of iron knives by lightning was published in 1735 [1]. In 1797, von Humboldt conjectured that certain patterns of terrestrial magnetism were due to lighting strikes (see p. 13 of [83], a historical review of magnetism). A somewhat indecisive experiment involving a voltaic pile and a compass was performed by Romagnosi in 1802 [80].
Historical commentaries on Ørsted's work include [59, 60, 88].

was remedied by Biot in 1821 and 1824 [13, 25] with a form that can be written in vector notation (and in SI units) as,

$$\mathbf{F} = \frac{\mu_0 \, p}{4\pi} \oint \frac{I \, d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \, , \tag{16}$$

where $\mathbf{r}$ is the distance from a current element $I \, d\mathbf{l}$ to the magnetic pole. There was no immediate interpretation of eq. (16) in terms of a magnetic field,[6] $\mathbf{B} = \mathbf{F}/p$,

$$\mathbf{B} = \frac{\mu_0}{4\pi} \oint \frac{I \, d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \, , \tag{17}$$

which expression is now commonly called the Biot-Savart law.

Biot and Savart did not discuss the force on an electric current, but the expression,

$$\mathbf{F} = \oint I \, d\mathbf{l} \times \mathbf{B}, \tag{18}$$

is now also often called the Biot-Savart law.

A.3 Ampère

Between 1820 and 1825 Ampère made extensive studies [7, 8, 10, 11, 12, 14, 15, 17, 18, 19, 20, 21, 23, 24, 26, 27, 28] of the magnetic interactions of electrical currents.[7] Already in 1820 Ampère came to the vision that all magnetic effects are due to electrical currents.[8,9]

In 1822-1823 (pp. 21-24 of [28]), Ampère examined the force between two circuits, carrying currents I_1 and I_2, and inferred that this could be written (here in vector notation) as,

$$\mathbf{F}_{\text{on 1}} = \oint_1 \oint_2 d^2\mathbf{F}_{\text{on 1}}, \quad d^2\mathbf{F}_{\text{on 1}} = \frac{\mu_0}{4\pi} I_1 I_2 [3(\hat{\mathbf{r}} \cdot d\mathbf{l}_1)(\hat{\mathbf{r}} \cdot d\mathbf{l}_2) - 2 \, d\mathbf{l}_1 \cdot d\mathbf{l}_2] \frac{\hat{\mathbf{r}}}{r^2} = -d^2\mathbf{F}_{\text{on 2}}, \tag{19}$$

where $\mathbf{r} = \mathbf{l}_1 - \mathbf{l}_2$ is the distance from a current element $I_2 \, d\mathbf{l}_2$ at $\mathbf{r}_2 = \mathbf{l}_2$ to element $I_1 \, d\mathbf{l}_1$ at $\mathbf{r}_1 = \mathbf{l}_1$..[10,11] The integrand $d^2\mathbf{F}_{\text{on 1}}$ of eq. (19) has the appeal that it changes sign if

[6] Although the concept of the magnetic field is latent in discussions of magnetic force by Michell, Coulomb, Poisson and Ørsted (and many other in the years 1820-45), the first use of the term "magnetic field" seems to be due to Faraday, Art. 2147 of [35].

[7] Discussion in English of Ampère's attitudes on the relation between magnetism and mechanics is given in [64, 79, 87]. Historical surveys of 19th-century electrodynamics are given in [58, 81], and studies with emphasis on Ampère include [61, 63, 67, 68, 69, 72, 73, 74, 88]. See also sec. IIA of [82].

[8] See, for example, [69].

[9] The confirmation that permanent magnetism, due to the magnetic moments of electrons, is Ampèrian (rather than Gilbertian = due to pairs of opposite magnetic charges) came only after detailed studies of positronium (e^+e^- "atoms") in the 1940's [66, 86].

[10] Ampère sometimes used the notation that the angles between $d\mathbf{l}_i$ and $\mathbf{r}$ are θ_i, and the angle between the plane of $d\mathbf{l}_1$ and $\mathbf{r}$ and that of $d\mathbf{l}_2$ and $\mathbf{r}$ is ω. Then, $d\mathbf{l}_1 \cdot d\mathbf{l}_2 = dl_1 \, dl_2 (\sin\theta_1 \sin\theta_2 \cos\omega + \cos\theta_1 \cos\theta_2)$, and the force element of eq. (19) can be written as,

$$d^2\mathbf{F}_{\text{on 1}} = \frac{\mu_0}{4\pi} I_1 I_2 \, dl_1 \, dl_2 (\cos\theta_1 \cos\theta_2 - 2 \sin\theta_1 \sin\theta_2 \cos\omega) \frac{\hat{\mathbf{r}}}{r^2} = -d^2\mathbf{F}_{\text{on 2}}. \tag{20}$$

[11] Ampère also noted the equivalents to,

$$d\mathbf{l}_1 = \frac{\partial \mathbf{r}}{\partial l_1} \, dl_1, \qquad \mathbf{r} \cdot d\mathbf{l}_1 = \mathbf{r} \cdot \frac{\partial \mathbf{r}}{\partial l_1} \, dl_1 = \frac{1}{2} \frac{\partial r^2}{\partial l_1} \, dl_1 = r \frac{\partial r}{\partial l_1} \, dl_1, \qquad d\mathbf{l}_2 = -\frac{\partial \mathbf{r}}{\partial l_2} \, dl_2, \qquad \mathbf{r} \cdot d\mathbf{l}_2 = -r \frac{\partial r}{\partial l_2} \, dl_2, \tag{21}$$

elements 1 and 2 are interchanged, and so suggests a force law for current elements that obeys Newton's third law. However, the integrand does not factorize into a product of terms in the two current elements, in contrast to Newton's gravitational force, and Coulomb's law for the static force between electric charges (and between static magnetic poles, whose existence Ampère doubted). As such, Ampère (correctly) hesitated to interpret the integrand as providing the force law between a pair of isolated current elements, *i.e.*, a pair of moving electric charges.[12]

An important qualitative consequence of eq. (19) is that parallel currents attract, and opposite current repel.

Around 1825, Ampère noted, p. 214 of [27], p. 29 of [28], p. 366 of the English translation in [87], that for a closed circuit, eq. (19) can be rewritten as,[13]

$$\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})\, d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2)\, \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} \oint_1 \oint_2 I_1 d\mathbf{l}_1 \times \frac{I_2\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}, \qquad (25)$$

in vector notation (which, of course, Ampère did not use). Ampère made very little comment on this result,[14] and certainly did not factorize it into the forms now related to the Biot-Savart law(s) (17)-(18).

Ampère performed an experiment in 1821-22 [18, 22, 30] that showed a weak effect of electromagnetic induction, which was largely disregarded at the time.[15]

A.4 Gauss

In 1867 Gauss posthumously published an analysis that he dated to 1835 (p. 609 of [48]), in which he stated that a time-dependent electric current leads to an electric field which is the

where l_1 and l_2 measure distance along the corresponding circuits in the directions of their currents. Then,

$$d\mathbf{l}_1 \cdot d\mathbf{l}_2 = -d\mathbf{l}_1 \cdot \frac{\partial \mathbf{r}}{\partial l_2}\, dl_2 = -\frac{\partial}{\partial l_2}(\mathbf{r} \cdot d\mathbf{l}_1)\, dl_2 = -\frac{\partial}{\partial l_2}\left(r \frac{\partial r}{\partial l_1}\right) dl_1\, dl_2 = -\left(\frac{\partial r}{\partial l_1}\frac{\partial r}{\partial l_2} + r \frac{\partial^2 r}{\partial l_1 \partial l_2}\right) dl_1\, dl_2, \qquad (22)$$

and eq. (19) can also be written in forms closer to those used by Ampère,

$$d^2\mathbf{F}_{\text{on }1} = \frac{\mu_0}{4\pi} I_1 I_2 dl_1 dl_2 \left[2r \frac{\partial^2 r}{\partial l_1 \partial l_2} - \frac{\partial r}{\partial l_1}\frac{\partial r}{\partial l_2}\right] \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi} 2 I_1 I_2 dl_1 dl_2 \frac{\partial^2 \sqrt{r}}{\partial l_1 \partial l_2} \frac{\hat{\mathbf{r}}}{\sqrt{r}} = -d^2\mathbf{F}_{\text{on }2}. \qquad (23)$$

[12]If we follow Ampère in defining a "current element" as being electrically neutral, which is a good (but not exact [84]) approximation for currents in electrical circuits, then a moving charge is not a "current element", and such elements cannot exist except in closed circuits (contrary to remarks such as in [75]).

[13]Note that for a fixed point 2, $d\mathbf{l}_1 = d\mathbf{r}$, and $dr = d\mathbf{r} \cdot \hat{\mathbf{r}} = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. Then, for any function $f(r)$, $df = (df/dr)\, dr = (df/dr)\, d\mathbf{l}_1 \cdot \hat{\mathbf{r}}$. In particular, for $f = -1/r$, $df = d\mathbf{l}_1 \cdot \hat{\mathbf{r}}/r^2$, so the first term of the first form of eq. (25) is a perfect differential with respect to $\mathbf{l}_1$. Hence, when integrating around a closed loop 1, the first term does not contribute, and it is sufficient to write (as first noted by Neumann, p. 67 of [38]),

$$\mathbf{F}_{\text{on }1} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r^2}\, \hat{\mathbf{r}} = -\mathbf{F}_{\text{on }2}. \qquad (24)$$

[14]As a consequence, the form (25) is generally attributed to Grassmann [37], as in [82], for example.

[15]Reviews of this experiment include [57, 63, 71, 72].

time derivative of what we now called the vector potential. English translation from [76]:

> The Law of Induction
> Found out Jan. 23, 1835, at 7 a.m. before getting up.
>
> 1. The electricity producing power, which is caused in a point P by a current-element $\boldsymbol{\gamma}$, at a distance from P, $= r$, is during the time dt the difference in the values of $\boldsymbol{\gamma}/r$ corresponding to the moments t and dt, divided by dt. where $\boldsymbol{\gamma}$ is considered both with respect to size and direction. This can be expressed briefly and clearly by
>
> $$-\frac{\mathrm{d}(\boldsymbol{\gamma}/r)}{\mathrm{d}t}\,. \tag{26}$$

Gauss' unpublished insight that electromagnetic induction is related to the negative time derivative of a scalar quantity was probably communicated in the late 1830's to his German colleagues, of whom Weber was the closest.

On p. 612 (presumably also from 1835), Gauss noted a relation (here transcribed into vector notation) between the vector $\mathbf{A} = \oint d\mathbf{l}/r$ and the magnetic scalar potential Ω of a circuit with unit electrical current (which he related to the solid angle subtended by the circuit on p. 611),

$$-\boldsymbol{\nabla}\Omega = \boldsymbol{\nabla} \times \mathbf{A}. \tag{27}$$

While we would identify eq. (27) with the magnetic field $\mathbf{H}$, Gauss called it the "electricity-generating force".

In any case, eq. (27) is the earliest appearance of the curl operator (although published later than MacCullagh's use of this, p. 22 of [32].

A.5 Grassmann

In 1845, Grassmann [37] remarked that although Ampère claimed [28] that his force law was *uniquement déduite de l'expérience*, it included the assumption that it obeyed Newton's third law. He noted that Ampère's law (19) implies that the force is zero for parallel current elements whose lie of centers makes angle $\cos^{-1}\sqrt{2/3}$ to the direction of the currents, which seemed implausible to him. Grassmann claimed that, unlike Ampère, he would make no "arbitrary" assumptions, but in effect he assumed that there is no magnetic force between collinear current elements, which leads to a force law,

$$\mathbf{F}_{\mathrm{on}\ 1} = \oint_1 \oint_2 d^2\mathbf{F}_{\mathrm{on}\ 1}, \qquad d^2\mathbf{F}_{\mathrm{on}\ 1} = \frac{\mu_0}{4\pi} I_1 d\mathbf{l}_1 \times \frac{I_2 d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}\,, \tag{28}$$

in vector notation (which Grassmann did not use in [37], although he invented the notion of an exterior product of vectors in an n-dimensional space [34]). While $d^2\mathbf{F}_{\mathrm{on}\ 1}$ is not equal and opposite to $d^2\mathbf{F}_{\mathrm{on}\ 2}$, Grassmann showed that the total force on circuit 1 is equal and opposite to that on circuit 2, $\mathbf{F}_{\mathrm{on}\ 1} = -\mathbf{F}_{\mathrm{on}\ 2}$.

Grassmann's result is now called the Biot-Savart force law, eqs. (17)-(18),

$$\mathbf{F}_{\mathrm{on}\ 1} = \oint_1 I_1\, d\mathbf{l}_1 \times \mathbf{B}_2, \qquad \mathbf{B}_2 = \frac{\mu_0}{4\pi} \oint_2 \frac{I_2\, d\mathbf{l}_2 \times \hat{\mathbf{r}}}{r^2}\,, \tag{29}$$

although Grassmann did not identify the quantity $\mathbf{B}_2$ as the magnetic field.

In 1844, Grassmann [34] invented linear algebra, which went largely unrecognized for 30 years. In 1877 [53], he related this to the algebra of Hamilton's quaternions [47], which together with a paper by Clifford (1878) [54], form the basis of contemporary geometric algebra.

A.6 Neumann

In 1845, Neumann [38] inferred from Lenz' law [31] that a conduction line segment $d\mathbf{s}$ that moves with velocity $\mathbf{v}$ near a magnet experiences a (scalar) electromotive force $d\mathcal{E}$ (with dimensions of energy) of the form (eq. (1), p. 15 of [38]),

$$d\mathcal{E} = E.Ds = -\epsilon v C.Ds, \tag{30}$$

where ϵ is a constant and C is a function of the magnet (and its geometric relation to the conducting line segment).

Although $\mathcal{E}$ is a scalar, Neumann had an intuition that the other quantities in his equation were not simply scalars but rather were what we now call vectors. For example, at the bottom of p. 66 he considered two line segments of lengths Ds and $D\sigma$ with components (Dx, Dy, Dz) and $D\xi, D\eta, D\zeta)$ with respect to rectangular axes, and mentioned the equality $DxD\xi + DyD\eta + DzD\zeta = \cos(D\sigma, Ds)D\sigma Ds$, we recognize as the scalar product $D\boldsymbol{\sigma} \cdot D\mathbf{s}$ of vectors $D\boldsymbol{\sigma}$ and $D\mathbf{s}$.

This has led many people to transcribe eq. (30) as,

$$d\mathcal{E} = \mathbf{E} \cdot D\mathbf{s}s = \mathbf{v} \times \mathbf{B} \cdot D\mathbf{s}, \tag{31}$$

and to credit Neumann with having been the first to identify the motional $\mathcal{EMF} = \oint_{\mathrm{loop}} \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$, although, like Ampère, Neumann had no concept of a magnetic (or electric) field.

On p. 67 of [38], Neumann discussed (what is now written as) the form (24), and verified that it gives the same total force between closed circuits as does Ampère's eq. (19).

Neumann did appreciate that electromagnetic induction occurs not only for moving conductors near a fixed magnet, but also for conductors at rest near a time-dependent magnet. This leads some people to credit him with also having stated the generalized flux law, $\mathcal{EMF} = -d\Phi_{\mathbf{B}}/dt$.

Following the examples of Lagrange, Laplace and Poisson in relating forces of gravity and electrostatics to potentials, Neumann sought a potential for Ampère's force law between two (closed) current loops. For this, he noted that this force law can be rewritten in the form (24), which permits us to write $\mathbf{F}_{\mathrm{on1}} = -\boldsymbol{\nabla} U$ where, U is the scalar potential (energy) given on p. 67 of [38],[16]

$$U = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} \tag{32}$$

in SI units. We now also write this as,

$$U = I_i \oint_i d\mathbf{l}_i \cdot \mathbf{A}_j = I_i \int d\mathbf{Area}_i \cdot \boldsymbol{\nabla} \times \mathbf{A}_j = I_i \int d\mathbf{Area}_i \cdot \mathbf{B}_j = I_i \Phi_{ij}, \tag{33}$$

[16]If we write eq. (32) as $U = I_1 I_2 M_{12}$, then M_{12} is the mutual inductance of circuits 1 and 2.

where Φ_{ij} is the magnetic flux through circuit i due to the current I_j in circuit j, and,

$$\mathbf{A}_j = \frac{\mu_0}{4\pi} \oint_j \frac{I_j \, d\mathbf{l}_j}{r} \,, \tag{34}$$

such that Neumann is often credited in inventing the vector potential $\mathbf{A}$, although he appears not to have written his eq. (32) in any of the forms of eq. (33). Yet, we can say that while Neumann had no concept of the magnetic field, he did emphasize a quantity with the significance of magnetic flux linked by a circuit.[17]

One application of his potential was given by Neumann in his eq. (4), p. 65 of [38]), which he expressed verbally on p. 68. This concerned the time integral (*Stromintegral*) of the current I_1 in one circuit, (with electrical resistance R_1, called $1/\epsilon'$ by Neumann), due to the motion of a second circuit whose current I_2 remains constant,

$$\int I_1 \, dt = I_2 \frac{V_i - V_f}{R_1} \,, \qquad \text{where} \qquad V = \frac{\mu_0}{4\pi} \oint_i \oint_j \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{r} \tag{36}$$

and $V_{i,f}$ are the initial and final potentials for unit currents (*i.e.*, the initial and final mutual inductances). This result has the implication that the $\mathcal{EMF}$ induced in circuit 1 due to the motion of circuit 2 has the form,

$$\mathcal{E} = -I_2 \frac{dV}{dt} = -\frac{d}{dt}(I_2 V) = -\frac{d\Phi_{1,2}}{dt} \,, \tag{37}$$

where $\Phi_{1,2}$ is the magnetic flux linked by circuit 1 due to the current in circuit 2. This was not explicitly stated by Neumann in [38], but apparently many astute readers did infer that the induced $\mathcal{EMF}$ is the negative time derivative of an appropriate version of Neumann's potential. Hence, Neumann is often credited with formulating an early version of what we now call Faraday's law.

A.7 Weber

The term unipolar induction for Faraday's homopolar dynamo is due to Weber (1839) [33].

Weber was perhaps the last major physicist who did not use electric and magnetic fields to describe electromagnetism, postulating instead an (instantaneous) action-at-a-distance formulation for the (central) force between charges (1846, p. 376 of [39], p. 144 of [56]),[18]

$$\mathbf{F}_{\text{Weber}} = -\frac{ee'}{r^2} \left[1 - \frac{a^2}{16} \left(\frac{\partial r}{\partial t} \right)^2 + \frac{a^2 r}{8} \frac{\partial^2 r}{\partial t^2} \right] \hat{\mathbf{r}} \tag{38}$$

[17]Note also that using our eq. (22), due to Ampère, the magnetic flux can be written as,

$$\Phi_{ij} = \frac{\mu_0 I_j}{4\pi} \oint_i \oint_j \frac{d\mathbf{l}_i \cdot d\mathbf{l}_j}{r} = -\frac{\mu_0 I_j}{4\pi} \oint_i \oint_j \frac{\partial r}{\partial l_1} \frac{\partial r}{\partial l_j} \frac{dl_i \, dl_j}{r} = -\frac{\mu_0 I_j}{4\pi} \oint_i \oint_j \frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})}{r} \,, \tag{35}$$

since the integral of $\partial^2 r / \partial l_i \partial l_j$ around a closed loop vanishes, and the third form follows from the second recalling eq. (21).

[18]For an extensive discussion of Weber's electrodynamics, see [78]. Maxwell gave a review of the German school of electrodynamics of the mid 19th century in the final chapter 23 of his *Treatise* [50].

where $\mathbf{r} = \mathbf{r}_{e'} - \mathbf{r}_e$. This was the first published force law for moving charges (which topic Ampère refused to speculate upon). The constant a has dimensions of velocity^{-1}, and was later (1856) written by Weber and Kohlsrausch, p. 20 of [46], as $4/C$, who noted that their C is the ratio of the magnetic units to electrical units in the description of static phenomenon, which they determined experimentally to have a value close to 4.4×10^8 m/s. Apparently, they regarded it as a coincidence that their C is roughly $\sqrt{2}$ times the speed c of light.

In 1848, p. 229 of [41], Weber related his force law (38) to the (velocity-dependent) potential,[19,20]

$$U_{\text{Weber}} = \frac{ee'}{r}\left[1 - \frac{a^2}{16}\left(\frac{dr}{dt}\right)^2\right]. \tag{40}$$

Weber showed that his eq. (38) can be used to deduce Ampère's force law, as well as Neumann's potential (32) (as also discussed by the latter in sec. 5 of his 1847 paper [40]).[21] In secs. 28 and 30 of [39], Weber consider the induction in a current element $d\mathbf{l}'$ due to a variable current I in element $d\mathbf{l}$, with both elements at rest, and deduced that,

$$d^2\mathcal{E}' = -\frac{\mu_0}{4\pi}\frac{(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})(d\mathbf{l}_1 \cdot \hat{\mathbf{r}})}{r}\frac{dI}{dt}, \quad \text{whose integral is} \quad \mathcal{E}' = -\frac{\mu_0}{4\pi}\oint\oint \frac{d\mathbf{l} \cdot d\mathbf{l}'}{r}\frac{dI}{dt} = -M\frac{dI}{dt}, \tag{41}$$

recalling eq. (35), and where M is the mutual inductance in the case the elements $d\mathbf{l}$ and $d\mathbf{l}'$ are in different circuits. Weber seemed little concerned with circuit analysis, did not mention integral form of eq. (41), and downplayed the merits of Neumann's "potential" compared to his own, eq. (40). However, it may be the Weber was the first to explicitly note that induced $\mathcal{EMF}$'s are proportional to a time derivative (although this result is implicit in Faraday's discussion, starting with that in [29]).

Weber's electrodynamics was more ambitious than that of Neumann's (which was tacitly restricted to quasistatic examples with low-velocity charges), as Weber sought to describe charges with arbitrary velocities and accelerations. Neumann's contributions, within their realm of applicability have aged well, while Weber's electrodynamics is largely forgotten as it does not contain electromagnetic radiation.

Weber's most lasting contribution to electrodynamics was his theory of paramagnetism [42, 44], developed following Faraday's studies of diamagnetism [35, 36] and paramagnetism [43] (which terms were coined by Faraday). Weber considered that paramagnetic "atoms" are objects with a permanent electric current circulating around a diameter.[22]

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[19]Weber had in effect deduced this potential in 1846, p. 375 of [39], but did not identify it as such.

[20]Weber's force law (38) has the form $\mathbf{F} = f(r,t), \hat{\mathbf{r}}$, where $\mathbf{r} = \mathbf{x} - \mathbf{x}'$ with $\mathbf{x}(\mathbf{x}')$ being the positions of charges $e(e')$. If the force can be deduced from a potential U according to $\mathbf{F} = -\boldsymbol{\nabla}U$, we expect that U is also a function only of r and t. In this case $[\boldsymbol{\nabla}U]_x = \partial V/\partial x = (\partial U/\partial r)\partial r/\partial x = (\partial U/\partial r)(x - x')/r$. That is, $-\boldsymbol{\nabla}U = -(\partial U/\partial r)\,\hat{\mathbf{r}}$. For Weber's potential (40), we have,

$$-\frac{\partial U_{\text{Weber}}}{\partial r} = \frac{ee'}{r^2}\left[1 - \frac{a^2}{16}\left(\frac{\partial r}{\partial t}\right)^2 + \frac{a^2}{8}r\frac{\partial r}{\partial t}\frac{\partial}{\partial r}\frac{\partial r}{\partial t}\right] \qquad \text{and} \qquad \frac{\partial}{\partial r}\frac{\partial r}{\partial t} = \frac{\partial t}{\partial r}\frac{\partial^2 r}{\partial t^2}, \tag{39}$$

so indeed $\mathbf{F}_{\text{Weber}} = -\boldsymbol{\nabla}U_{\text{Weber}}$.

[21]A comparison of the theories of Neumann and Weber is given in sec. 2 of [77], which also includes a review of Helmholtz' later theories [51].

[22]This view was discussed by Faraday (1854) [45], and by Maxwell in Art. 843 of [50].

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